强子分子态

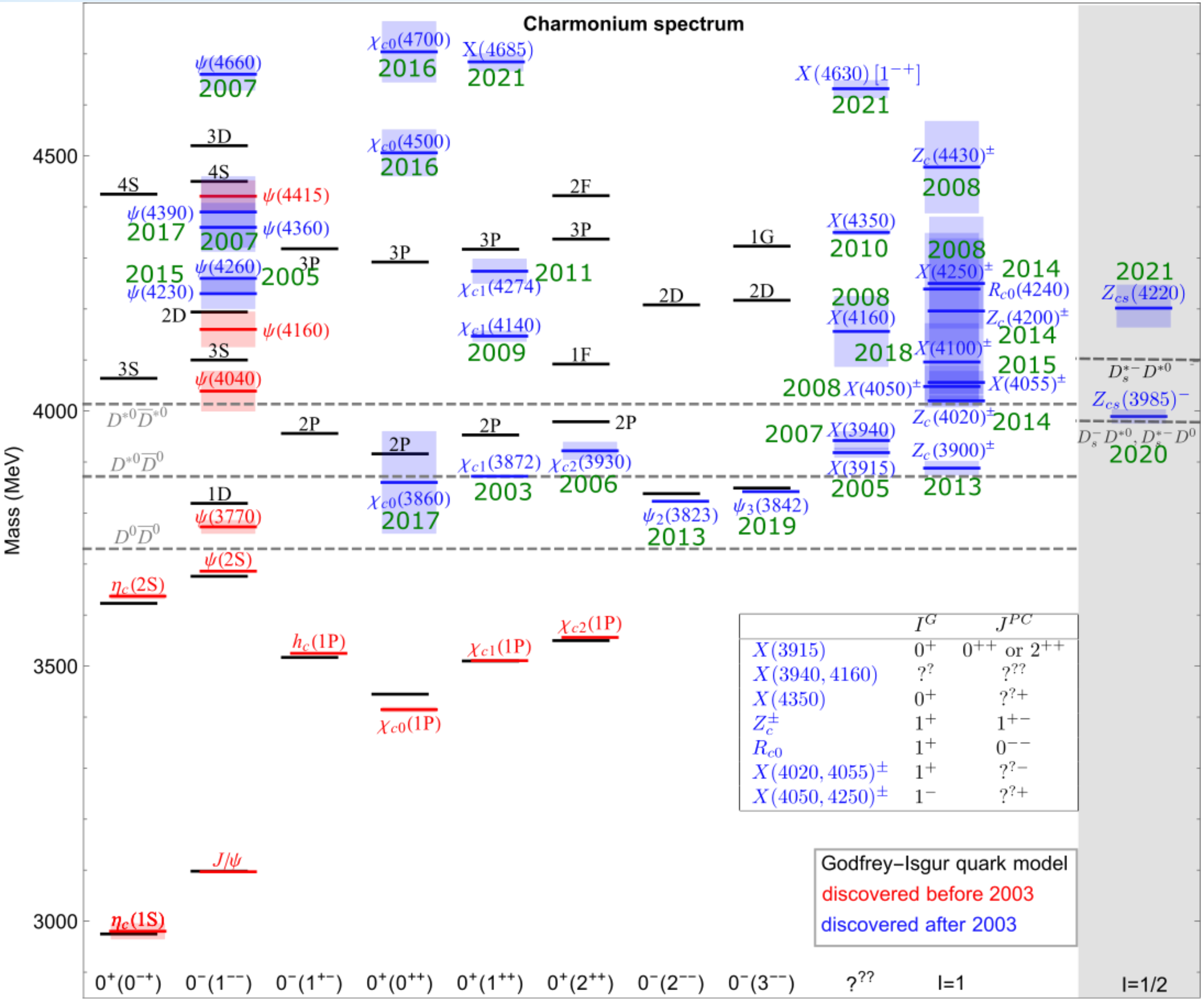
郭奉坤

中国科学院理论物理研究所

BEPCII升级暨BESIII物理联合研讨会

25 April 2021

Charmonium-like structures



	I^G	J^{PC}
$X(3915)$	0^+	0^{++} or 2^{++}
$X(3940, 4160)$	$?^?$	$?^{??}$
$X(4350)$	0^+	$?^{?+}$
$Z_c^\pm$	1^+	1^{+-}
R_{c0}	1^+	0^{--}
$X(4020, 4055)^\pm$	1^+	$?^{?-}$
$X(4050, 4250)^\pm$	1^-	$?^{?+}$

Charmonium-like structures

$B \to J/\psi\phi K$

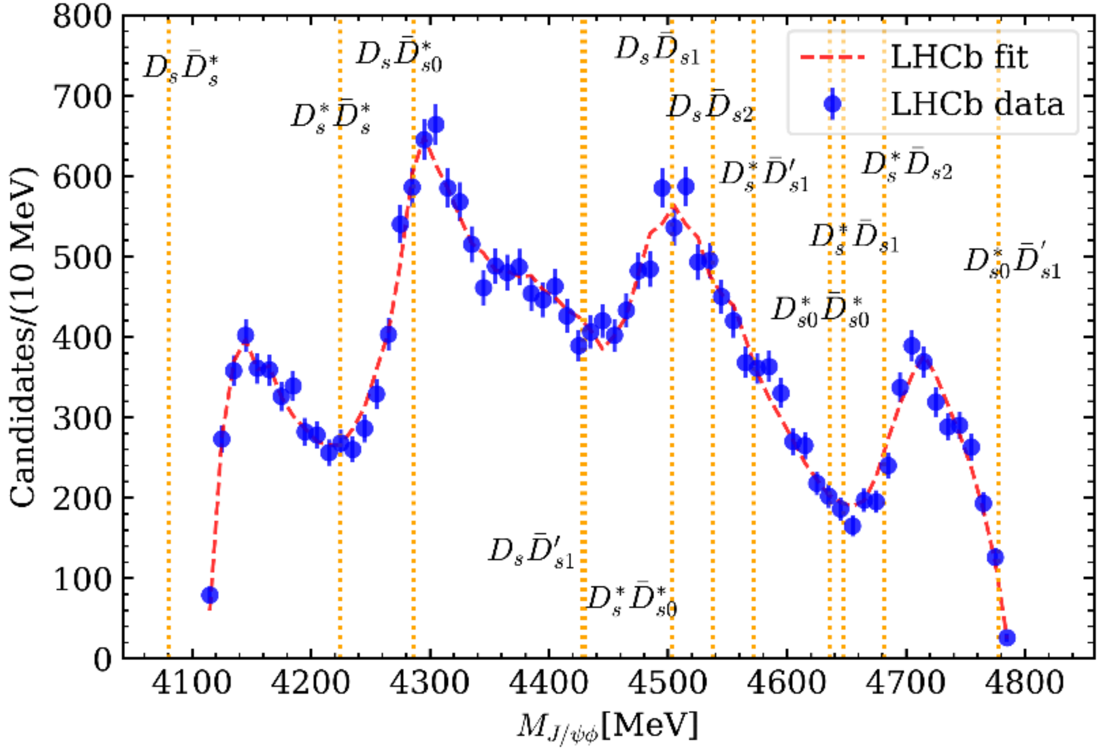


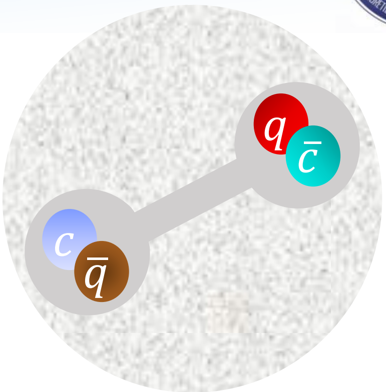
Figure modified from
Dong, FKG, Zou, 2101.01021
w/ new LHCb data, arXiv:2103.01803

Some recent reviews

Lots of new hadron resonances and resonance-like structures were found since 2003

- H.-X. Chen et al., *The hidden-charm pentaquark and tetraquark states*, Phys. Rept. 639 (2016) 1 [arXiv:1601.02092]
- A. Hosaka et al., *Exotic hadrons with heavy flavors — X, Y, Z and related states*, Prog. Theor. Exp. Phys. 2016, 062C01 [arXiv:1603.09229]
- R. F. Lebed, R. E. Mitchell, E. Swanson, *Heavy-quark QCD exotica*, Prog. Part. Nucl. Phys. 93 (2017) 143 [arXiv:1610.04528]
- A. Esposito, A. Pilloni, A. D. Polosa, *Multiquark resonances*, Phys. Rept. 668 (2017) 1 [arXiv:1611.07920]
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004 [arXiv:1705.00141]
- S. L. Olsen, T. Skwarnicki, *Nonstandard heavy mesons and baryons: Experimental evidence*, Rev. Mod. Phys. 90 (2018) 015003 [arXiv:1708.04012]
- M. Karliner, J. L. Rosner, T. Skwarnicki, *Multiquark states*, Ann. Rev. Nucl. Part. Sci. 68 (2018) 17 [arXiv:1711.10626]
- C.-Z. Yuan, *The XYZ states revisited*, Int. J. Mod. Phys. A 33 (2018) 1830018 [arXiv:1808.01570]
- N. Brambilla et al., *The XYZ states: experimental and theoretical status and perspectives*, Phys. Rept. 873 (2020) 154 [arXiv:1907.07583]
- F.-K. Guo, X.-H. Liu, S. Sakai, *Threshold cusps and triangle singularities in hadronic reactions*, Prog. Part. Nucl. Phys. 112 (2020) 103757 [arXiv:1912.07030]

Hidden-charm states



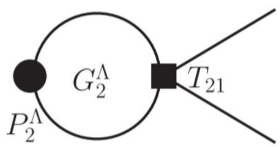
- Masses of excited hadrons:
  - Radial excitations?
  - Excitation of light quark-antiquark pairs?
  - Hadron-hadron pairs?
    - Indication from large N_c analysis: $\frac{V_{qq[\bar{3}]}}{V_{q\bar{q}[1]}} = \frac{1}{N_c - 1}$ Lucha et al., 2012.02542
    - Indication from functional method (DS and BS equations) Eichmann et al., 2008.10240
    - Implication of confinement (large-size systems in favor of color-singlet clusters)?
    - Experimental evidence? Hidden-charm states

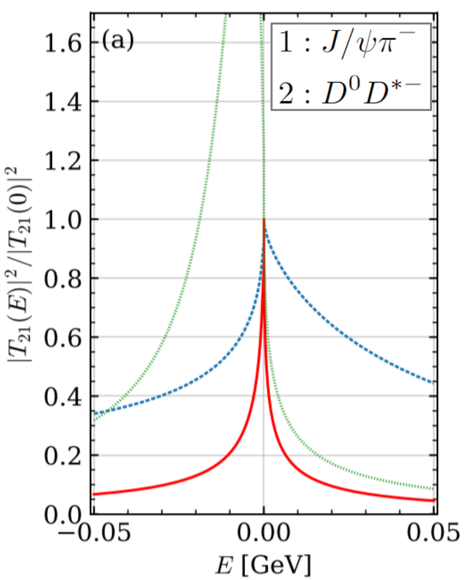
- Indication of the importance of hadron-hadron channels in charmonium-like states
  - Higher statistics for energy region below ~4.8 GeV (BESIII, B decays)
  - Higher energies covering charmed baryon-antibaryon thresholds (up to 5.6 GeV)

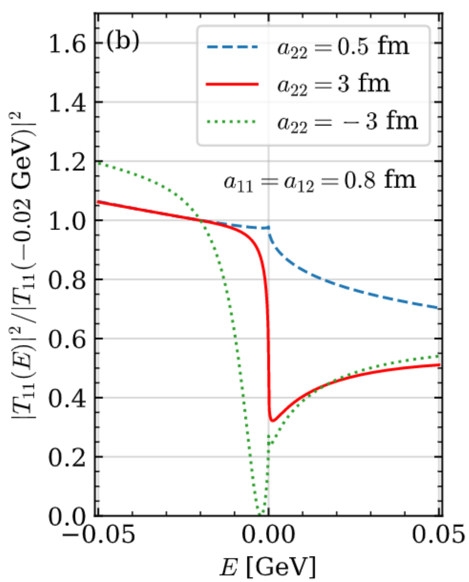
(Near-)threshold structures

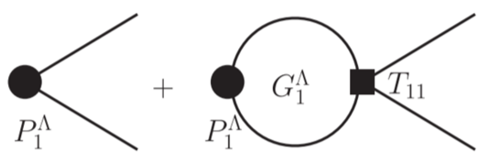
X.-K. Dong, FKG, B.-S. Zou, Phys. Rev. Lett. 126 (2021) 152001

- (Near-)threshold structures (S-wave)
  - There must be nontrivial (near-)threshold structures for attractive interaction
  - Either threshold cusp or below-threshold peak
  - Peak more pronounced for heavier hadrons and stronger interaction
  - That’s the why many (near-)threshold structures were observed in hidden-charm and hidden-bottom spectra
  - Structures are process dependent









Poles in complex momentum plane:

$(0.37 - i0.08)$GeV

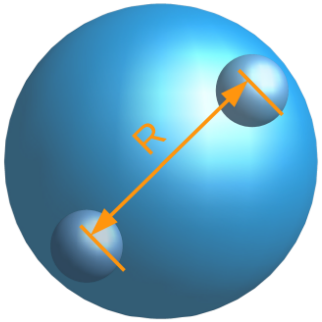
$(0.04 - i0.08)$GeV

$(-0.09 - i0.08)$GeV

Hadronic molecules

- Hadronic molecule: analogues of deuteron and other light nuclei
  dominant component is a composite state of 2 or more hadrons; extended
- Concept at large distances, so that can be approximated by system of multi-hadrons at low energies
  Consider a 2-body bound state with a mass $M = m_1 + m_2 - E_B$

  size: $R \sim \frac{1}{\sqrt{2\mu E_B}} \gg r_{\text{hadron}}$



  $\Rightarrow$ well-separated scales, nonrelativistic EFT
- Only narrow hadrons can be considered as components of hadronic molecules,
  $\Gamma_h \ll 1/r$, r: range of forces

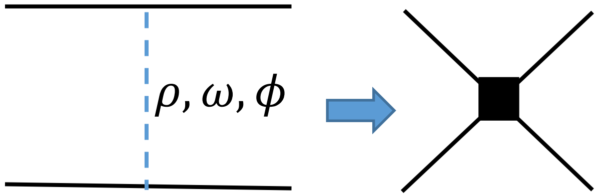
FKG, Meißner, PRD84(2011)014013; see also Filin *et al.*, PRL105(2010)019101

Survey of hidden-charm hadronic molecules

X.-K. Dong, FKG, B.-S. Zou, 《物理学进展》41 (2021) 65 [arXiv:2101.01021]

- Approximations:
  - Constant contact terms (V) saturated by light-vector-meson exchange, similar to the VMD in the resonance saturation of the low-energy constants in CHPT

    G. Ecker, J. Gasser, A. Pich, E. de Rafael, NPB321(1989)311
  - Single channels
  - Neglecting mixing with normal charmonia



- Resummation:

$$T = \frac{V}{1 - VG}$$

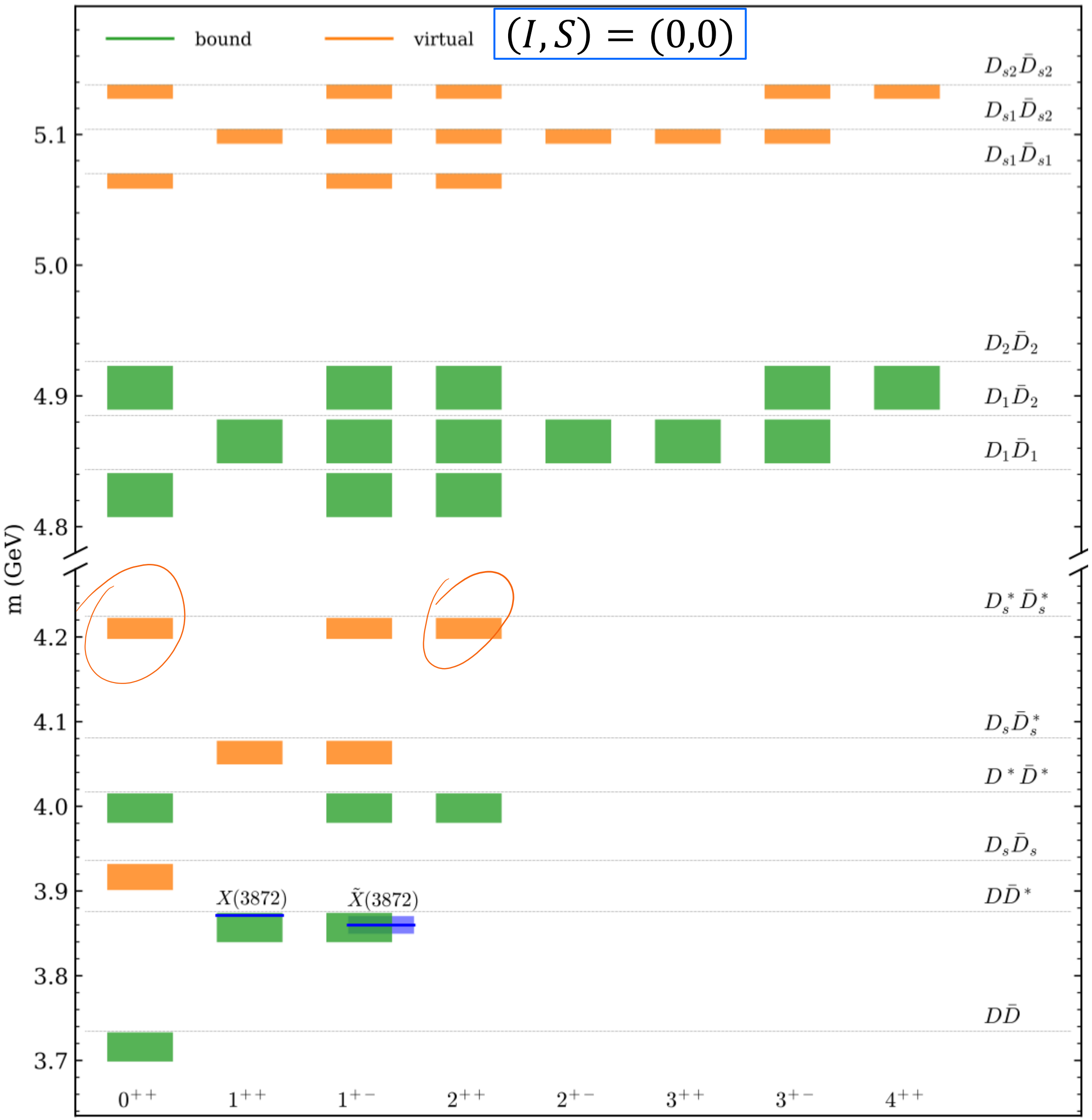
G: two-point scalar loop integral regularized using dim.reg. with a subtraction constant matched to a Gaussian regularized G at threshold

$$G(E) = \frac{1}{16\pi^2}\left\{a(\mu) + \log\frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s}\log\frac{m_2^2}{m_1^2} + \frac{k}{E}\log\frac{(2kE + s)^2 - m_1^2 + m_2^2}{(2kE - s)^2 - m_1^2 + m_2^2}\right\}$$

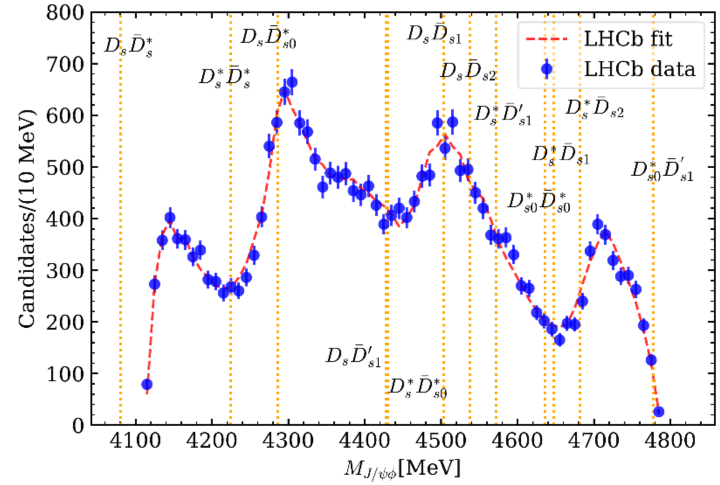
$$G(E) = \int \frac{l^2 dl}{4\pi^2}\frac{\omega_1 + \omega_2}{\omega_1\omega_2}\frac{e^{-2l^2/\Lambda^2}}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon} \quad \text{with } \omega_i = \sqrt{m_i^2 + l^2}$$

- Hadronic molecules appear as bound or virtual state poles of the T matrix

X(3872) and related states



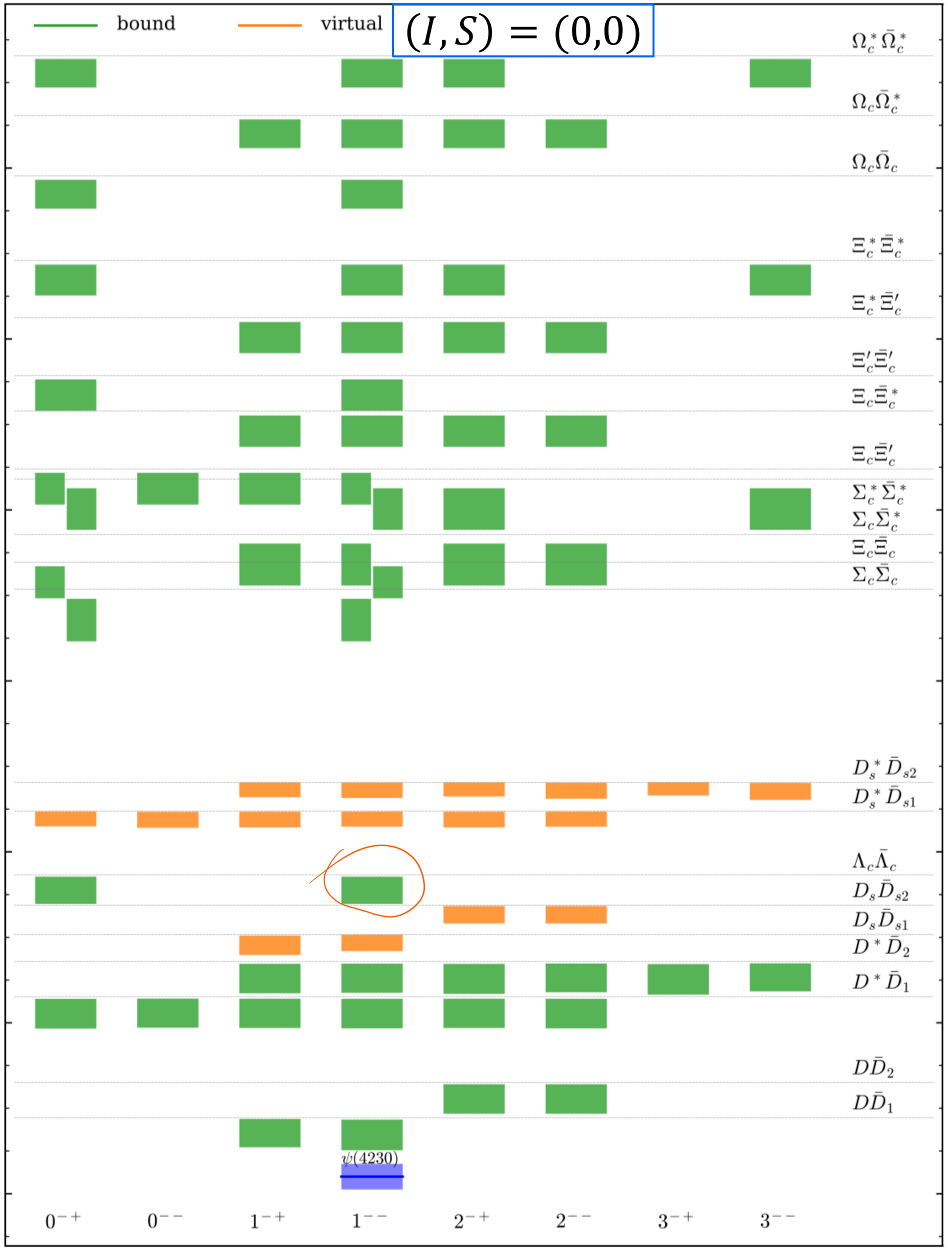
- ✓ $X(3872)$ as a $\bar{D}D^*$ bound state
- ✓ Negative-C parity partner observed by COMPASS PLB783(2018)334
- ✓ $\bar{D}D$ bound state predicted with lattice Prelovsek et al., 2011.02542
- ✓ Evidence for a $D_s^*\bar{D}_s^*$ virtual state in LHCb data?



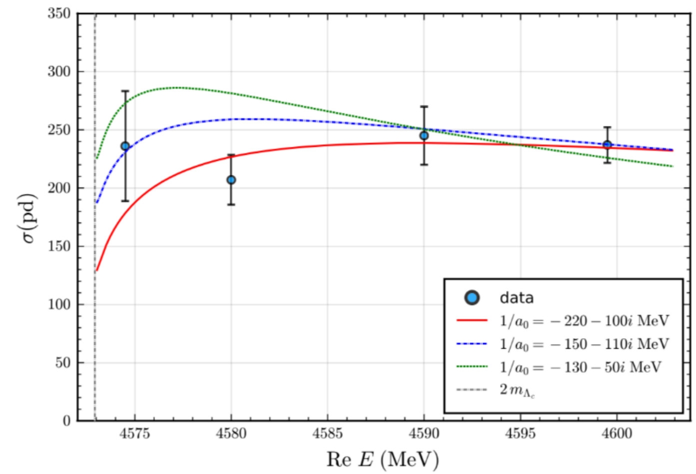
LHCb data, arXiv:2103.01803

- ✓ $e^+e^- \to J/\psi\phi\phi$, $D_s^{(*)}\bar{D}_s^{(*)}\phi$ up to 5.6 GeV covers [4.12, 4.58] GeV

Isoscalar vectors and related states



- ✓ $Y(4260)/\psi(4230)$ as a $\bar{D}D_1$ bound state
- ✓ Vector charmonia around 4.4 GeV unclear
- ✓ Evidence for 1^{--} $\Lambda_c\bar{\Lambda}_c$ bound state in BESIII data
  - Sommerfeld factor
  - Near-threshold pole
  - Different from $Y(4630/4660)$



Data taken from BESIII, PRL120(2018)132001

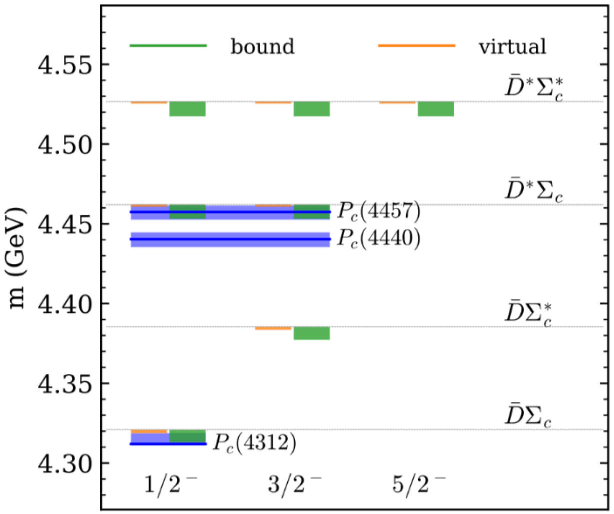
- ✓ Many 1^{--} states in [4.8, 5.6] GeV

Isoscalar vectors and related states

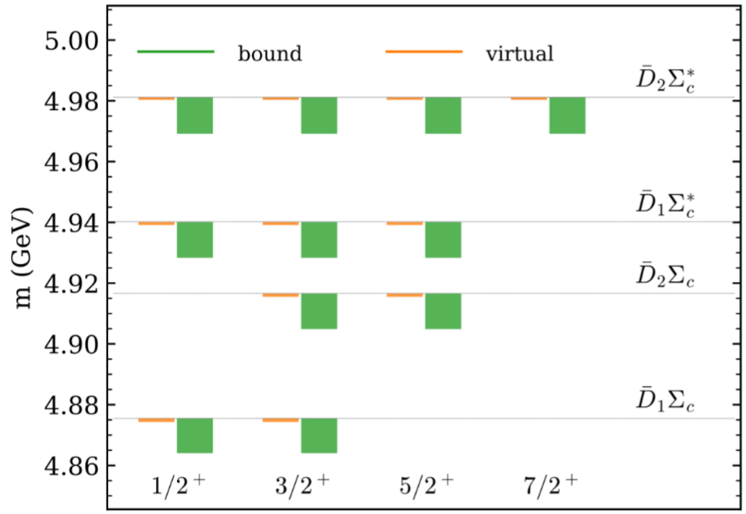
	Threshold [MeV]	J^{PC}	Binding energy [MeV] $\Lambda = [0.5,$	$1.0]$ GeV
$\Lambda_c\bar{\Lambda}_c$	4573	$0^{-+}, 1^{--}$	1.98	33.8
$\Sigma_c\bar{\Sigma}_c$	4907	$0^{-+}, 1^{--}$	11.1	60.8
$\Xi_c\bar{\Xi}_c$	4939	$0^{-+}, 1^{--}$	4.72	42.2
$\Sigma_c^*\bar{\Sigma}_c$	4972	$1^{-\pm}, 2^{-\pm}$	11.0	60.1
$\Sigma_c^*\bar{\Sigma}_c^*$	5036	$(0,2)^{-+}, (1,3)^{--}$	10.9	59.5
$\Xi_c\bar{\Xi}_c'$	5048	$0^{-\pm}, 1^{-\pm}$	4.79	41.9
$\Xi_c\bar{\Xi}_c^*$	5115	$1^{-\pm}, 2^{-\pm}$	4.84	41.6
$\Xi_c'\bar{\Xi}_c'$	5158	$0^{-+}, 1^{--}$	4.87	41.5
$\Xi_c^*\bar{\Xi}_c'$	5225	$1^{-\pm}, 2^{-\pm}$	4.91	41.3
$\Xi_c^*\bar{\Xi}_c^*$	5292	$(0,2)^{-+}, (1,3)^{--}$	4.95	41.0
$\Omega_c\bar{\Omega}_c$	5390	$0^{-+}, 1^{--}$	4.17	38.0
$\Omega_c^*\bar{\Omega}_c$	5461	$1^{-\pm}, 2^{-\pm}$	4.22	37.8
$\Omega_c^*\bar{\Omega}_c^*$	5532	$(0,2)^{-+}, (1,3)^{--}$	4.26	37.6

Hidden-charm pentaquarks

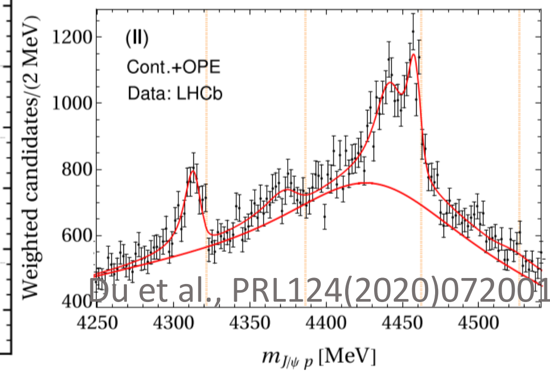
$(I,S) = (1/2,0)$



$(I,S) = (1/2,0)$

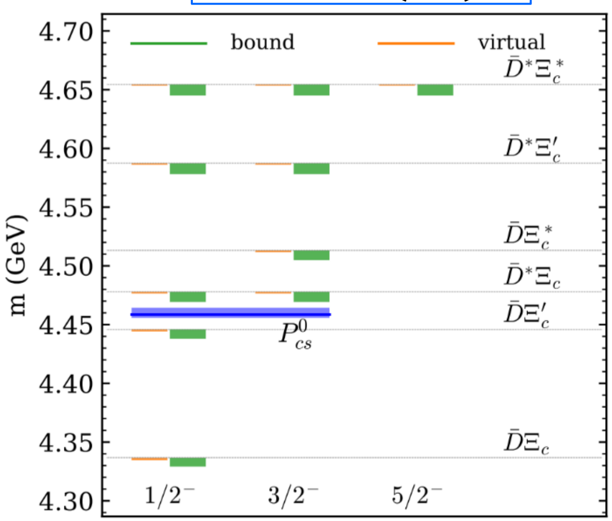


- ✓ The LHCb P_c states as $\bar{D}^{(*)}\Sigma_c$ molecules
- ✓ $\bar{D}\Sigma_c^*$ molecule: hint in the LHCb data

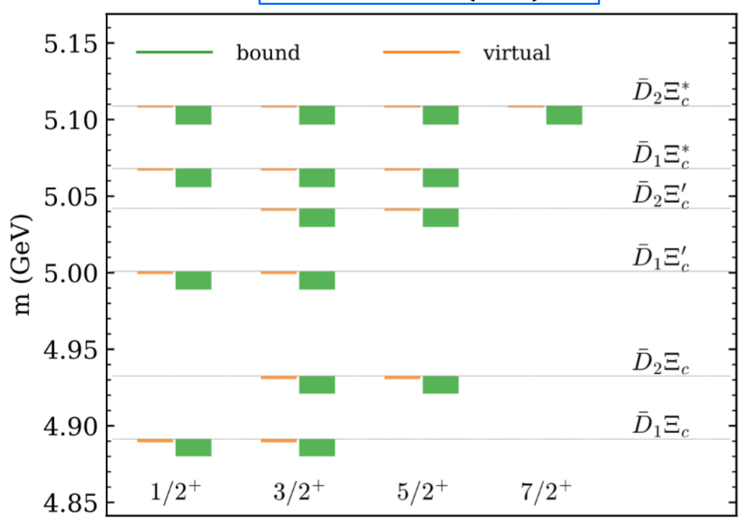


Du et al., PRL124(2020)072001

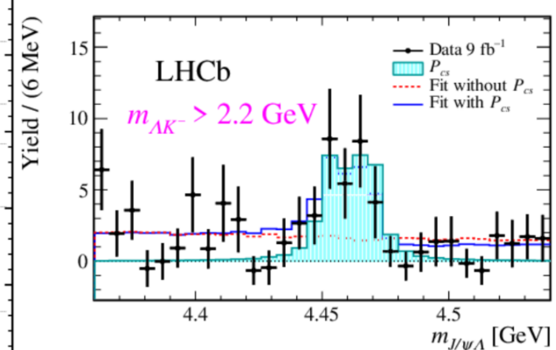
$(I,S) = (0,1)$



$(I,S) = (0,1)$



- ✓ The $P_{cs}(4459)$ could be two $\bar{D}^*\Xi_c$ molecules



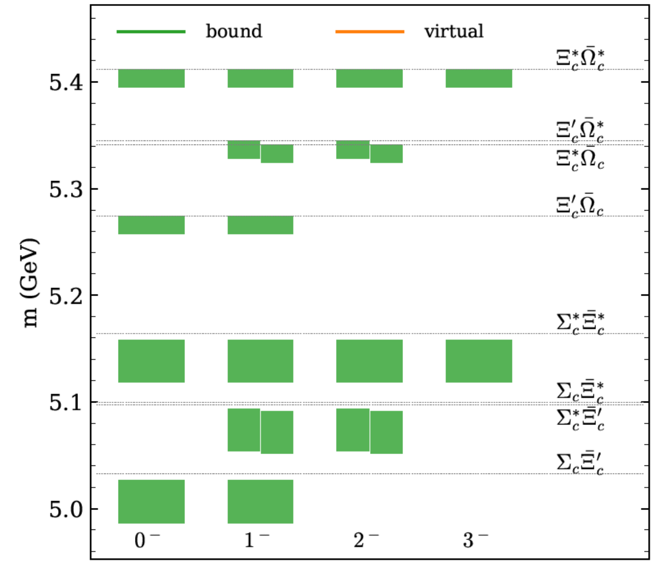
LHCb, 2012.10380

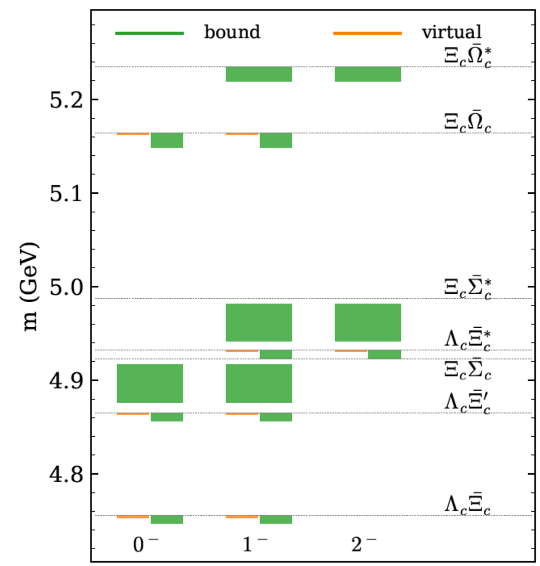
$$e^+e^- \to J/\psi p\bar{p}, \Lambda_c\bar{D}^{(*)}p, J/\psi\Lambda\bar{\Lambda}, \Sigma_c^{(*)}\bar{D}^{(*)}p, \ldots$$

More states with exotic quantum numbers

$(I,S) = (1,0)$

$(I,S) = (1/2,1)$



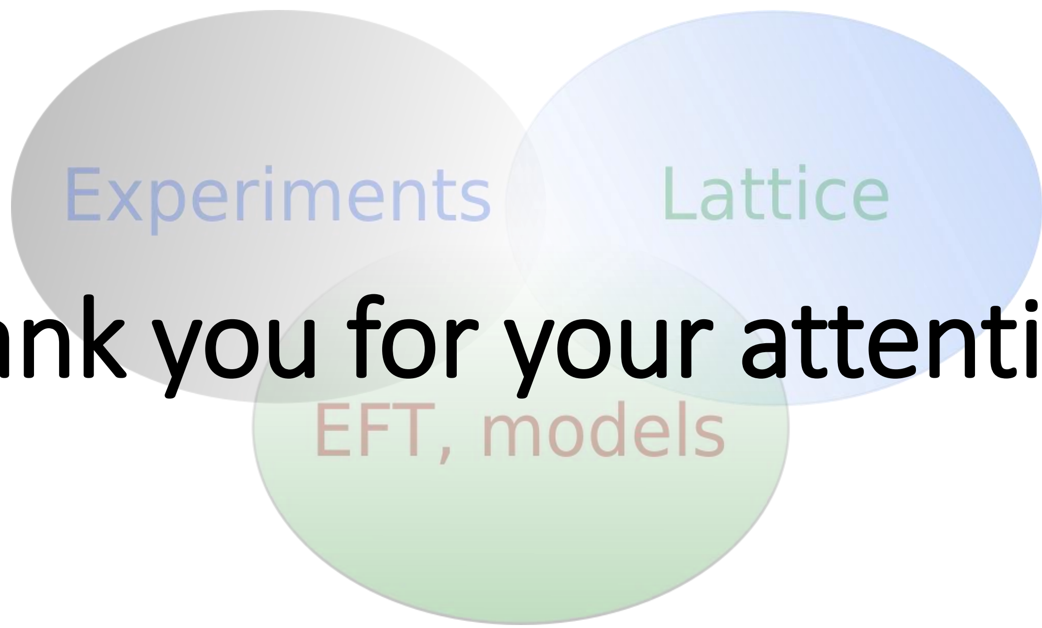


- ✓ Many baryon-antibaryon molecular states above 4.7 GeV, beyond the current exp. region

$J/\psi\pi, J/\psi K, \ldots$

Conclusion

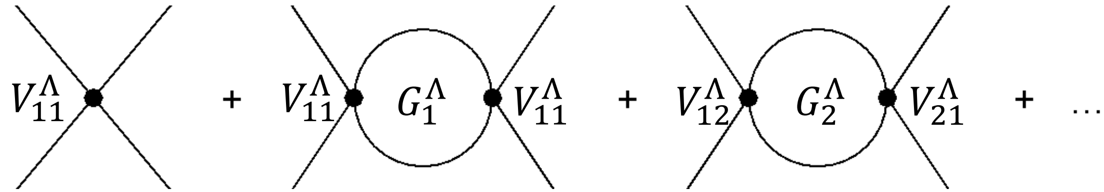
- e^+e^- up to 5.6 GeV highly desirable for understanding hidden-charm spectrum
  - (Near-)threshold structures expected for any pair with S-wave attraction
  - Many hidden-charm molecules are expected:
    - Meson-meson pairs via emission of light meson: $e^+e^- \to M_1M_2(\phi, \omega, \rho, \pi, \eta, K^{(*)})$
    - Many baryon-antibaryon pairs above 4.8 GeV: $e^+e^- \to B_1\bar{B}_2$



Thank you for your attention!

Coupled channels

- Full threshold structure needs to be measured in a lower channel ⇨ coupled channels
- Consider a two-channel system, construct a nonrelativistic effective field theory (NREFT)
  - ➢ Energy region around the higher threshold, Σ_2
  - ➢ Expansion in powers of $E = \sqrt{s} - \Sigma_2$
  - ➢ Momentum in the lower channel can also be expanded



$$T(E) = V + VG(E)V + VG(E)VG(E)V + \cdots = \frac{1}{V^{-1} - G(E)}$$

$$G_1^{\Lambda}(E) = i\int^{\Lambda_1} \frac{d^4q}{(2\pi)^2} \frac{1}{(q^2 - m_{1,1}^2 + i\epsilon)[(P-q)^2 - m_{1,2}^2 + i\epsilon]} = R(\Lambda_1) - i\frac{k_1}{8\pi\sqrt{s}}$$

$$G_2^{\Lambda}(E) = -\frac{1}{4m_{2,1}m_{2,2}} \int^{\Lambda_2} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{2\mu_2}{\mathbf{q}^2 - 2\mu_2 E - i\epsilon} = \frac{1}{8\pi\Sigma_2}\left[-\frac{2\Lambda_2}{\pi} + \boxed{\sqrt{-2\mu_2 E - i\epsilon}} + \mathcal{O}\left(\frac{k_2^2}{\Lambda_2}\right)\right]$$

- ➢ Λ dependence absorbed by V^{-1}

Nonanalyticity only from here

NREFT at LO

- Very close to the higher threshold, LO:

$$T(E) = 8\pi\Sigma_2 \begin{pmatrix} -\frac{1}{a_{11}} + ik_1 & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & -\frac{1}{a_{22}} - \sqrt{-2\mu_2 E - i\epsilon} \end{pmatrix}^{-1}$$

$$= -\frac{8\pi\Sigma_2}{\det} \begin{pmatrix} \frac{1}{a_{22}} + \sqrt{-2\mu_2 E - i\epsilon} & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & \frac{1}{a_{11}} - ik_1 \end{pmatrix},$$

$$\det = \left(\frac{1}{a_{11}} - i\,k_1\right)\left(\frac{1}{a_{22}} + \sqrt{-2\mu_2 E - i\epsilon}\right) - \frac{1}{a_{12}^2}$$

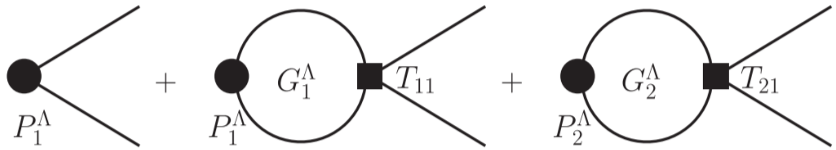
Effective scattering length with open-channel effects becomes complex, $\mathrm{Im}\,\frac{1}{a_{22,\mathrm{eff}}} \leq 0$

$$T_{22}(E) = -\frac{8\pi}{\Sigma_2}\left[\frac{1}{a_{22,\mathrm{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}$$

$$\frac{1}{a_{22,\mathrm{eff}}} = \frac{1}{a_{22}} - \frac{a_{11}}{a_{12}^2(1 + a_{11}^2 k_1^2)} - i\frac{a_{11}^2 k_1}{a_{12}^2(1 + a_{11}^2 k_1^2)}.$$

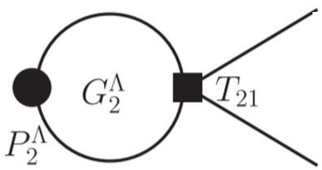
NREFT at LO

- Consider a production process, must go through final-state interaction (unitarity)



$$
\begin{aligned}
&P_1^\Lambda[1+G_1^\Lambda T_{11}(E)] + P_2^\Lambda G_2^\Lambda(E) T_{21}(E) \\
&= P_1^\Lambda (V_{11}^\Lambda)^{-1} T_{11}(E) + \left[P_1^\Lambda (V_{11}^\Lambda)^{-1} V_{12}^\Lambda + P_2^\Lambda\right] G_2^\Lambda T_{21}(E) \\
&\equiv P_1 T_{11}(E) + P_2 T_{21}(E)
\end{aligned}
$$

- All nontrivial energy dependence are contained in $T_{11}(E)$ and $T_{21}(E)$
- Case-1: dominated by $T_{21}(E)$,



$$
T_{21}(E) = \frac{-8\pi\Sigma_2}{a_{12}(1/a_{11} - ik_1)} \left[\frac{1}{a_{22,\text{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}.
$$

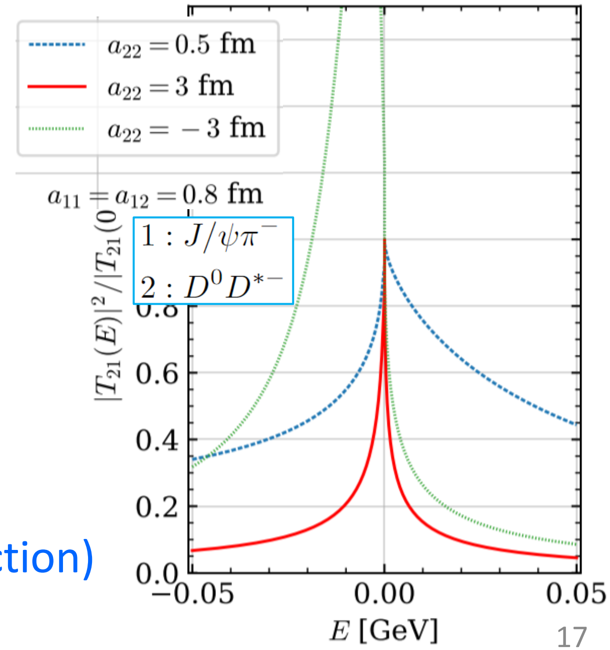
Poles in complex momentum plane:

$(0.37 - i0.08)\text{GeV}$

$(0.04 - i0.08)\text{GeV}$

$(-0.09 - i0.08)\text{GeV}$

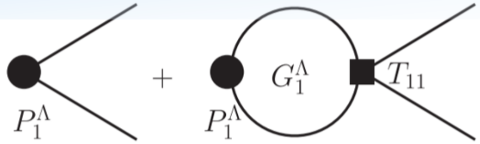
$$
|T_{21}(E)|^2 \propto |T_{22}(E)|^2 \propto
\begin{cases}
\left[\left(\text{Re}\,\frac{1}{a_{22,\text{eff}}}\right)^2 + \left(\text{Im}\,\frac{1}{a_{22,\text{eff}}} - \sqrt{2\mu E}\right)^2\right]^{-1} & \text{for } E \geq 0 \\
\left[\left(\text{Im}\,\frac{1}{a_{22,\text{eff}}}\right)^2 + \left(\text{Re}\,\frac{1}{a_{22,\text{eff}}} + \sqrt{-2\mu E}\right)^2\right]^{-1} & \text{for } E < 0
\end{cases}
$$



- ➢ Cusp at threshold (E=0)
- ➢ Maximal at threshold for positive $\text{Re}(a_{22,\text{eff}})$ (attraction)
- ➢ Peaking at pole for negative $\text{Re}(a_{22,\text{eff}})$

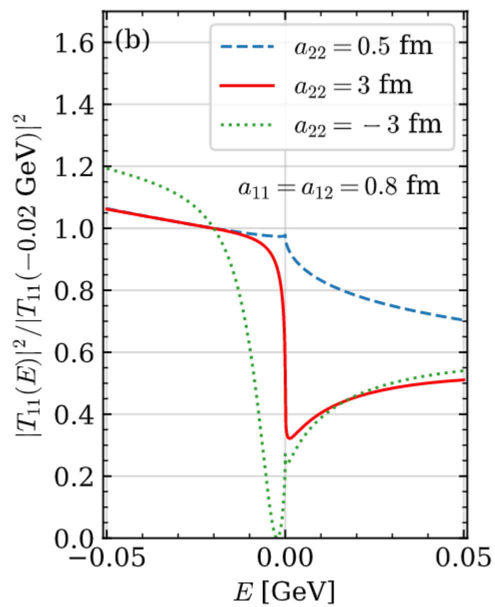
NREFT at LO

- Case-2: dominated by $T_{11}(E)$



$$T_{11}(E) = \frac{-8\pi\Sigma_2\left(\frac{1}{a_{22}} - i\sqrt{2\mu_2 E}\right)}{\left(\frac{1}{a_{11}} - i\,k_1\right)\left[\frac{1}{a_{22,\mathrm{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]}$$

- Cusp at threshold (E=0)
- One pole and one zero
- For strongly interacting channel-2 (large a_{22}), there must be a dip around threshold
- Abrupt drop if there is a nearby pole



Poles in complex momentum plane:

$(0.37 - i0.08)$GeV

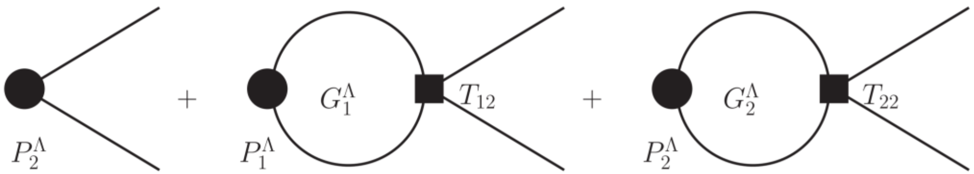
$(0.04 - i0.08)$GeV

$(-0.09 \ - i0.08)$GeV

- More complicated line shape if both channels are important for the production

NREFT at LO

- Case-3: final states in channel-2



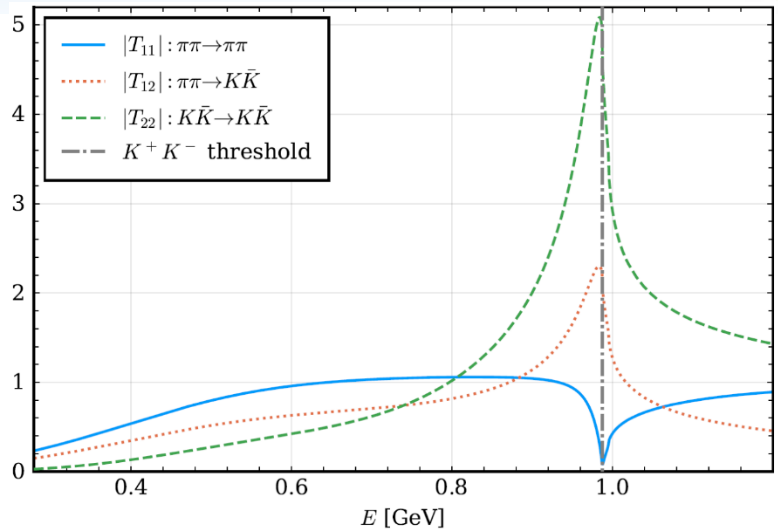
$$P_1 T_{12}(E) + P_2 T_{22}(E) \propto \left[\frac{1}{a_{22,\text{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}$$

- Suppression due to phase space
- Peak just above threshold would require the pole to be nearby

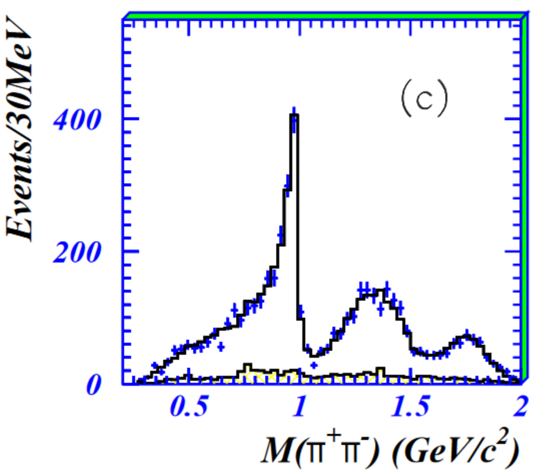
Phenomenology

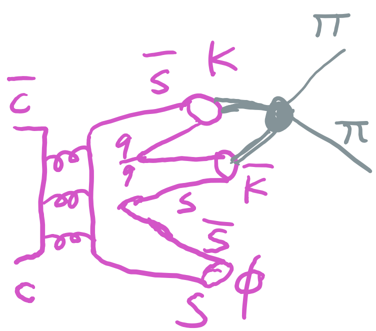
- T-matrix for $\pi\pi$ and $K\bar{K}$ coupled channels

  with the T-matrix from
  L.-Y. Dai, M. R. Pennington, PRD90(2014)036004



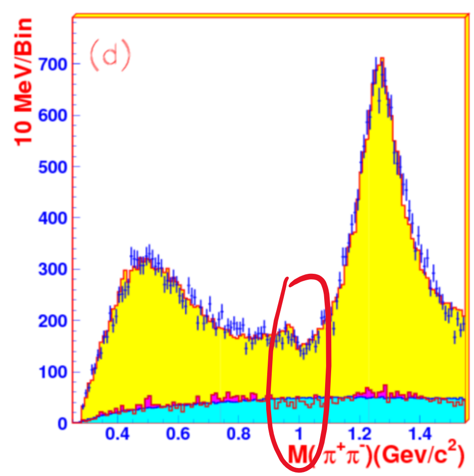
- $f_0(980)$ in $J/\psi \to \phi\pi^+\pi^-$ and $J/\psi \to \omega\pi^+\pi^-$ Channels: $\pi\pi$ and $K\bar{K}$

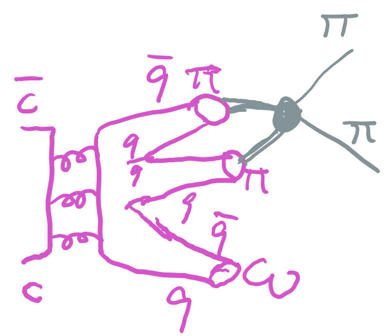




BES, PLB607(2005)243

Driving channel: $K\bar{K}$





BES, PLB598(2004)149

Driving channel: $\pi\pi$

Model estimate of near-th. interactions

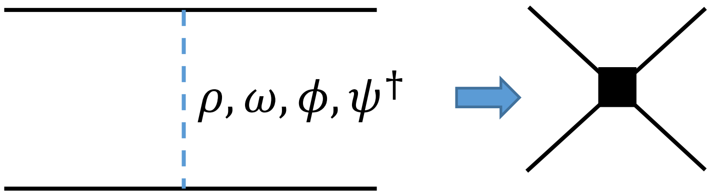
- Constant contact terms saturated by light-vector-meson exchange, similar to VMD in the resonance saturation of the low-energy constants in CHPT

V, A, S, S_1 and η_1 contributions to the coupling constants L_i^r in units of 10^{-3}.

	$L_i^r(M_\rho)$	V	A	S	S_1	η_1	Total
L_1^r	0.7 ± 0.3	0.6	0	−0.2	0.2 [b]	0	0.6
L_2^r	1.3 ± 0.7	1.2	0	0	0	0	1.2
L_3	-4.4 ± 2.5	−3.6	0	0.6	0	0	−3.0
L_4^r	-0.3 ± 0.5	0	0	−0.5	0.5 [b]	0	0.0
L_5^r	1.4 ± 0.5	0	0	1.4 [a]	0	0	1.4
L_6^r	-0.2 ± 0.3	0	0	−0.3	0.3 [b]	0	0.0
L_7	-0.4 ± 0.15	0	0	0	0	−0.3	−0.3
L_8^r	0.9 ± 0.3	0	0	0.9 [a]	0	0	0.9
L_9^r	6.9 ± 0.7	6.9 [a]	0	0	0	0	6.9
L_{10}^r	-5.2 ± 0.3	−10.0	4.0	0	0	0	−6.0

[a] Input.
[b] Large-N_C estimate.

Ecker, Gasser, Pich, de Rafael, NPB321(1989)311



- List of attractive pairs

$H\bar{H}$	$D^{(*)}\bar{D}^{(*)}[0, 1^\dagger]$; $X(3872), Z_c(3900, 4020)$ $Z_b(10610, 10650)$	$D_s^{(*)}\bar{D}^{(*)}$ $[\frac{1}{2}^\dagger]$; $Z_{cs}(3985)$	$D_s^{(*)}\bar{D}_s^{(*)}$ $[0]$ $X(4140)$
$\bar{H}T$	$\bar{D}^{(*)}\Xi_c$ $[0]$; $P_{cs}(4459)$	$\bar{D}_s^{(*)}\Lambda_c$ $[0^\dagger]$	
$\bar{H}S$	$\bar{D}^{(*)}\Sigma_c^{(*)}$ $[\frac{1}{2}]$; $P_c(4312, 4440, 4457)$ $\bar{D}^{(*)}\Omega_c^{(*)}$ $[\frac{1}{2}^\dagger]$	$\bar{D}_s^{(*)}\Sigma_c^{(*)}$ $[1^\dagger]$;	$\bar{D}^{(*)}\Xi_c'^{(*)}$ $[0]$;
$T\bar{T}$	$\Lambda_c\bar{\Lambda}_c$ $[0]$;	$\Lambda_c\bar{\Xi}_c$ $[\frac{1}{2}]$;	$\Xi_c\bar{\Xi}_c$ $[0, 1]$
$T\bar{S}$	$\Lambda_c\bar{\Sigma}_c^{(*)}$ $[1]$; $\Xi_c\bar{\Sigma}_c^{(*)}$ $[\frac{3}{2}^\dagger, \frac{1}{2}]$;	$\Lambda_c\bar{\Xi}_c'^{(*)}$ $[\frac{1}{2}]$; $\Xi_c\bar{\Xi}_c'^{(*)}$ $[1, 0]$;	$\Lambda_c\bar{\Omega}_c^{(*)}$ $[0^\dagger]$; $\Xi_c\bar{\Omega}_c^{(*)}$ $[\frac{1}{2}]$
$S\bar{S}$	$\Sigma_c^{(*)}\bar{\Sigma}_c^{(*)}$ $[2^\dagger, 1, 0]$; $\Xi_c'^{(*)}\bar{\Xi}_c'^{(*)}$ $[1, 0]$;	$\Sigma_c^{(*)}\bar{\Xi}_c'^{(*)}$ $[\frac{3}{2}^\dagger, \frac{1}{2}]$; $\Xi_c'^{(*)}\bar{\Omega}_c^{(*)}$ $[\frac{1}{2}]$;	$\Sigma_c^{(*)}\bar{\Omega}_c^{(*)}$ $[0^\dagger]$; $\Omega_c^{(*)}\bar{\Omega}_c^{(*)}$ $[0]$

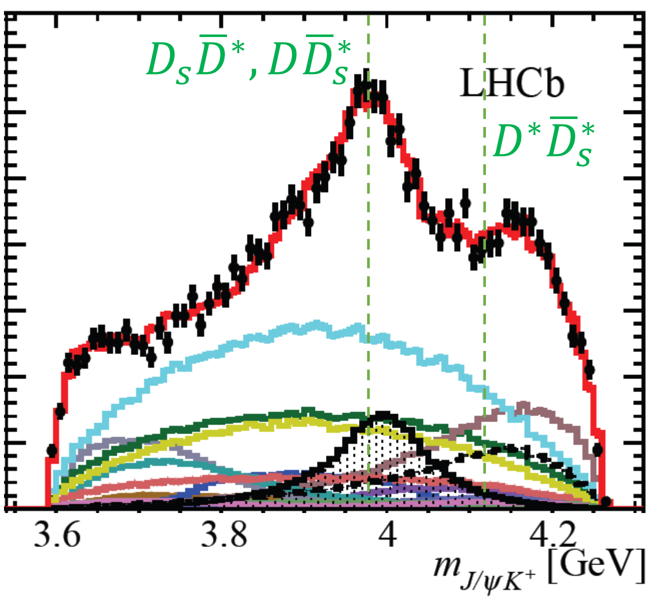
Comments on Z_c and Z_{cs}

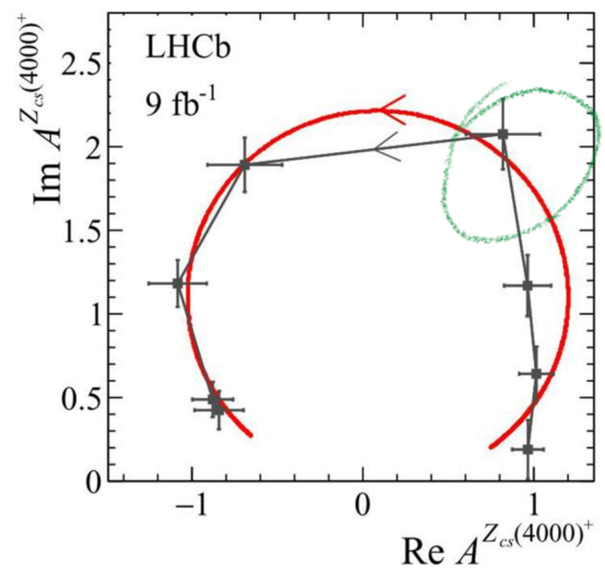
- ✓ Isovector interaction between $D^{(*)}\bar{D}^{(*)}$ from light vector exchange vanishes
- ✓ Charmonia exchange could be important here: F.Aceti, M.Bayar, E.Oset et al., PRD90(2014)016003

  no mass hierarchy, a series of charmonia can be exchanged Dong, FKG, Zou, arXiv:2101.01021

  axial-vector meson exchange considered in Yan, Peng, Sanchez Sanchez, Pavon Valderrama 2102.13058
- ✓ $Z_c(3900,4020)$ as $\bar{D}^{(*)}D^*$ virtual states
- ✓ $Z_{cs}(3985)$ as $D_s\bar{D}^*, D\bar{D}_s^*$ virtual state; there should also be a $D^*\bar{D}_s^*$ state around 4.1 GeV

  Z. Yang, X. Cao, FKG, J. Nieves, M. Pavon Valderrama, arXiv:2011.08725





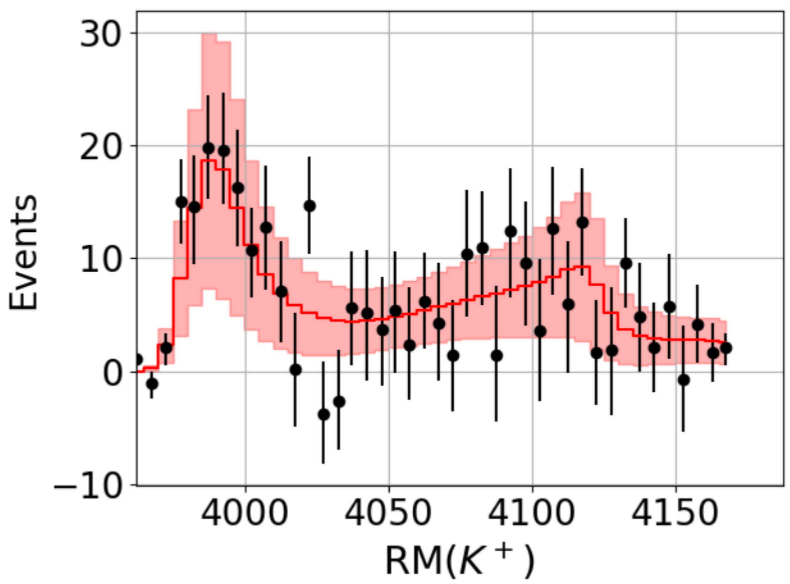
Threshold behavior with a strong coupling

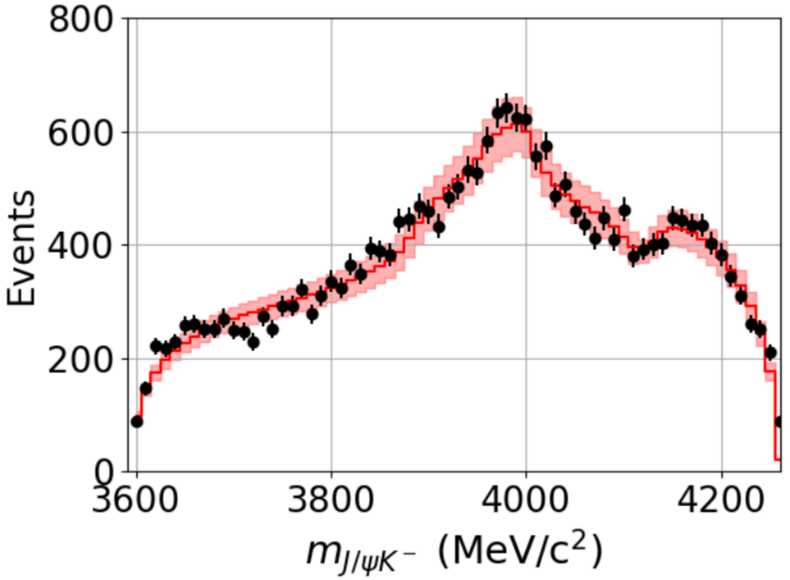
LHCb, arXiv: 2103.01803

Comments on Z_c and Z_{cs}

✓ Simultaneous fit to the BESIII and LHCb Z_{cs} data: Z_{cs} as virtual states

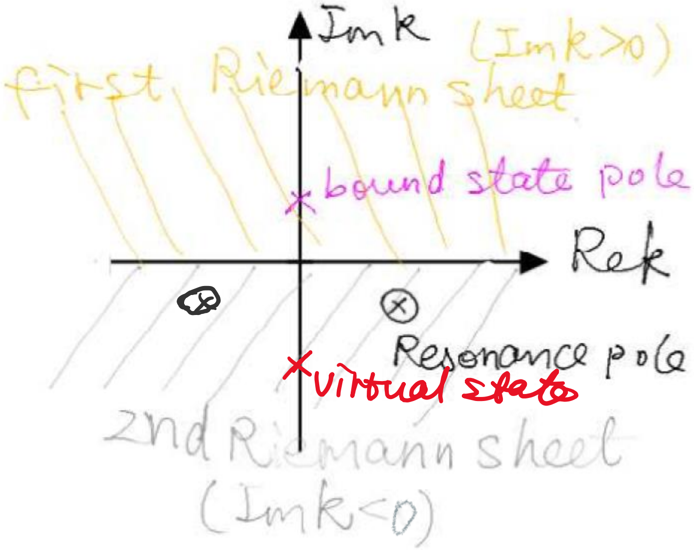
Ortega, Entem, Fernandez, 2103.07781

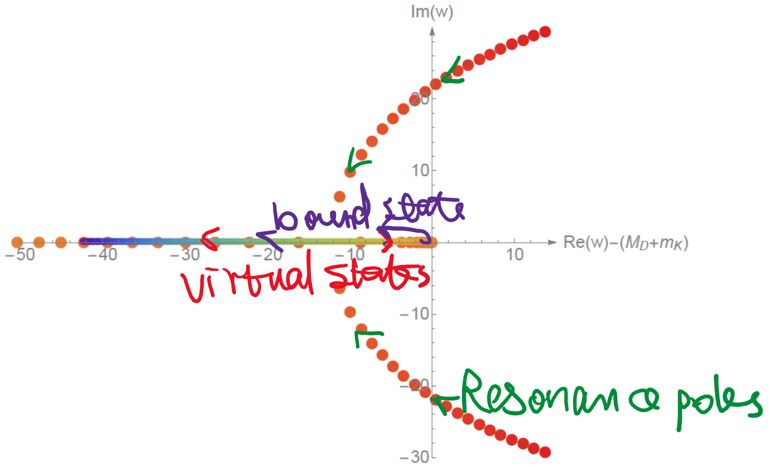


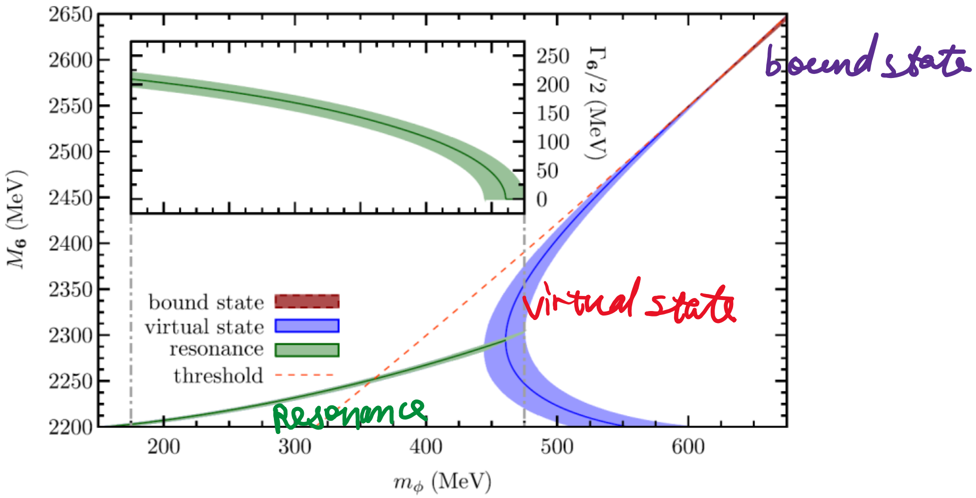


Bound state, virtual state and resonance

- Bound state: pole below threshold on real axis of the first Riemann sheet of complex energy plane
- Virtual state: pole below threshold on real axis of the second Riemann sheet
- Resonance: pole in the complex plane on the second Riemann sheet







Plots from Matuschek, Baru, FKG, Hanhart, 2007.05329; M.-L. Du et al., PRD98(2018)094018

For $\dfrac{1}{1/a_0 - i\,k}$, only bound or virtual state poles are possible