Charmonium-like hybrids from lattice QCD

Ying Chen

Institute of High Energy Physics,
Chinese Academy of Sciences, China
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Outline

I. Introduction
II. BS Wave functions and spectrum of charmonium-like hybrids
III. Mass decomposition of charmonium-like states
IV. Summary
As an extension of the constituent quark model, it is conjectured that gluons can be building blocks to make hadrons since they carry color charges.

**Glueballs**: $gg, ggg, ...$

**Hybrid mesons**: $q_1\bar{q}_2g$

### 1. Hybrid mesons

1. **Introduction**

   - Hybrid mesons
   - Phenomenological descriptions of hybrid mesons

   - Glueballs: $gg$, $ggg$, ...
   - Hybrid mesons: $q_1\bar{q}_2g$

   **Constituent gluon model**
   **Flux-tube model——Born-Oppenheimer approx.**


   **Gluons**: light and fast degrees of freedom
   **effect as an flux-tube along the $Q\bar{Q}$ axis**
3. Latest lattice QCD results of charmonium

Courtesy to S. Prelovsek

Red blocks $\left(1^{--}, (0,1,2)^{-+}\right)$ super-multiplet
Intuitively, the coupling of this kind of operators to conventional charmonia can be suppressed from two aspects:

a) spin states of the $c$-$\bar{c}$bar (spin flipping is suppressed by the heavy quark mass.

b) center-of-mass motion (to the leading order of NR, there is no center-of-mass motions for conventional charmonia.)

This is equivalent to give a $c$-$\bar{c}$bar center of mass motion, which describes the recoil of the $c$-$\bar{c}$bar against additional degrees of freedom.
Intuitively, the coupling of this kind of operators to conventional charmonia can be suppressed from two aspects:

a) spin states of the c-cbar (spin flipping is suppressed by the heavy quark mass.

b) the block \((\vec{c}^a \Gamma B^{ab})\) can be viewed as a dressed charm anti-quark.

We use this kind of operator to do a self-consistent check.

Type II operator: \(O(\vec{r}) = (\vec{c}^a \Gamma B^{ab})(0)c(\vec{r})\) (Coulomb gauge)
1. Lattice setup

- Quenched approximation, anisotropic lattices;
- At least, the $1^{-+}$ hybrids are well defined in QA.
- $(1^{--}, (0, 1, 2)^{-+})$ supermultiplet is focused on

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\xi$</th>
<th>$a_s$(fm)</th>
<th>$L a_s$(fm)</th>
<th>$L^3 \times T$</th>
<th>$N_{conf}$</th>
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<td>0.222(2)</td>
<td>3.55</td>
<td>$16^3 \times 160$</td>
<td>500</td>
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<tr>
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<td>0.138(1)</td>
<td>3.31</td>
<td>$24^3 \times 192$</td>
<td>200</td>
</tr>
</tbody>
</table>

2. Numerical strategy

$$C(r, t) = \left\langle O(\vec{r}, t) O^{(w)+}(0) \right\rangle$$

$$= \sum_i W_i(r) e^{-m_i t}$$

$$\Phi_n(r) = \langle \Omega | O(r) | i \rangle \propto W_i(r)$$: Bethe-Salpeter wave function

Operator prototypes:

\begin{align*}
O_{0_{\nu} +}(t) &= \sum_x \bar{c}^a(x, t) \gamma_i c^b(x, t) B_i^{ab}(x, t), \\
O_{1_{\nu} +}^k(t) &= \sum_x \bar{c}^a(x, t) \gamma_i c^b(x, t) B_j^{ab}(x, t) \epsilon_{ijk}, \\
O_{2_{\rho} +}^k(t) &= \sum_x \bar{c}^a(x, t) \gamma_i c^b(x, t) B_j^{ab}(x, t) \epsilon_{ijk}, \\
O_{1_{\rho} -}^k(t) &= \sum_x \bar{c}^a(x, t) \gamma_5 c^b(x, t) B_k^{ab}(x, t),
\end{align*}

In practice, for the type-I and II operators, the masses and the BS functions can be derived simultaneously from a joint fit to $C(r, t)$
A. The $1^{-+}$ states

**Spectrum**

- BS wave functions from type-I operator ($c\bar{c} - g$)

<table>
<thead>
<tr>
<th>#node</th>
<th>$m(1^{-})$ (GeV)</th>
<th>$m(0^{+-})$ (GeV)</th>
<th>$m(1^{-+})$ (GeV)</th>
<th>$m(2^{-+})$ (GeV)</th>
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<tbody>
<tr>
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<td>3.109(5)</td>
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<td>0</td>
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- BS wave functions from type-II operator ($\bar{c}g - c$)
B. The $2^{-+}$ states

- $2^{-+} \ c\bar{c} \ (1^1D_2, \eta_{c2})$ mass $\sim 3.83$ GeV
- $c\bar{c}g$ operator couples almost exclusively with states higher than 4 GeV.
- The states observed might be members of $\{1^{--}, (0,1,2)^{-+}\}$ supermultiplets.
C. The $0^{-+}$ states

The $0^{-+}$ states are conventional for quarkonium. Conventional charmonia appear in this channel. The lowest two states can be identified to be $\eta_c(1S)$ and $\eta_c(2S)$. This is supported by the masses and w.f. from type-II operators. The 4-5th states have almost degenerate masses and similar w.f. to their $(1,2)^{-+}$ counterparts. They may be members of the set $\{1^{--}, (0,1,2)^{-+}\}$ supermultiplets. These are tentative interpretations.

\[
C(r, t) = \sum_i W_i(r) e^{-m_i t}
\]

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BS w.f. from type-I operator ($c\bar{c} - g$)  
BS w.f. from type-II operator ($\bar{c}g - c$)

3-mass-term fit  
4-mass-term fit  
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4-mass-term fit
B. The $0^{-+}$ states

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- These are tentative interpretations.

**BS w. f. from type-I operator** $(c\bar{c} - g)$

\[
C(r, t) = \sum_i W_i (r) e^{-m_i t}
\]

**BS w. f. from type-II operator** $(\bar{c}g - c)$

Comp. with $1^{-+}$

4-mass-term fit
D. The $1^{--}$ states

- $1^{--}$ is conventional for quarkonium.
- Conventional charmonia appear in this channel.
- The lowest two states can be identified to be $J/\psi(1S)$ and $\psi(2S)$. This is supported by the masses and w.f. from type-II operators.
- The 4-5th states have almost degenerate masses and similar w.f. to their $(1,2)^{--}$ counterparts. They may be members of \{$1^{--}, (0,1,2)^{--}$\} super-multiplets.
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- 4-mass-term fit

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D. The $1^{--}$ states

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BS w. f. from type-I operator $(c\bar{c} - g)$

Comp. with $1^{-+}$

4-mass-term fit

BS w. f. from type-II operator $(\bar{c}g - c)$

Comp. with $1^{-+}$

4-mass-term fit
3. Discussion

\[ \Phi_{\eta}(r) / \Phi_{\eta}(0) \]

\[ \beta = 2.4 \]

\[ \Phi_{\eta}(r) / \Phi_{\eta}(0) \]

\[ \beta = 2.8 \]

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A. Phenomenological picture of the hybrid-like charmonium

“Halo charmonium”:

The spatial displacement between the $c\bar{c}$ and the gluonic degree of freedom seems a physically relevant dynamical variable in the description of the internal structure of hybrid-like charmonia.

A relatively localized kernel of color octet $c\bar{c}$ surrounded by a gluonic cloud.

The gluonic cloud can be easily hadronized into light hadrons by emitting or absorbing a soft gluon.

Consequently halo charmonium has large branching ratios to decay into a conventional charmoium by emitting light hadrons.
In analogy with bi-atomic molecule:
\[ \Lambda^\epsilon \] ——cylinder symmetry along the \( Q\bar{Q} \) axis
\[ |\vec{n} \cdot K| = 0, 1, 2, \ldots \Rightarrow \Lambda = \Sigma, \Pi, \Delta, \ldots \]
\[ \eta: CP \) of the flux tube, \( CP = +, - (g, u) \]
\[ \epsilon: \) reflection symmetry with respect to the plane including the \( Q\bar{Q} \) axis

Spin of the meson : \( J = L + S \)
\[ L = L_{Q\bar{Q}} + K \]
\[ L^2_{Q\bar{Q}} = L^2 - 2\Lambda^2 + K^2 + \ldots \]

\( PC \) of the meson : \( P = \epsilon(-1)^{L+\Lambda+1}, C = \eta\epsilon(-1)^{L+\Lambda+S} \)
Radial Schrödinger Equation: \[ \left( J^2_{\text{light}} \right) \approx 2 \]

\[
\left[ -\frac{1}{m_Q} \left( \frac{d}{dr} \right)^2 + \frac{L(L+1) - 2\Lambda^2 + \langle J^2_{\text{light}} \rangle_{\Gamma,r}}{m_Q r^2} + V_{\Gamma}(r) \right] r R(r) = E r R(r)
\]

<table>
<thead>
<tr>
<th>quarkonia and hybrids</th>
<th>(\Gamma(nL))</th>
<th>(S = 0)</th>
<th>(S = 1)</th>
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<tr>
<td>(\Sigma^+_g(1S))</td>
<td>0(^{-+})</td>
<td>1(^{-+})</td>
<td></td>
</tr>
<tr>
<td>(\Sigma^+_g(1P))</td>
<td>1(^{+-})</td>
<td>(0, 1, 2)(^{++})</td>
<td></td>
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<tr>
<td>(\Sigma^+_g(1D))</td>
<td>2(^{++})</td>
<td>(1, 2, 3)(^{--})</td>
<td></td>
</tr>
<tr>
<td>(\Pi^+_u(1P))</td>
<td>1(^{-+})</td>
<td>(0, 1, 2)(^{+-})</td>
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<tr>
<td>(\Pi^+_u(1F))</td>
<td>3(^{++})</td>
<td>(2, 3, 4)(^{+-})</td>
<td></td>
</tr>
<tr>
<td>(\Pi^-_u(1P))</td>
<td>2(^{-+})</td>
<td>(0, 1, 2)(^{+-})</td>
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<td>(1, 2, 3)(^{+-})</td>
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</tbody>
</table>

Figure 2: Hybrid static energies, \(E_{\Gamma}\), in lattice units, from [14].

Lightest hybrid super-multiplet: \(L = 1, \Lambda = 1, \eta = u, \epsilon = +\)
The non-relativistic potential model solutions for heavy quarkonium-like hybrids in BO approximation (E. Braaten et al., PRD 90 (2014)014044)

In comparison with our results:

\[ \Delta M(2P - 1P) \sim 380\text{MeV} \]

\[ \Delta M(2P - 1P) \sim 200\text{MeV} \]

Spectrum of \( c\bar{c} \) hybrids

Spectrum of \( b\bar{b} \) hybrids

\[ M(\psi_H(2S) - \psi_H(1S)) \approx 1200\text{MeV} \approx 2 M(\psi(2S) - \psi(1S)) \]
C. Static potential from lattice QCD

- The color singlet static potential between two color charge in $D$ and $D^*$ representation can be parameterized as

$$V_D(r) = V_{D,0} - C_D \frac{\alpha_s}{r} + \sigma_D r,$$

where $C_D$ is the Casimir operator of $D$.

- In the plot, the filled lines are plotted assuming the relation $\sigma_D = \frac{C_D}{C_F} \sigma$.

- This behavior is called Casimir Scaling.

- Even in the quenched approximation, the flux tube between adjoint color charge is expected to break easily due to the excitation of two soft gluons.
If the energy splitting is insensitive to the constituent masses, for example,

\[
M(\Upsilon(2S) - \Upsilon(1S)) = 563 \text{ MeV} \\
M(\psi(2S) - \psi(1S)) = 589 \text{ MeV}
\]

Then the mass splitting is possibly dependent only of the string tension, say, \( \sigma_D = \frac{C_D}{C_F} \sigma \).

\[
M(\psi_H(2S) - \psi_H(1S)) \approx 1200 \text{ MeV} \approx 2 \, M(\psi(2S) - \psi(1S))
\]

\[
\sigma_A = \frac{C_A}{C_F} \sigma = 2.25 \sigma
\]
D. Implication to the decay of charmonium-like hybrids

If the excitation of hybrid charmonium can be understood as the bound state energy levels in the adjoint potential $V_A(r)$, then the number of the energy levels is much less than that of the fundamental potential.
Understanding the Y(4260) decays in the “halo-charmonium” picture

\(J/\psi \pi^+ \pi^-\) mode: relative S-wave between \(J/\psi\) and \(\pi^+ \pi^-\)

\(\chi_{c0} \omega\) mode: relative S-wave between \(\chi_{c0}\) and \(\omega\)

\(h_c \pi^+ \pi^-\) mode: relative P-wave between \(h_c\) and \(\pi^+ \pi^-\)

The \(c\bar{c}\) in the halo charmonium is spin singlet (S=0),

\(J/\psi \pi^+ \pi^-\) mode: \(J/\psi\) (S=1), spin flipping, \(m_c\) suppressed, no centrifugal barrier

\(\chi_{c0} \omega\) mode: \(\chi_{c0}\) (S=1), spin flipping, \(m_c\) suppressed, no refugal barrier

\(h_c \pi^+ \pi^-\) mode: \(h_c\) (S=0), no spin flipping, but suppressed by the centrifugal barrier.

In this picture, it is understandable that the above three modes have similar cross section at \(\sqrt{s} \sim 4.22\, GeV\)
Possible decay modes of is charmonium-like $(0, 1, 2)^{-+}$ hybrids,

$\chi_c \eta (\eta')$ mode: $\chi_c (S=1)$, no spin flipping, no centrifugal barrier

$J/\psi \omega (\phi)$ mode: $J/\psi (S=1)$, no spin flipping, $P$ – wave centrifugal barrier

These decay modes can be searched by LHCb and Belle II.
III. Mass decomposition of charmonium-like states

W. Sun et al. (χQCD Collab., PRD 103 (2021)094503

- **Mass structure:** Another aspect to explore the difference of internal structure of conventional charmonia and exotic charmonia

- Conventional charmonia involved in this study
  - **1S states:** η_c, J/ψ
  - **1P states:** χ_{cJ}, h_c
  - **1D state:** η_{c2}

- Charmonium-like state with exotic quantum number
  - J^{PC} = 1^{-+}: \bar{c}c - g or \bar{c}c - q_1q_2

- Lattice setup: \( N_f = 2 + 1 \) configurations generated by RBC/UKQCD Collaboration

<table>
<thead>
<tr>
<th>ensemble</th>
<th>( L^3 \times T )</th>
<th>( a ) (fm)</th>
<th>( m_\pi ) (MeV)</th>
<th>( N_{cfg} )</th>
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<td>48IF</td>
<td>( 48^3 \times 96 )</td>
<td>0.0711(3)</td>
<td>278</td>
<td>100</td>
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1. Theoretical framework

- Hadron mass components can be investigated through the energy-momentum tensor of QCD (X. Ji, Phys. Rev. Lett. 74, 1071 (1995)).

- Euclidean space:

\[ T_{\mu\nu} = \frac{1}{4} \bar{\psi} \gamma_{(\mu} \bar{D}_{\nu)} \psi + F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} \delta_{\mu\nu} F^2 \]

\[ H_{QCD} \equiv - \int d^3 \vec{x} \, T_{44} (\vec{x}) = H_q + H_g + \frac{1}{4} (H^a_g + H^\gamma_m) \]

\[ = H_m + H_E + H_g + \frac{1}{4} H_a \]

\[ H_q = - \sum_{u,d,s,\ldots} \int d^3 \vec{x} \bar{\psi} (D_4 \gamma_4) \psi = \sum_{u,d,s,\ldots} \int d^3 \vec{x} \left( \bar{\psi} \left( \bar{D} \cdot \gamma \right) + m \right) \psi \]

\[ H_m = \sum_{u,d,s,\ldots} \int d^3 \vec{x} \, m \bar{\psi} \psi \quad H_E = \sum_{u,d,s,\ldots} \int d^3 \vec{x} \bar{\psi} \left( \bar{D} \cdot \gamma \right) \psi \]

\[ H_g = \int d^3 \vec{x} \frac{1}{2} (B^2 - E^2) \quad H_a = H^a_g + H^\gamma_m \]

\[ H^a_g = - \int d^3 \vec{x} \frac{\beta (g)}{g} (E^2 + B^2) \quad H^\gamma_m = \sum_{u,d,s,\ldots} \int d^x \vec{x} \, \gamma_m m \bar{\psi} \psi \]
\[ M = \frac{\left\langle P \left| - \int d^3 \vec{x} \ T_{44}(0, \vec{x}) \right| P \right\rangle}{\left\langle P | P \right\rangle} \equiv \left\langle T^{00} \right\rangle \]

**Hadron mass:**
\[ P = (M, \vec{0}) \]

**Charm quark momentum fraction of charmonium(-like) states:**

\[ \langle x \rangle_c = \left( \frac{4}{3} \frac{\left\langle H | \bar{T}_{44}^c | H \right\rangle}{M_H \left\langle H | H \right\rangle} \right) \]

\[ \bar{T}_{44}^c = \frac{1}{2} \int d^3 \vec{x} \ \bar{c}(x) \left( \gamma_4 \vec{D}_4 - \frac{1}{4} \gamma_\mu \vec{D}_\mu \right) c(x) \]
• Mass spectrum (from two-point functions)
• Quark mass contribution to charmonium(-like) states $\langle H_m \rangle$;

For all these states, 

$$\langle H_m \rangle \approx 2 \left( \bar{m}_c (3 \text{GeV}) \sim \bar{m}_c (\bar{m}_c) \right)$$
• Charm quark momentum fraction $\langle x \rangle^R_c$

- $\langle x \rangle^R_c$ is almost the same for conventional charmonia;
- $\langle x \rangle^R_c$ of $1^{-+}$ charmonium-like state is obviously smaller.
- Additional degrees of freedom in the $1^{-+}$ state? Gluons or light quarks?
V. Summary

- The internal structure of hybrid charmonia can be reflected by the Bethe-Salpeter wave function with respect to the spatial displacement between the gluonic degree of freedom and the $c\bar{c}$.

- “Halo-charmonium” picture for charmonium-like hybrids.

- Lattice results do not support the Born-Oppenheimer flux-tube model for heavy quarkonium-like hybrids.

- Present results are obtained from Quenched LQCD. Full-QCD lattice studies are desired and decay properties should be considered.

- The mass structure of charmonium-like states are investigated by $2+1$ full lattice QCD. The charm quark momentum fraction of the $1^{-+}$ charmonium-like state is obviously smaller than that of conventional charmonia.
Thanks!