

Box 图奇点研究

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Paper is preparing.....

第七届手征有效场论研讨会

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东南大学 (online)



中国科学院大学
University of Chinese Academy of Sciences

Content

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- From Triangle Singularity to Box Singularity
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Background of Triangle Singularity

L. D. Landau, Nucl. Phys. 13, no.1, 181-192 (1960)

S. Coleman, R.E. Norton, Nuovo Cim. **1965**, 38, 438–442,

R. Karplus, C.M. Sommerfield, E.H. Wichmann, PR **1958**, 111, 1187–1190.

J.D. Bjorken, Ph.D. Thesis, Stanford University, Stanford, CA, USA, 1959.

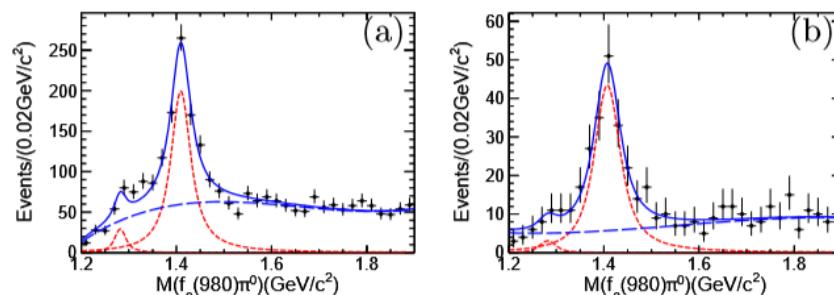
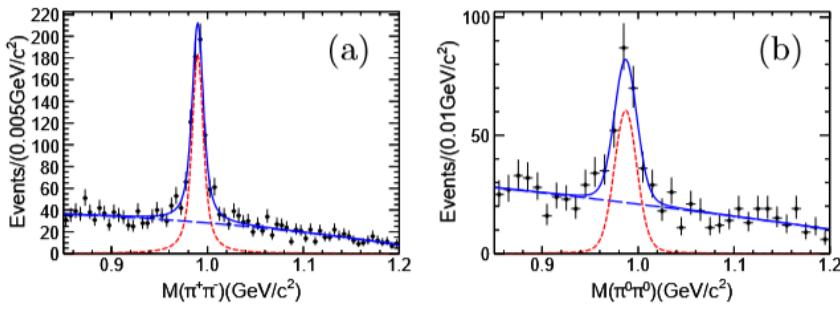
C. Schmid, Phys. Rev. **1967**, 154, 1363,

BESIII collaboration,

Phys. Rev. Lett. 2012, 108, 182001

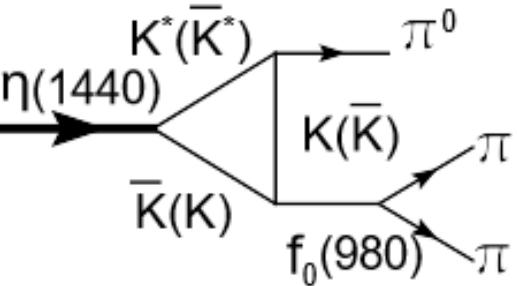
Wu, J.J.; Liu, X.H.; Zhao, Q.; Zou, B.S.

Phys. Rev. Lett. 2012, 108, 081803



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F. K. Guo, X. H. Liu, S. Sakai
PPNP 2020, 112, 103757



Structures	Processes	Loops	I/F	Refs.
2.1 GeV [41]	$\gamma p \rightarrow N^*(2030) \rightarrow K^+ \Lambda(1405)$	$K^* \Sigma \pi$	I	[142]
2.1 GeV	$\pi^- p \rightarrow K^0 \Lambda(1405), pp \rightarrow p K^+ \Lambda(1405)$	$K^* \Sigma \pi$	I	[143]
1.88 GeV	$\Lambda_c^+ \rightarrow \pi^+ \pi^0 \eta \Sigma$	$K^* N \bar{K}$	I	[144, 145] ^a
$N(1700)$ [10]	$N(1700) \rightarrow \pi \Delta$	$\rho N \pi$	I	[146]
$N(1875)$ [10]	$N(1875) \rightarrow \pi N(1535)$	$\Sigma^* K \Lambda$	I	[147]
$\Delta(1700)$ [148–150]	$\gamma p \rightarrow \Delta(1700) \rightarrow \pi N(1535) \rightarrow p \pi^0 \eta$	$\Delta \eta \rho$	I	[151]
2.2 GeV [152]	$\Lambda_c^+ \rightarrow \pi^0 \phi p$	$\Sigma^* K^* \Lambda$	F	[153]
1.66 GeV [154, 155]	$\Lambda_c^+ \rightarrow \pi^+ K^- p$	$a_0 \Lambda \eta, \Sigma^* \eta \Lambda$	F	[156]
$P_c(4450)$ [35]	$\Lambda_b^0 \rightarrow K^- J/\psi p$	$\Lambda(1890) \chi_{c1} p$	F	[157–160] ^b
		$N(1900) \chi_{c1} p$	F	[159]
		$D_{sJ} \Lambda_c^{(*)} \bar{D}^{(*)}$	F	[36, 158]

Structures	Processes	Loops	I/F	Refs.
$\rho(1480)$ [78, 79]	$\pi^- p \rightarrow \phi \pi^0 n$	$K^* \bar{K} K$	I	[80, 81]
$\eta(1405/1475)$ [82, 86]	$\eta(1405/1475) \rightarrow \pi f_0$	$K^* \bar{K} K$	I	[87, 91] ^{a,b}
$f_1(1285)$ [92]	$f_1(1285) \rightarrow \pi a_0/\pi f_0$	$K^* \bar{K} K$	I	[89, 93–95] ^b
$a_1(1260)$ [96, 97]	$a_1(1260) \rightarrow f_0 \pi \rightarrow 3\pi$	$K^* \bar{K} K$	I	[97, 99]
1.4 GeV [100]	$J/\psi \rightarrow \phi \pi^0 \eta/\phi \pi^0 \pi^0$	$K^* \bar{K} K$	I	[101] ^b
1.42 GeV	$B^- \rightarrow D^0 \pi^- f_0(a_0), \tau \rightarrow \nu_\tau \pi^- f_0(a_0)$	$K^* \bar{K} K$	I	[102, 103]
	$D_s^+ \rightarrow \pi^+ \pi^0 f_0(a_0), \bar{B}_s^0 \rightarrow J/\psi \pi^0 f_0(a_0)$	$K^* \bar{K} K$	I	[104, 105]
$f_2(1810)$ [10]	$f_2(1810) \rightarrow \pi \pi \rho$	$K^* \bar{K}^* K$	I	[106]
1.65 GeV	$\tau \rightarrow \nu_\tau \pi^- f_1(1285)$	$K^* \bar{K}^* K$	I	[107]
1515 MeV	$J/\psi \rightarrow K^+ K^- f_0(a_0)$	$\phi \bar{K} K$	I	[108]
2.85 GeV, 3.0 GeV	$B^- \rightarrow K^- \pi^- D_{s0}^* / K^- \pi^- D_{s1}$	$K^{*0} D^{(*)0} K^+$	I	[109, 110]
5.78 GeV	$B_s^+ \rightarrow \pi^0 \pi^+ B_s^0$	$K^{*0} B \bar{K}$	F	[111]
[4.01, 4.02] GeV	$[D^{*0} D^0] \rightarrow \gamma X$	$D^{*0} \bar{D}^0 D^0$	I	[112]
4015 MeV	$e^+ e^- \rightarrow \chi X$	$D^{*0} \bar{D}^0 D^0$	I	[113, 114]
4015 MeV	$B \rightarrow K X \pi, pp/p\bar{p} \rightarrow X \pi + \text{anything}$	$D^{*0} \bar{D}^{*0} D^0$	I	[115, 116]
$\Upsilon(1020)$ [117, 118]	$e^+ e^- \rightarrow Z_b \pi$	$B_1(5721) BB^*$	I	[119, 120]
3.73 GeV	$X \rightarrow \pi^0 \pi^+ \pi^-$	$D^{*0} \bar{D}^0 D^0$	F	[121]
[4.22, 4.24] GeV	$e^+ e^- \rightarrow \gamma J/\psi \phi/\pi^0 J/\psi \eta$	$D_{s0(s)}^* \bar{D}_s^* D_s^0$	F	[122]
[4.08, 4.09] GeV	$e^+ e^- \rightarrow \pi^0 J/\psi \eta$	$D_{s0(s)}^* \bar{D}_s^* D_s^0$	F	[122]
$Z_c(3900)$ [31, 32]	$e^+ e^- \rightarrow J/\psi \pi^+ \pi^-$	$D_1 \bar{D} D^*$	I	[119, 123–127] ^c
		$D_0^* (2400) D^* D$	F	[128, 129]
		$D_{1(2)} \bar{D}^{(*)} D^{(*)}$	F	[125]
$Z_c(4020, 4030)$ [33, 130]	$e^+ e^- \rightarrow \pi^+ \pi^- h_c(\psi')$	$K_1(1650) \phi'/\phi$	F	[133]
$X(4700)$ [131, 132]	$B^+ \rightarrow K^+ J/\psi \phi$	$K^{*0} \psi(4260) \pi^+$	F	[135]
$Z_c(4430)$ [30, 134]	$B^0 \rightarrow K^- \pi^+ J/\psi$	$\bar{K}^* \psi(3770) \pi^+$	F	[135]
$Z_c(4200)$ [136, 137]	$\bar{B}^0 \rightarrow K^- \pi^+ \psi(2S)$	$N^* \psi(3770) \pi^-$	F	[135]
	$\Lambda_b^0 \rightarrow p \pi^- J/\psi$	$N^* \psi(3770) \pi^-$	F	[135]
$X(4050)^{\pm}$ [138]	$B^0 \rightarrow K^- \pi^+ \chi_{c1}$	$\bar{K}^{*0} X \pi^+$	F	[139]
$X(4250)^{\pm}$ [138]	$B^0 \rightarrow K^- \pi^+ \chi_{c1}$	$\bar{K}_2^* \psi(3770) \pi^+$	F	[139]
$Z_b(10610)$ [34]	$e^+ e^- \rightarrow \Upsilon(1S) \pi^+ \pi^-$	$B_s^* \bar{B}^* B$	F	[128]



Why is Triangle Singularity interesting ?

1. It happens at a pure kinematic point

-> Model independent

2. The effect of Loop

-> Understand hadronic Loop contribution

3. Provide a peak structure

-> May mixing with resonance

4. To extract the nature of hadron

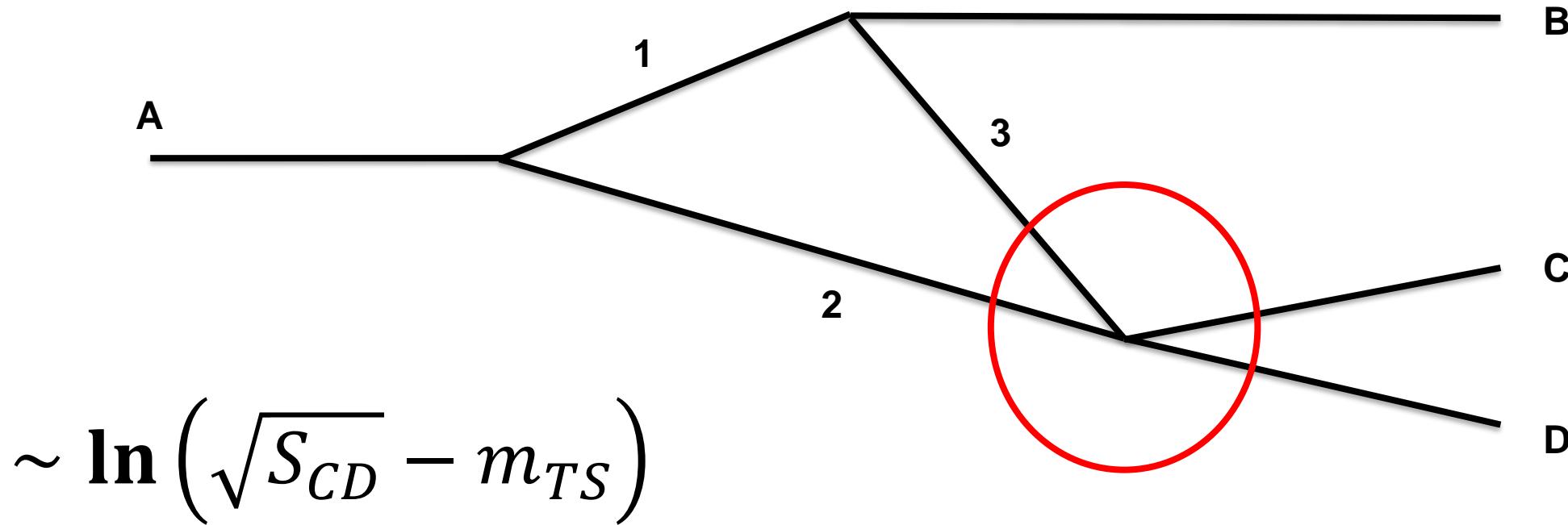
-> Study the coupling in the special point

5.

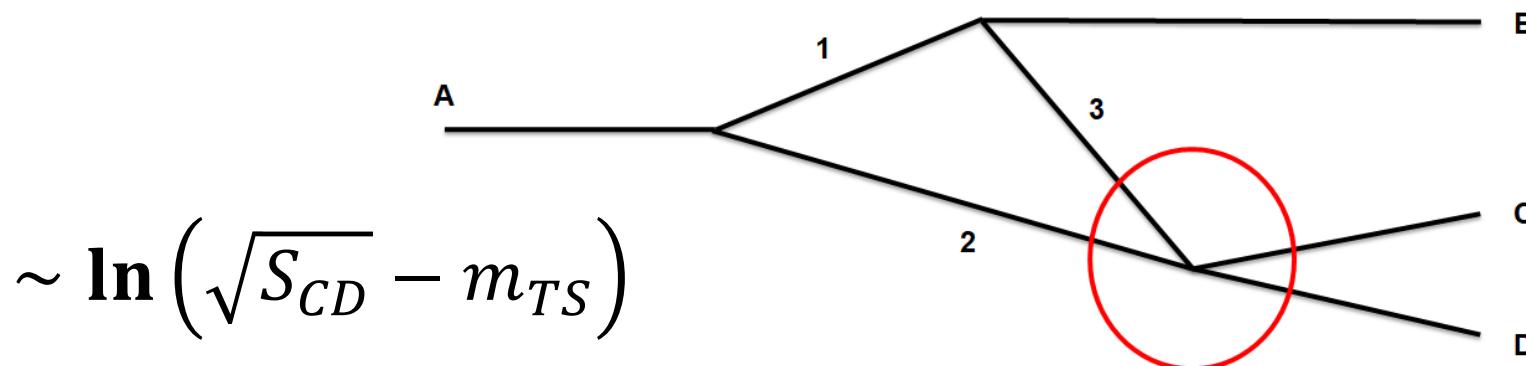


Why Triangle Singularity interesting ?

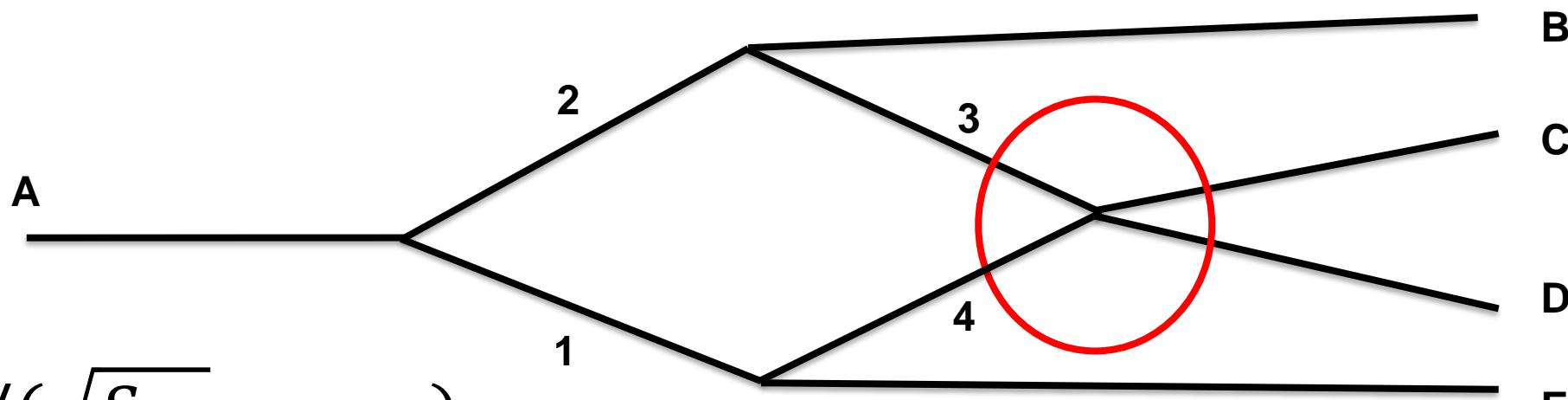
Personally, the most important thing is that we can extract the information of $2 \rightarrow 2$ scattering process.



From Triangle Singularity to Box Singularity



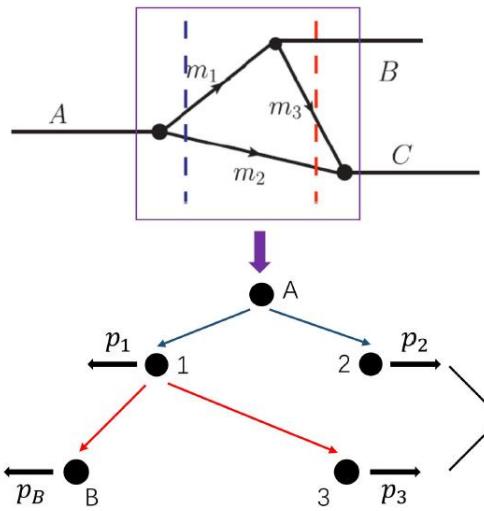
$$\sim \ln \left(\sqrt{S_{CD}} - m_{TS} \right)$$



$$\sim 1 / (\sqrt{S_{CD}} - m_{TS})$$



Triangle Singularity



Actually Happened Process

- (1) All particles are on mass-shell;
- (2) Particle 3 catch up particle 2.

If Integral Divergence,
it should require the pole at

$$q_b = q_{on} - i\epsilon'$$

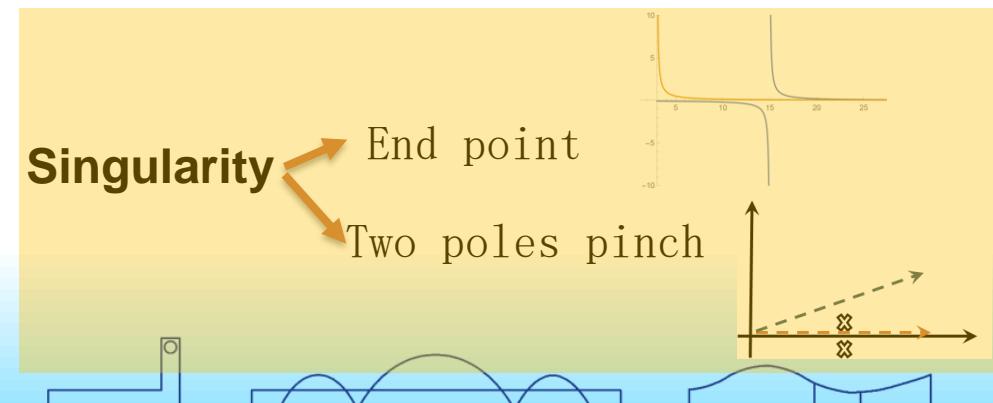
$$\cos \theta = -1 \text{ or } 1$$

$$E_C - \sqrt{q_{on}^2 + m_2^2} - \sqrt{\vec{p}_C^2 + q_{on}^2 + m_3^2 - 2|\vec{p}_C|q_{on}(-1 \text{ or } 1)} = 0$$

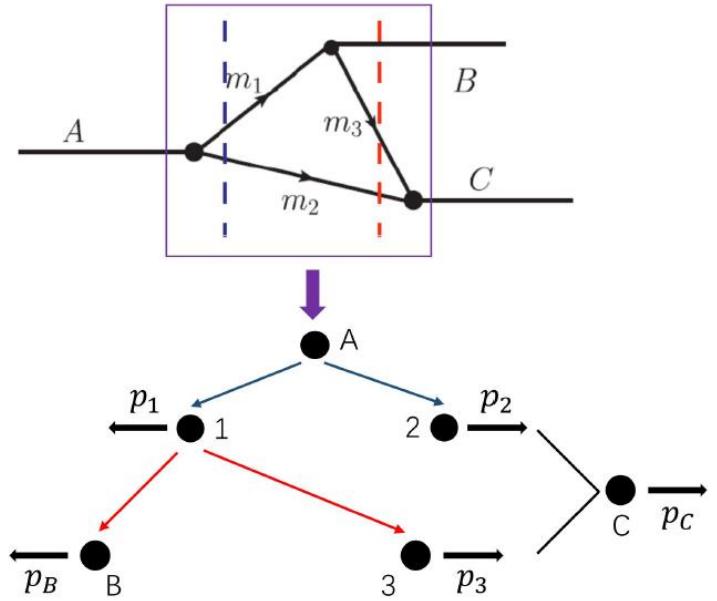
$$\left(\frac{q_{on}}{\omega_2} + \frac{q_{on} - |\vec{p}_C|(-1 \text{ or } 1)}{\omega_3} \right) i\epsilon' + i\epsilon = 0$$

$$q_{on} = \frac{\sqrt{(m_A^2 - (m_1 + m_2)^2)(m_A^2 - (m_1 - m_2)^2)}}{2m_A}$$

$$(v_2 - v_3) < 0$$



Triangle Singularity: Physics & Math



Actually Happened Process

- (1) All particles are on mass-shell;
- (2) Particle 3 catch up particle 2.

$$(v_2 - v_3) < 0$$

$$\sqrt{(E_C - \omega_2)^2 - m_3^2} - (|\vec{p}_C| - |q_{on}|) = 0$$

$$\& \frac{|q_{on}|}{\omega_2} < \frac{|\vec{p}_C| - |q_{on}|}{E_C - \omega_2}$$

with

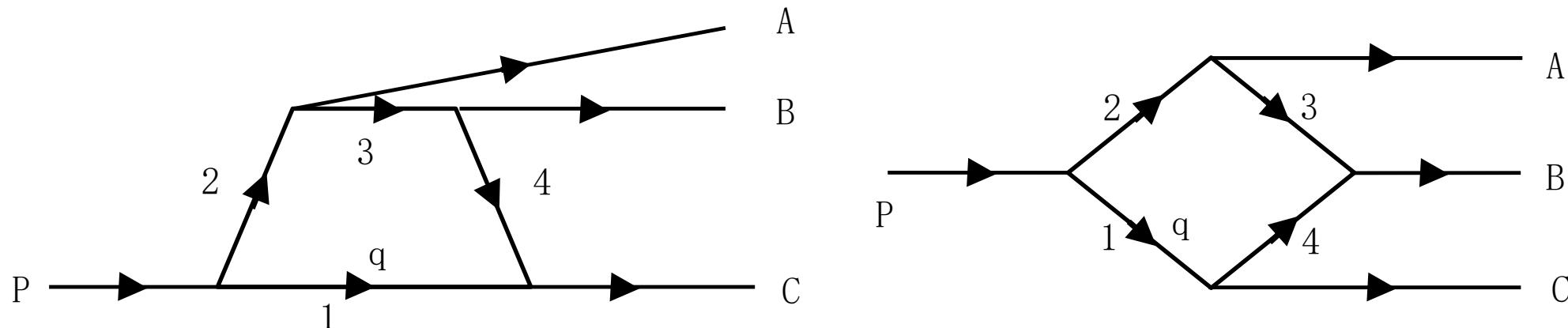
$$E_C = \frac{m_A^2 + m_C^2 - m_B^2}{2m_A^2} = |\vec{p}_C|^2 + m_C^2$$

$$\omega_2 = \frac{m_A^2 + m_2^2 - m_1^2}{2m_A^2} = |q_{on}|^2 + m_2^2$$

The condition for six particles.



How about Box Singularity



$$I(P, p_A, p_C) = \int d^4q \frac{1}{q^2 - m_1^2 + i\epsilon} \frac{1}{(P - q)^2 - m_2^2 + i\epsilon} \frac{1}{(P - p_A - q)^2 - m_3^2 + i\epsilon} \frac{1}{(q - p_C)^2 - m_4^2 + i\epsilon}$$

$$P = (M, 0, 0, 0)$$

$$p_C = (E_C, 0, 0, p_C) \equiv k_2$$

$$p_A \equiv P - k_1$$

$$k_1 = (P - E_A, p_A \sin \theta_1, 0, p_A \cos \theta_1)$$

$$q = (q_0, q \sin \theta \cos \phi, q \sin \theta \sin \phi, q \cos \theta)$$

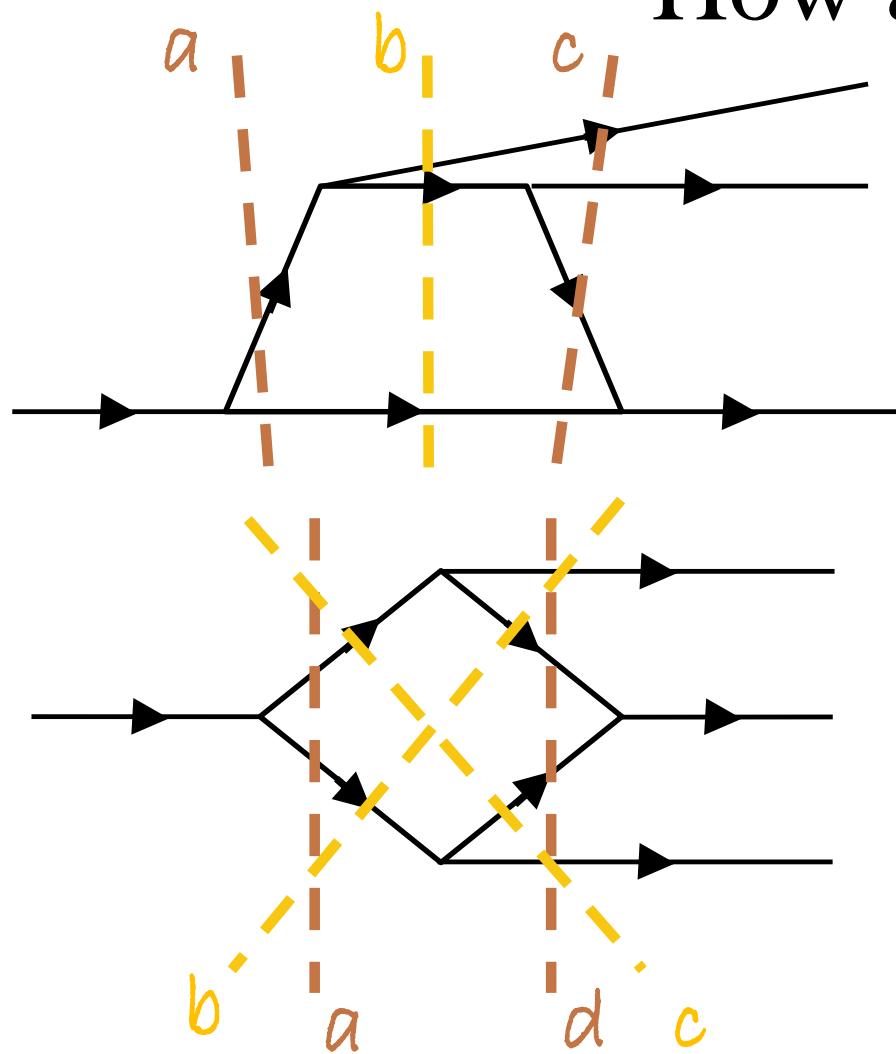
q^0 的奇点: $q^0 = -\omega_1 + i\epsilon$ $P^0 - \omega_2 + i\epsilon$ $k_1^0 - \omega_3 + i\epsilon$ $k_2^0 - \omega_4 + i\epsilon$

$$\omega_1 - i\epsilon P^0 + \omega_2 - i\epsilon k_1^0 + \omega_3 - i\epsilon k_2^0 + \omega_4 - i\epsilon$$

$$\omega_1 = \sqrt{\vec{q}^2 + m_1^2}, \omega_2 = \sqrt{\vec{q}^2 + m_2^2}, \omega_3 = \sqrt{(\vec{k}_1 - \vec{q})^2 + m_3^2}, \omega_4 = \sqrt{(\vec{q} - \vec{k}_2)^2 + m_4^2}.$$



How about Box Singularity



$$\sim \int d^3 \vec{q} \frac{1}{abc}$$

$$\sim \int d^3 \vec{q} \frac{a+d}{abcd}$$

$$a + d = b + c$$

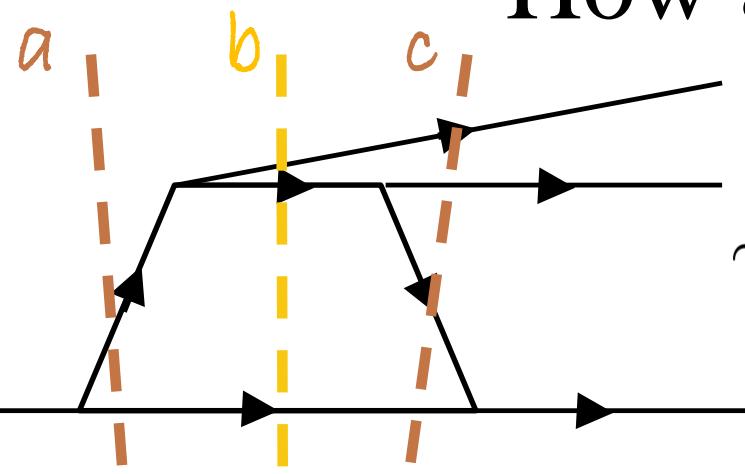
a = $P^0 - \omega_1 - \omega_2 + i\epsilon \rightarrow q$
b = $k_1^0 - \omega_1 - \omega_3 + i\epsilon \rightarrow q, \theta, \phi$
c = $k_2^0 - \omega_1 - \omega_4 + i\epsilon \rightarrow q, \theta$

a = $P^0 - \omega_1 - \omega_2 + i\epsilon \rightarrow q$
b = $k_1^0 - \omega_1 - \omega_3 + i\epsilon \rightarrow q, \theta, \phi$
c = $P^0 - k_2^0 - \omega_2 - \omega_4 + i\epsilon \rightarrow q, \theta$
d = $k_1^0 - k_2^0 - \omega_3 - \omega_4 + i\epsilon \rightarrow q, \theta, \phi$



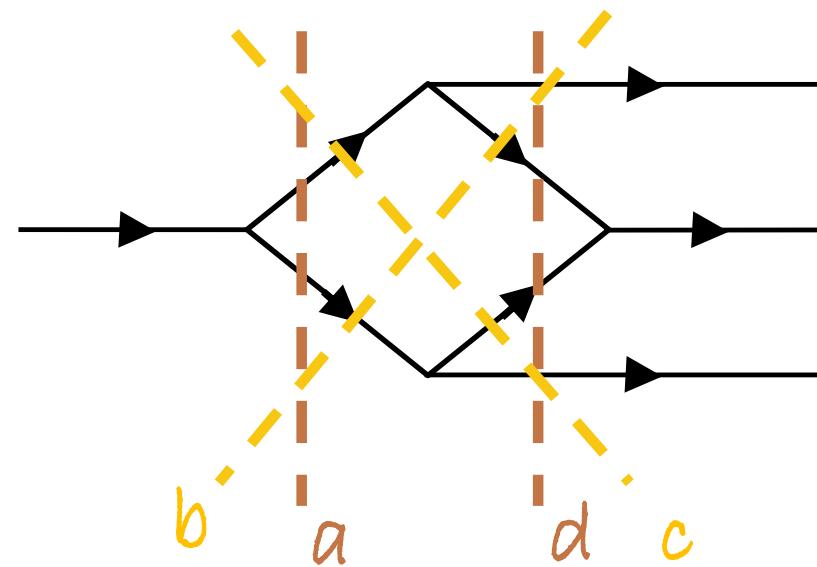
preliminary

How about Box Singularity



$$\sim \int d^3 \vec{q} \frac{1}{abc}$$

$a, b, c \rightarrow 0???$



$$\sim \int d^3 \vec{q} \frac{a+d}{abcd}$$

$a+d = b+c$

$a, b, c, d \rightarrow 0???$

$$a, b \rightarrow 0 \& c \not\rightarrow 0$$

$$a, c \rightarrow 0 \& b \not\rightarrow 0$$

$$b, c \rightarrow 0 \& a \not\rightarrow 0$$

$$a, b \rightarrow 0 \& c, d \not\rightarrow 0$$

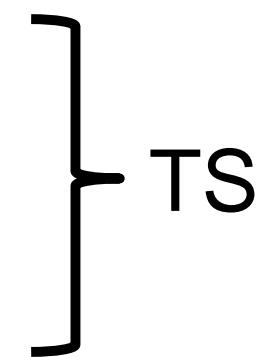
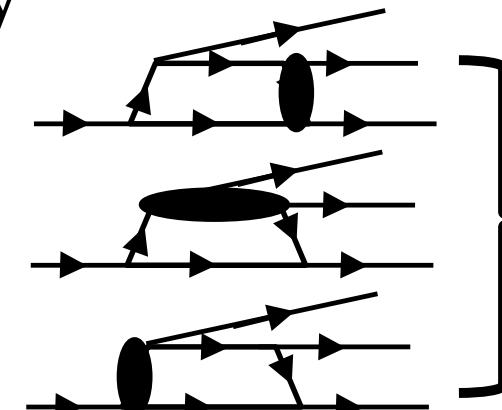
$$a, c \rightarrow 0 \& b, d \not\rightarrow 0$$

$$b, d \rightarrow 0 \& a, c \not\rightarrow 0$$

$$c, d \rightarrow 0 \& a, b \not\rightarrow 0$$

$$a, d \rightarrow 0 \& b, c \not\rightarrow 0$$

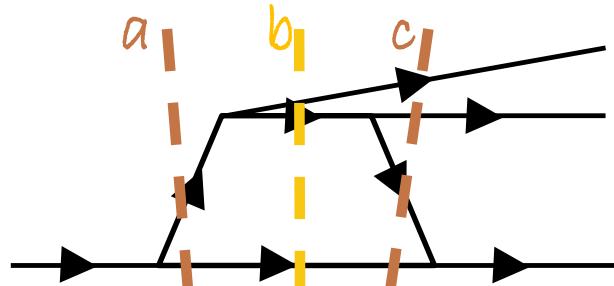
$$b, c \rightarrow 0 \& a, d \not\rightarrow 0$$



Convergence!!!



How about Box Singularity



$$\sim \int d^3 \vec{q} \frac{1}{a b c}$$

$$\begin{aligned} & \int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \\ & \int_{-1}^1 d\cos\theta \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2qk_2 \cos\theta + i\varepsilon)} \\ & \int_0^{2\pi} d\phi \frac{1}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} - 2k_1 q (\sin\theta_1 \sin\theta \cos\phi + \cos\theta_1 \cos\theta) + i\varepsilon)} \end{aligned}$$

The discussion of three integrated variables:

1. q : one singularity is on $q_{on} + i\varepsilon$. Since the range of q is $[0, \infty)$, thus it requires for another pinch singularity, i.e., $q_{on} - i\varepsilon$, which is similar as Triangle Singularity.
2. $\cos\theta$ will have two singularity cases: end point & pinch
3. For ϕ , actually it is only depend on $\cos\phi$, and it is a 1st singularity. Thus, after integration, its divergences only happen at the end point, i.e., $\phi = 0$ or π , which means that box singularity only happens when all momenta on the same plane. Correspondingly, in Triangle Singularity case, all momenta are on the on line.

As distinguished by rely on ϕ or not, we can divide our discussion into two cases:

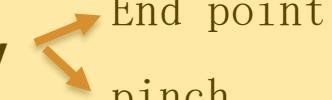
$\theta_1 = 0/\pi$: nothing about ϕ , only q and $\cos\theta$ are considered.

$0 < \theta_1 < \pi$: $\phi = 0$ or π all momenta are on one plane.

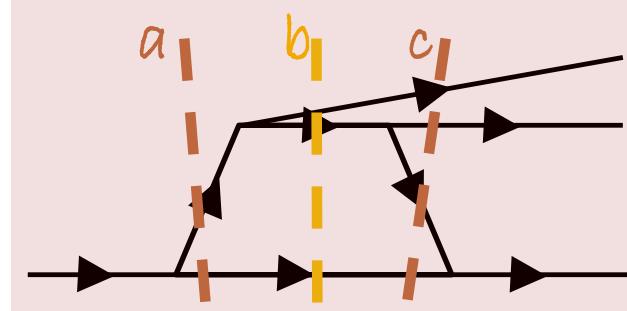
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How about Box Singularity

Singularity



If $\theta_1 = 0/\pi$, it will be independent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc} \frac{\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \int_{-1}^1 \frac{d\cos\theta}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} \mp 2k_1 q \cos\theta + i\varepsilon)} \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2q k_2 \cos\theta + i\varepsilon)}}{1}$$

Find pole of $\cos\theta$

$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 \frac{d\cos\theta}{\cos\theta \mp (Y(q) - i\varepsilon)} \frac{1}{\cos\theta - (X(q) - i\varepsilon)}$$

$$X(q) = \frac{m_4^2 - m_1^2 - m_C^2 + 2k_2^0 \sqrt{q^2 + m_1^2}}{2k_2 q}$$

$$Y(q) = \frac{m_3^2 - m_1^2 - m_{BC}^2 + 2k_1^0 \sqrt{q^2 + m_1^2}}{2k_1 q}$$

Pinch point for $q, q_{on} - i\varepsilon$ should be a singularity pole of terms related to $X(q)$ and $Y(q)$.

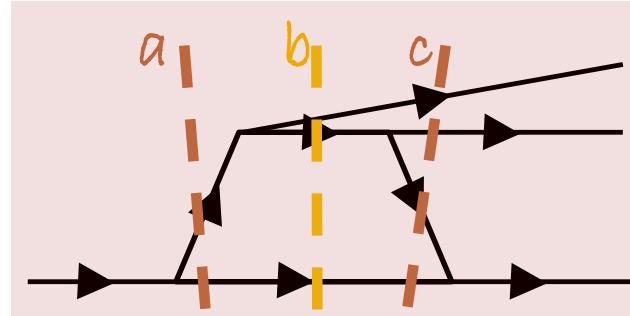


preliminary

How about Box Singularity



If $\theta_1 = 0/\pi$, it will be independent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc} \frac{\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \int_{-1}^1 \frac{d\cos\theta}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} \mp 2k_1 q \cos\theta + i\varepsilon)} \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2q k_2 \cos\theta + i\varepsilon)}}{1}$$

Find pole of $\cos\theta$

$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 \frac{d\cos\theta}{\cos\theta \mp (Y(q) - i\varepsilon)} \frac{1}{\cos\theta - (X(q) - i\varepsilon)}$$

$$X(q) = \frac{m_4^2 - m_1^2 - m_C^2 + 2k_2^0 \sqrt{q^2 + m_1^2}}{2k_2 q} \quad Y(q) = \frac{m_3^2 - m_1^2 - m_{BC}^2 + 2k_1^0 \sqrt{q^2 + m_1^2}}{2k_1 q}$$

Pinch point for q , $q_{on} - i\varepsilon$ should be a singularity pole of terms related to $X(q)$ and $Y(q)$.

$\theta_1 = 0$ $\cos\theta - (Y(q_{on}) - i\varepsilon') - i\varepsilon = 0$ and $\cos\theta - (X(q_{on}) - i\varepsilon') - i\varepsilon = 0$ only end point divergent

- (1) $\cos\theta = +1 \Leftrightarrow X(q_{on}) = Y(q_{on}) = +1$ (For real part) & ($v_4 > v_1$ or $v_3 > v_1$) (For imagery part)
- (2) $\cos\theta = -1 \Leftrightarrow X(q_{on}) = Y(q_{on}) = -1$ (For real part) but imagery part can not be satisfied

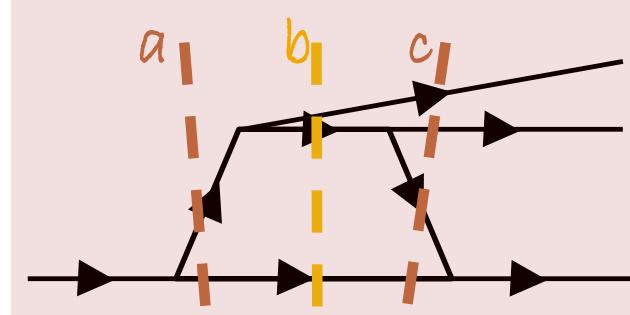


preliminary

How about Box Singularity



If $\theta_1 = 0/\pi$, it will be independent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc} \frac{\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)}}{\int_{-1}^1 d\cos\theta \frac{1}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} \mp 2k_1 q \cos\theta + i\varepsilon)} \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2q k_2 \cos\theta + i\varepsilon)}}$$

Find pole of $\cos\theta$

$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 d\cos\theta \frac{1}{\cos\theta \mp (Y(q) - i\varepsilon)} \frac{1}{\cos\theta - (X(q) - i\varepsilon)}$$

$$X(q) = \frac{m_4^2 - m_1^2 - m_C^2 + 2k_2^0 \sqrt{q^2 + m_1^2}}{2k_2 q}$$

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Pinch point for q , $q_{on} - i\varepsilon$ should be a singularity pole of terms related to $X(q)$ and $Y(q)$.

$\theta_1 = 0$ $\cos\theta - (Y(q_{on}) - i\varepsilon') - i\varepsilon = 0$ and $\cos\theta - (X(q_{on}) - i\varepsilon') - i\varepsilon = 0$ only end point diversion

(1) $\cos\theta = +1 \Leftrightarrow X(q_{on}) = Y(q_{on}) = +1$ (For real part) & ($v_4 > v_1$ or $v_3 > v_1$) (For imagery part)

(2) $\cos\theta = -1 \Leftrightarrow X(q_{on}) = Y(q_{on}) = -1$ (For real part) but imagery part can not be satisfied

$\theta_1 = \pi$ $\cos\theta + (Y(q_{on}) - i\varepsilon') - i\varepsilon = 0$ and $\cos\theta - (X(q_{on}) - i\varepsilon') - i\varepsilon = 0$ end point and pinch divergent

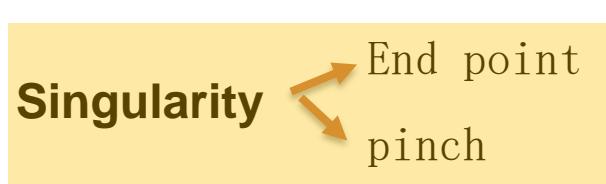
(1) $\cos\theta = +1 \Leftrightarrow X(q_{on}) = -Y(q_{on}) = +1$ (For real part) & ($v_4 > v_1$) (For imagery part)

(2) $\cos\theta = -1 \Leftrightarrow X(q_{on}) = -Y(q_{on}) = -1$ (For real part) & ($v_3 > v_1$) (For imagery part)

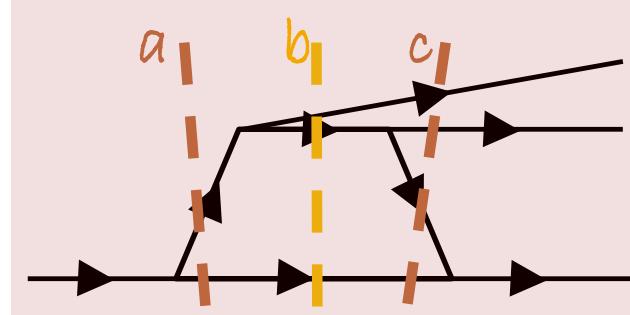


preliminary

How about Box Singularity



If $\theta_1 = 0/\pi$, it will be independent on ϕ



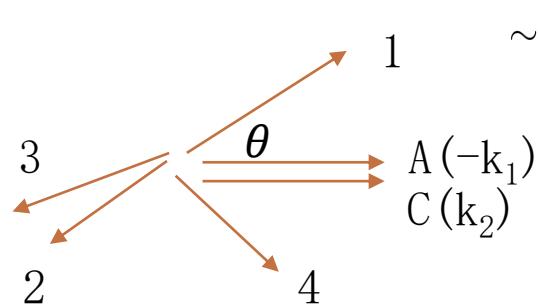
$$\sim \int d^3 \vec{q} \frac{1}{abc} \frac{\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \int_{-1}^1 \frac{dcos\theta}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} \mp 2k_1 q cos\theta + i\varepsilon)} \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2q k_2 cos\theta + i\varepsilon)}}{1}$$

$\theta_1 = \pi \Rightarrow$ Pinch point

$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 \frac{dcos\theta}{\cos\theta + (Y(q) - i\varepsilon)} \frac{1}{\cos\theta - (X(q) - i\varepsilon)}$$

$$X(q) = \frac{m_4^2 - m_1^2 - m_C^2 + 2k_2^0 \sqrt{q^2 + m_1^2}}{2k_2 q}$$

$$Y(q) = \frac{m_3^2 - m_1^2 - m_{BC}^2 + 2k_1^0 \sqrt{q^2 + m_1^2}}{2k_1 q}$$



$$\sim \int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \frac{1}{-Y(q) - X(q) + i\varepsilon} \int_{-1}^1 \frac{dcos\theta}{\left(\frac{1}{\cos\theta + (Y(q) - i\varepsilon)} - \frac{1}{\cos\theta - (X(q) - i\varepsilon)} \right)}$$

Finite Value

$$-Y(q_{on} - i\varepsilon') - X(q_{on} - i\varepsilon') + i\varepsilon = 0 \quad \& \quad -Y(q_{on}) = X(q_{on}) = \cos\theta$$

$$-(-i\varepsilon') \frac{d(Y(q) + X(q))}{dq} \Big|_{q=q_{on}} + i\varepsilon = 0 \quad \Rightarrow \quad \frac{d(Y(q) + X(q))}{dq} \Big|_{q=q_{on}} < 0$$

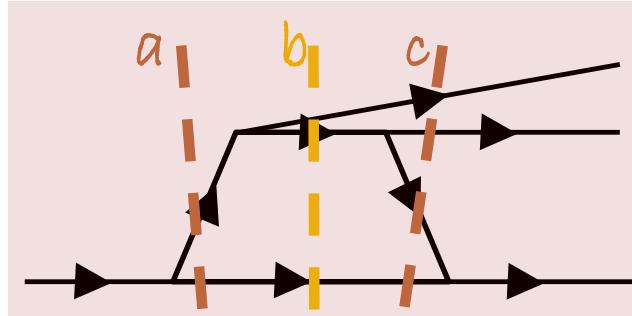
$$\frac{d(Y(q) + X(q))}{dq} \Big|_{q=q_{on}} = \frac{k_2^0}{k_2 \omega_1(q_{on})} - \frac{\cos\theta}{q_{on}} + \frac{k_1^0}{k_1 \omega_1(q_{on})} + \frac{\cos\theta}{q_{on}} = \frac{1}{\omega_1(q_{on})} \left(\frac{k_2^0}{k_2} + \frac{k_1^0}{k_1} \right) > 0$$

No Box
Singularity



How about Box Singularity

If $\theta_1 = 0/\pi$, it will be independent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc} \frac{\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \int_{-1}^1 \frac{d\cos\theta}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} \mp 2k_1 q \cos\theta + i\varepsilon)} \frac{1}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2q k_2 \cos\theta + i\varepsilon)}}{}$$

$\theta_1 = 0 \Rightarrow$ End point



Only $\cos\theta = 1$, physically, $\hat{\vec{v}}_1 = \hat{\vec{v}}_C$.

$$v_4 - v_1 > 0$$

$$v_3 - v_1 > 0$$

3 and 4
both can
catch with 1

$$v_4 - v_1 > 0$$

$$v_3 - v_1 < 0$$

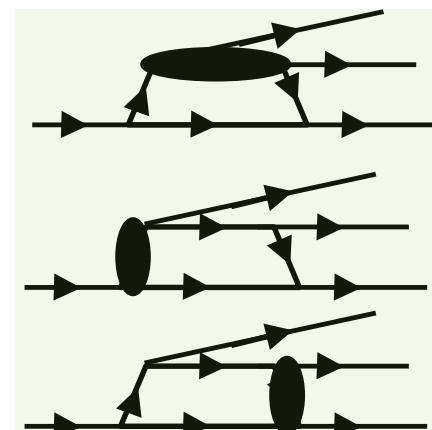
only 4 catch
with 1

$$1 = Y(q_{on}) = X(q_{on})$$

$$v_4 - v_1 < 0$$

$$v_3 - v_1 > 0$$

only 3 catch
with 1



$\theta_1 = \pi \Rightarrow$ End point

$$\cos\theta = 1 \quad v_4 > v_1$$

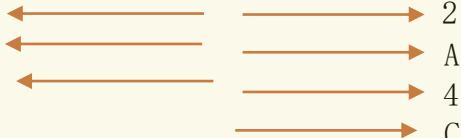
$$1 = -Y(q_{on}) = X(q_{on})$$



only 4 catch with 1

$$\cos\theta = -1 \quad v_3 > v_1$$

$$1 = Y(q_{on}) = -X(q_{on})$$



only 3 catch with 1

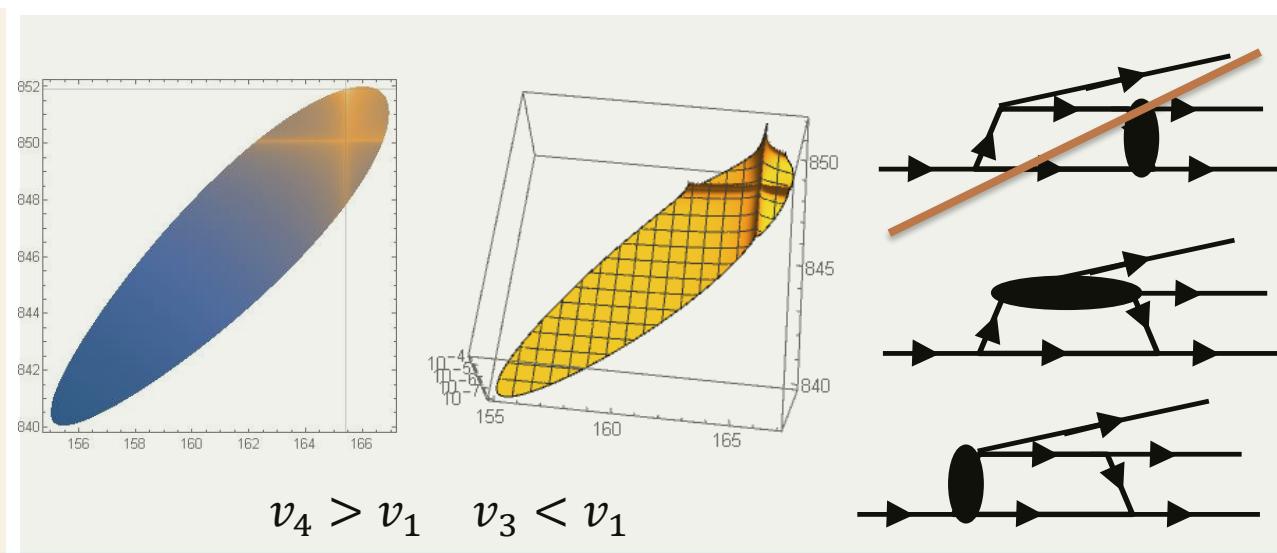
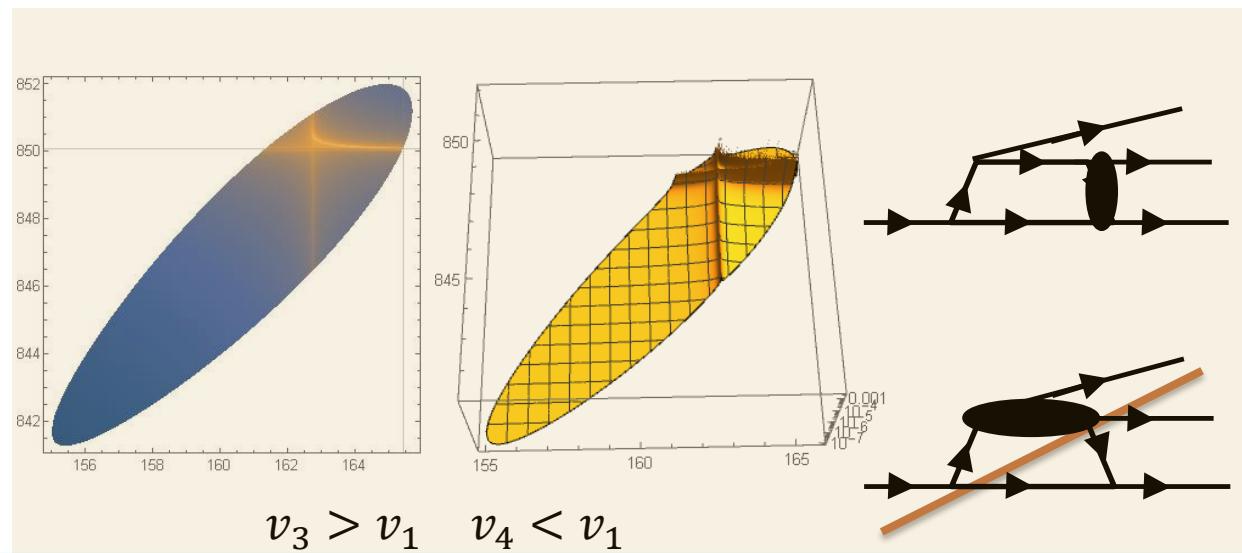
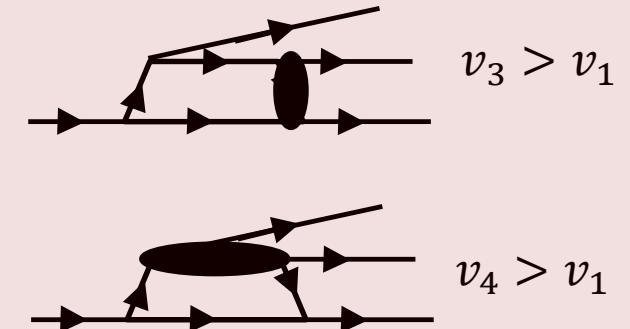
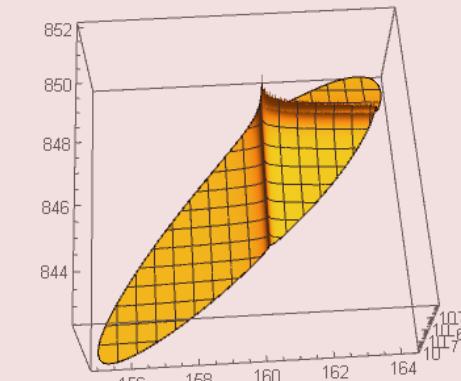
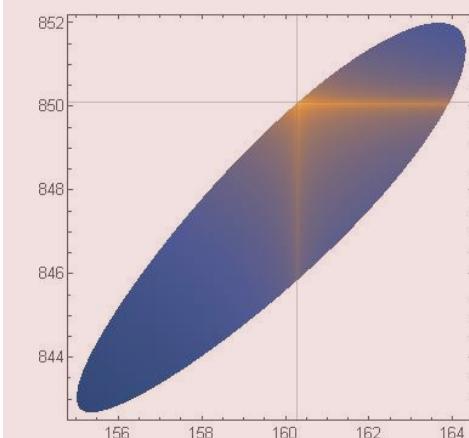
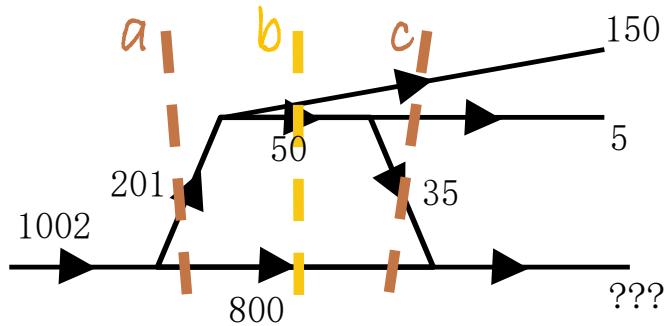
TS + On shell



preliminary

How about Box Singularity

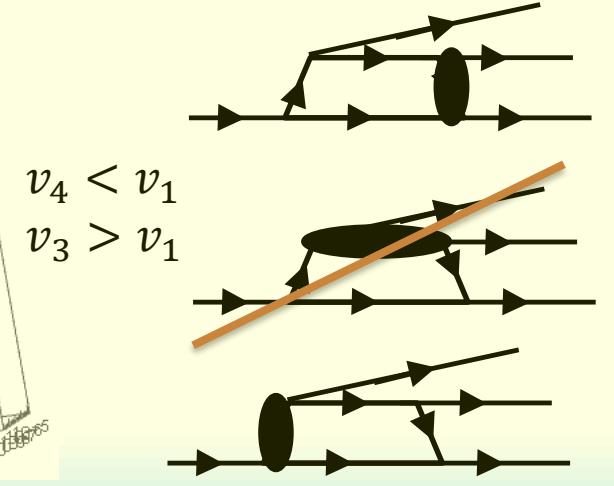
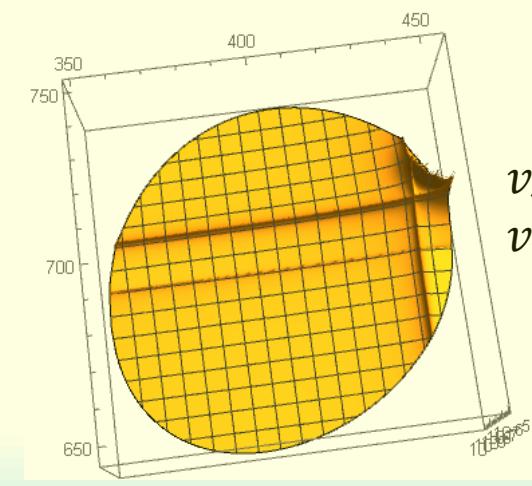
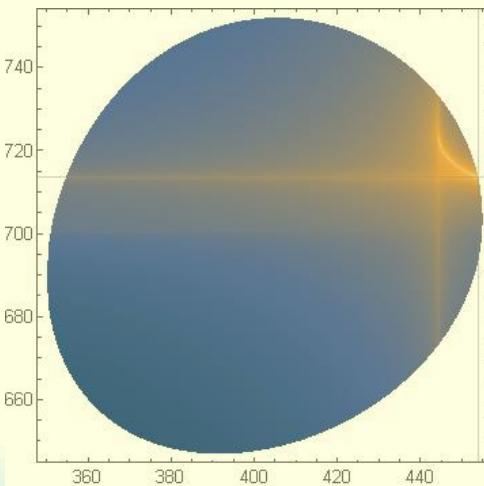
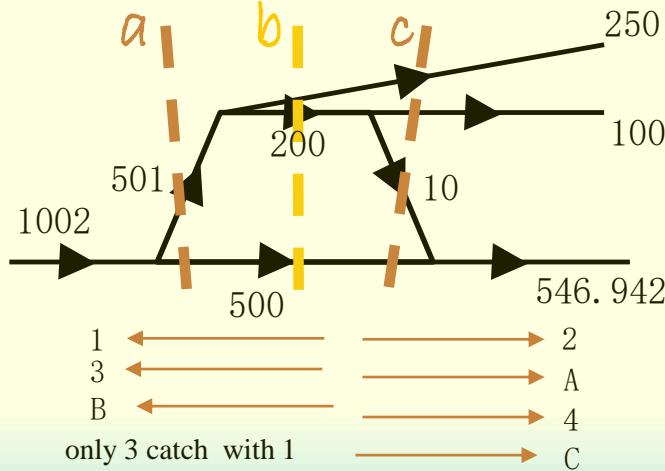
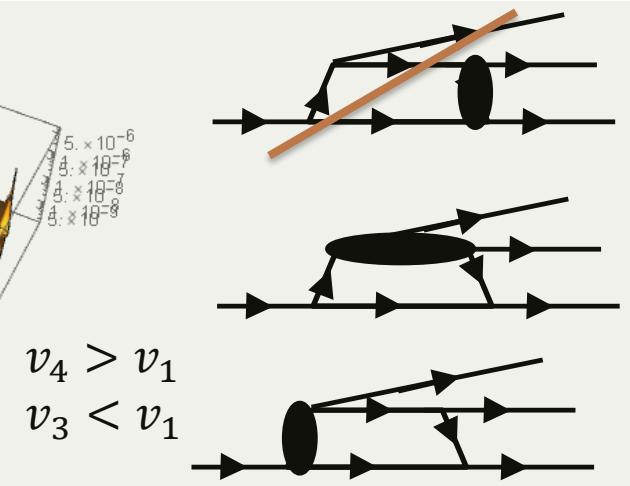
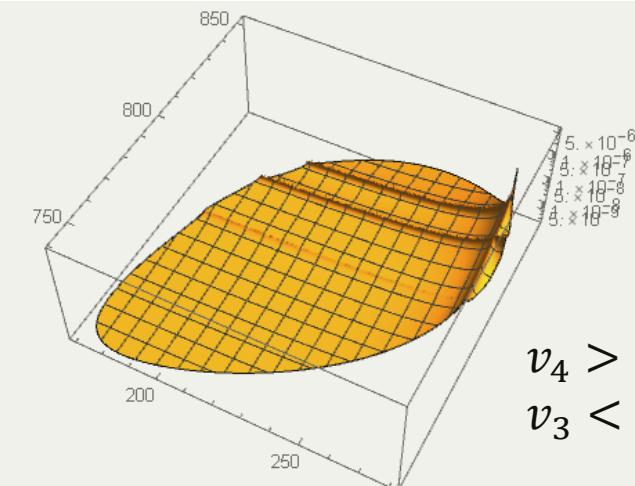
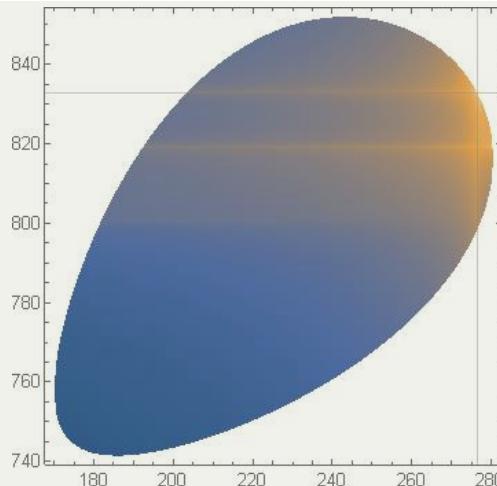
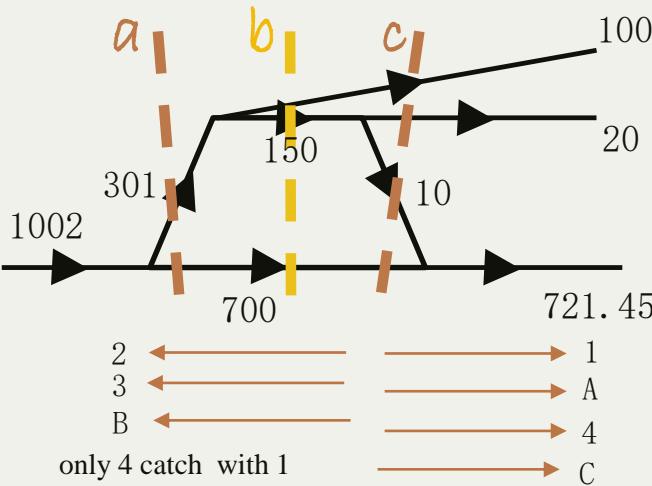
For $\theta_1 = 0$, some examples



preliminary

How about Box Singularity

For $\theta_1 = \pi$, two examples



How about Box Singularity

If $0 < \theta_1 < \pi$, it will be dependent on ϕ

$$\sim \int d^3 \vec{q} \frac{1}{a b c} \int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\epsilon)} \int_{-1}^1 \frac{d\cos\theta}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2qk_2\cos\theta + i\epsilon)} \int_0^{2\pi} \frac{d\phi}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} - 2k_1q(\sin\theta_1\sin\theta\cos\phi + \cos\theta_1\cos\theta) + i\epsilon)}$$

End point
of $\cos\phi$

For ϕ , it is only dependent on the $\cos\phi$, only end point diversion need to consider, thus, we take $\phi = 0$ or π . In physically, it means that all momenta of the particles are **in the one plane**, while the Triangle Singularity happen that all momenta are in the one line !

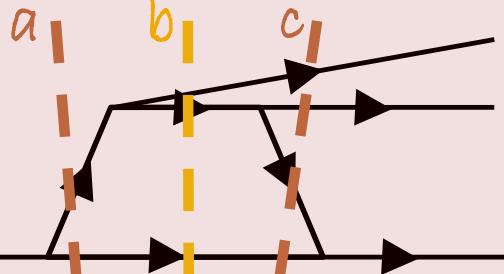
$$\sim \int_0^\infty \frac{dq}{q - q_{on} - i\epsilon} \int_{-1}^1 \frac{d\cos\theta}{\cos\theta - (X(q) - i\epsilon)} \frac{1}{\cos(\theta_1 \mp \theta) - (Y(q) - i\epsilon)} \quad \text{for } \phi = 0 \text{ or } \pi$$

Pinch / End divergent of $\cos\theta$ At $q_{on} - i\epsilon$



How about Box Singularity

If $0 < \theta_1 < \pi$, it will be dependent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc} \int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)} \int_{-1}^1 \frac{d\cos\theta}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2qk_2\cos\theta + i\varepsilon)} \int_0^{2\pi} \frac{d\phi}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} - 2k_1q(\sin\theta_1 \sin\theta \cos\phi + \cos\theta_1 \cos\theta) + i\varepsilon)}$$

End point
of $\cos\phi$

$$\sim \int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 \frac{d\cos\theta}{\cos\theta - (X(q) - i\varepsilon)} \frac{1}{\cos(\theta_1 \mp \theta) - (Y(q) - i\varepsilon)}$$

for $\phi = 0$ or π

Pinch/End divergent of $\cos\theta$ At $q_{on} - i\varepsilon$

- $\theta = 0 \rightarrow X(q) = +1 \& Y(q) = +\cos\theta_1 \& (\nu_4 > \nu_1 \text{ or } \nu_{3\parallel 1} > \nu_1)$

$$\theta_1 = 0/\pi$$

- $\theta = \pi \rightarrow X(q) = -1 \& Y(q) = -\cos\theta_1 \& (\nu_{3\parallel 1} > \nu_1)$

$$\theta_1 = \pi$$

- $\phi = 0 \quad \theta = \theta_1 \rightarrow X(q) = +\cos\theta_1 \& Y(q) = +1 \& (\nu_3 > \nu_1 \text{ or } \nu_{4\parallel 1} > \nu_1)$

- $\phi = \pi \quad \theta = \pi - \theta_1 \rightarrow X(q) = -\cos\theta_1 \& Y(q) = -1 \& (\nu_{4\parallel 1} > \nu_1)$

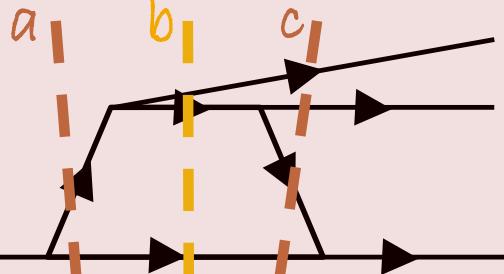
These two cases can be combined with three cases in $\theta_1 = 0/\pi$.

When $\cos(\theta_1 \mp \theta) = 1$ or -1 , it similar as θ has two poles with $\theta_1 \pm \varepsilon$



How about Box Singularity

If $0 < \theta_1 < \pi$, it will be dependent on ϕ



$$\sim \int d^3 \vec{q} \frac{1}{abc}$$

$$\int_0^\infty \frac{dq}{(P^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + m_2^2} + i\varepsilon)}$$

$$\int_{-1}^1 \frac{d\cos\theta}{(k_2^0 - \sqrt{q^2 + m_1^2} - \sqrt{q^2 + k_2^2 + m_4^2} - 2qk_2\cos\theta + i\varepsilon)}$$

$$\int_0^{2\pi} \frac{d\phi}{(k_1^0 - \sqrt{q^2 + m_1^2} - \sqrt{k_1^2 + q^2 + m_3^2} - 2k_1q(\sin\theta_1\sin\theta\cos\phi + \cos\theta_1\cos\theta) + i\varepsilon)}$$

End point
of $\cos\phi$

$$\sim \int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 \frac{d\cos\theta}{\cos\theta - (X(q) - i\varepsilon)} \frac{1}{\cos(\theta_1 \mp \theta) - (Y(q) - i\varepsilon)}$$

for $\phi = 0$ or π

Pinch/End divergent of $\cos\theta$ At $q_{on} - i\varepsilon$

$$\cos(\theta_1 \mp \theta) = Y(q) - i\varepsilon$$

$$\cos\theta = \cos((\theta \mp \theta_1) \pm \theta_1) \rightarrow \cos\theta = Y(q)\cos\theta_1 \mp \sin(\theta \mp \theta_1)\sin\theta_1 - \frac{\sin\theta}{\sin(\theta \mp \theta_1)}i\varepsilon$$

$$\sin(\theta \mp \theta_1) = \begin{cases} +\sqrt{1 - Y^2(q)} & \theta - \theta_1 > 0 \\ -\sqrt{1 - Y^2(q)} & \theta - \theta_1 < 0 \end{cases} \quad \begin{cases} \theta + \theta_1 < \pi \\ \theta + \theta_1 > \pi \end{cases}$$

$$\phi = 0 \quad \cos\theta = Y(q)\cos\theta_1 + \sqrt{1 - Y^2(q)}\sin\theta_1 + i\varepsilon \equiv \tilde{Y}^+(q) + i\varepsilon \quad \theta - \theta_1 < 0$$

$$\phi = \pi \quad \cos\theta = Y(q)\cos\theta_1 - \sqrt{1 - Y^2(q)}\sin\theta_1 + i\varepsilon \equiv \tilde{Y}^-(q) + i\varepsilon \quad \theta + \theta_1 > \pi$$

Physics Condition

$$\vec{p}_{3\perp 1} \cdot \vec{p}_{4\perp 1} < 0$$

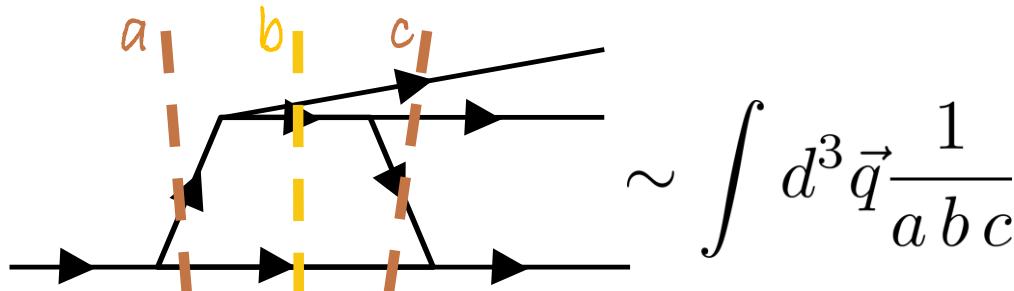
$$v_{4\perp 1} \cdot v_{3\perp 1} < 0$$

3 leave 4 back



How about Box Singularity

If $0 < \theta_1 < \pi$, it will be dependent on ϕ



$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 d\cos\theta \frac{1}{\cos\theta - (X(q) - i\varepsilon)} \frac{1}{\cos\theta - (\tilde{Y}(q) + i\varepsilon)} \sim \int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \frac{1}{X(q) - \tilde{Y}(q) - i\varepsilon}$$

$$\longrightarrow X(q_{on} - i\varepsilon') - \tilde{Y}(q_{on} - i\varepsilon') - i\varepsilon = 0$$

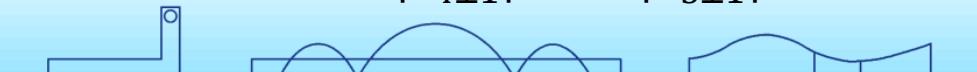
Real $\longrightarrow X(q_{on}) = \tilde{Y}(q_{on}) \longrightarrow$ On-shell

Imagery $\frac{dX(q)}{dq} \Big|_{q=q_{on}} - \frac{d\tilde{Y}(q)}{dq} \Big|_{q=q_{on}} < 0$

😭 Hard

$$\frac{v_1 - v_{4\perp 1}}{|v_{4\perp 1}|} + \frac{v_1 - v_{3\perp 1}}{|v_{3\perp 1}|} < 0$$

Real Collision between 4 and 1



Pinch/End divergent of $\cos\theta$ At $q_{on} - i\varepsilon$

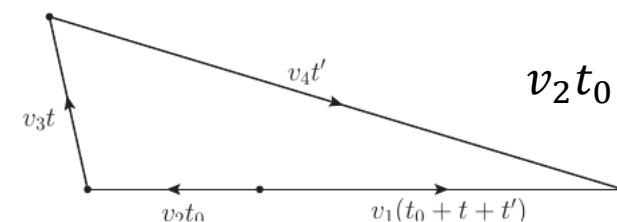
$$\int_0^\infty \frac{dq}{q - q_{on} - i\varepsilon} \int_{-1}^1 d\cos\theta \frac{1}{\cos\theta - (X(q) - i\varepsilon)} \frac{1}{\cos(\theta_1 \mp \theta) - (Y(q) - i\varepsilon)}$$

$$\phi = 0 \quad \cos\theta = \tilde{Y}^+(q) + i\varepsilon \quad \theta - \theta_1 < 0$$

$$\phi = \pi \quad \cos\theta = \tilde{Y}^-(q) + i\varepsilon \quad \theta + \theta_1 > \pi$$

Finite Value

$$\left(\frac{1}{\cos\theta - (X(q) - i\varepsilon)} - \frac{1}{\cos\theta - (\tilde{Y}(q) + i\varepsilon)} \right)$$



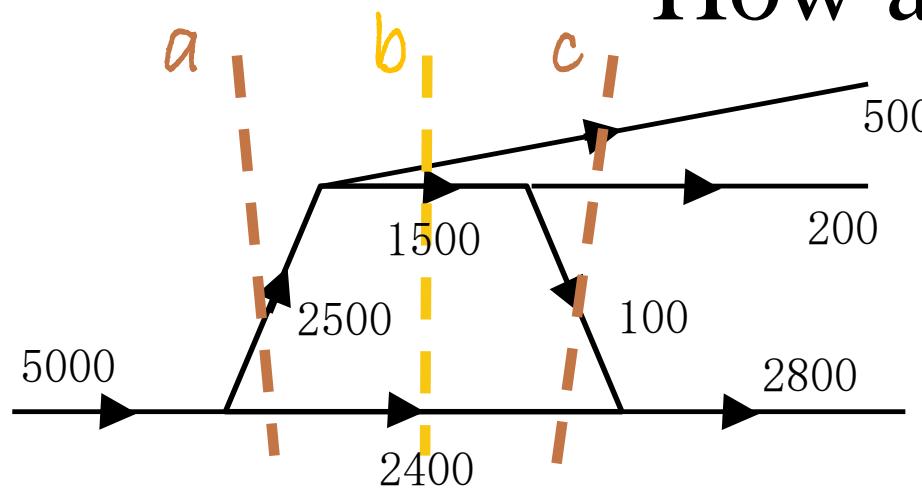
$$v_2 t_0 + v_{3\parallel 1} t + v_{4\parallel 1} t' = v_1 (t_0 + t + t')$$

$$h = |v_{3\perp 1}| t = |v_{4\perp 1}| t'$$

$$(v_1 - v_2)t_0 = (v_{3\parallel 1} - v_1)t + (v_{4\parallel 1} - v_1)t' \\ = - \left(\frac{v_1 - v_{4\parallel 1}}{|v_{4\perp 1}|} + \frac{v_1 - v_{3\parallel 1}}{|v_{3\perp 1}|} \right) h > 0$$



How about Box Singularity

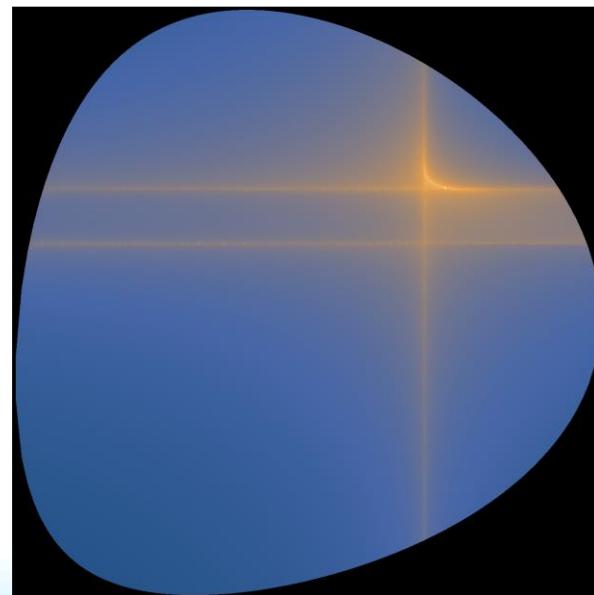
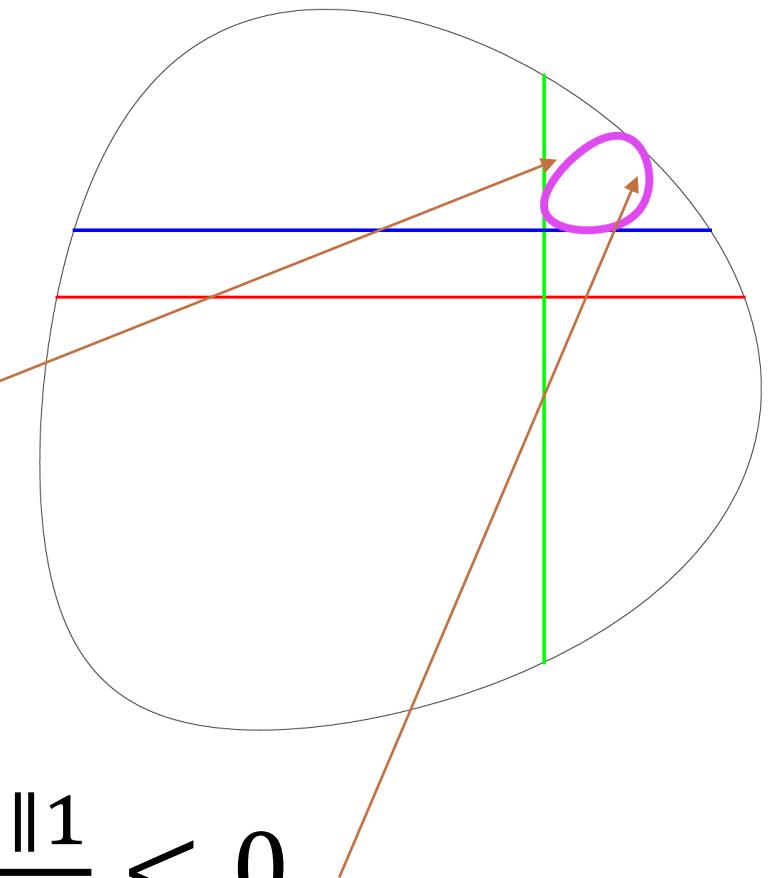


$$\sim \int d^3\vec{q} \frac{1}{abc}$$

a, b, c = 0

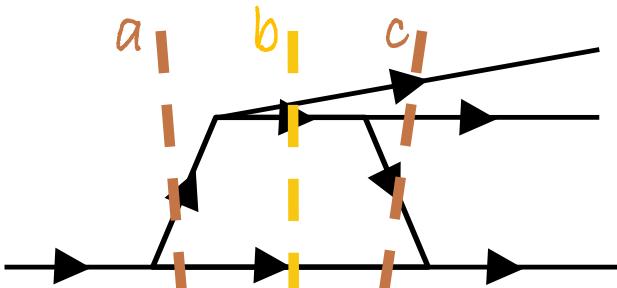
$$\nu_{4\perp 1} \cdot \nu_{3\perp 1} < 0$$

$$\frac{\nu_1 - \nu_{4\parallel 1}}{|\nu_{4\perp 1}|} + \frac{\nu_1 - \nu_{3\parallel 1}}{|\nu_{3\perp 1}|} < 0$$



How about Box Singularity

Summary of



$$q_{on} = \frac{m}{2} \sqrt{\left(1 - \left(\frac{m_1+m_2}{m}\right)^2\right) \left(1 - \left(\frac{m_1-m_2}{m}\right)^2\right)}, \quad \omega_1(q_{on}) = \sqrt{q_{on}^2 + m_1^2}$$

$$k_2^0 = E_C = \frac{m^2 - m_c^2 + m_{AB}^2}{m}, \quad k_2 = |\vec{p}_C| = \sqrt{(k_2^0)^2 - m_c^2}$$

$$k_1^0 = m - E_A = m - \frac{m^2 - m_A^2 + m_{BC}^2}{m}, \quad k_1 = |\vec{p}_A| = \frac{m}{2} \sqrt{\left(1 - \left(\frac{m_A+m_{BC}}{m}\right)^2\right) \left(1 - \left(\frac{m_A-m_{BC}}{m}\right)^2\right)}$$

$$\cos \theta_1 = -\frac{2E_c E_A - (m^2 + m_B^2 - m_{AB}^2 - m_{BC}^2)}{2|\vec{p}_C||\vec{p}_A|}$$

(2) Additional physics condition:

$$0 < \theta < \pi \quad v_{4\perp 1} \cdot v_{3\perp 1} < 0$$

$$\frac{v_1 - v_{4\parallel 1}}{|v_{4\perp 1}|} + \frac{v_1 - v_{3\parallel 1}}{|v_{3\perp 1}|} < 0$$

All vertices
real happen

$$v_{4\perp 1}=0$$

$$v_{3\perp 1}=0$$

1. $\theta = 0$: ($v_4 > v_1$ or $v_{3\parallel 1} > v_1$)

2. $\theta = \pi$: ($v_{3\parallel 1} > v_1$)

TS+on-shell & Special case

Not necessary real happen

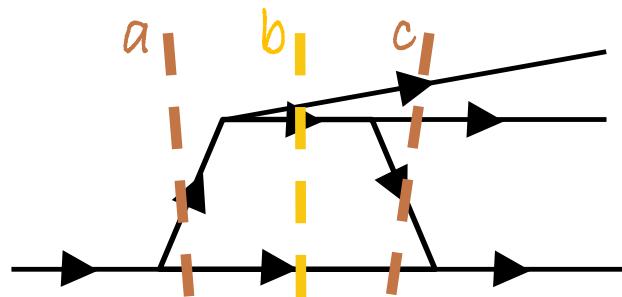
3. $\phi = 0$ $\theta = \theta_1$: ($v_3 > v_1$ or $v_{4\parallel 1} > v_1$)

4. $\phi = \pi$ $\theta = \pi - \theta_1$: ($v_{4\parallel 1} > v_1$)



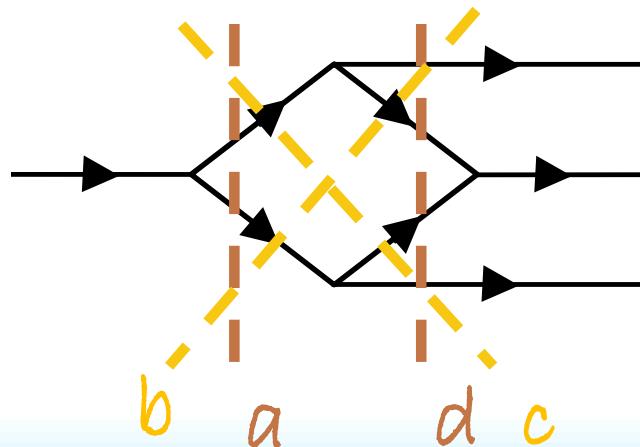
Summary

- We discuss the box singularity condition.



$$\begin{aligned} v_{4\perp 1} \cdot v_{3\perp 1} &< 0 \\ \frac{v_1 - v_{4\parallel 1}}{|v_{4\perp 1}|} + \frac{v_1 - v_{3\parallel 1}}{|v_{3\perp 1}|} &< 0 \\ \dots\dots \end{aligned}$$

All particles
on-shell at
one plane.



$$\begin{aligned} v_{4\perp 1} \cdot v_{3\perp 1} &> 0 \\ \frac{v_1 - v_{4\parallel 1}}{|v_{4\perp 1}|} + \frac{v_1 - v_{3\parallel 1}}{|v_{3\perp 1}|} &< 0 \quad \text{or} \quad \frac{v_2 - v_{4\parallel 1}}{|v_{4\perp 1}|} + \frac{v_2 - v_{3\parallel 1}}{|v_{3\perp 1}|} < 0 \\ |v_{4\perp 1}| &> |v_{3\perp 1}| \quad |v_{3\perp 1}| > |v_{4\perp 1}| \\ \dots\dots \end{aligned}$$

Some cases are divergent
but not means the collide
or decay really happen.



Backup



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University of Chinese Academy of Sciences



$$\sim \int d^3\vec{q} \left[\frac{1}{m_A - \omega_1(\vec{q}) - \omega_2(\vec{q}) + i\epsilon} \frac{1}{E_c - \omega_2(\vec{q}) - \omega_3(\vec{p}_C - \vec{q}) + i\epsilon} f(\vec{q}) + h(\vec{q}) \right]$$

$q_a = q_{on} + i\epsilon$ $E_C - \sqrt{q^2 + m_2^2} - \sqrt{\vec{p}_C^2 + q^2 + m_3^2 - 2|\vec{p}_C|q \cos \theta} + i\epsilon = 0$

If Integral Divergence,
it should require the pole at

$q_b = q_{on} - i\epsilon'$
 $\cos \theta = -1 \text{ or } 1$

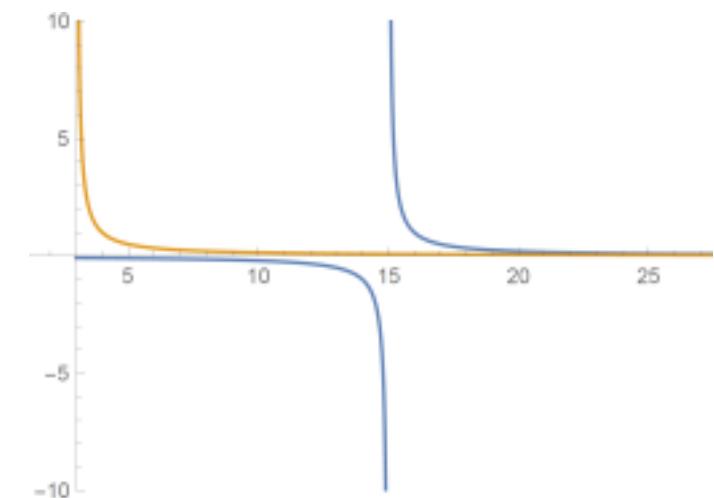
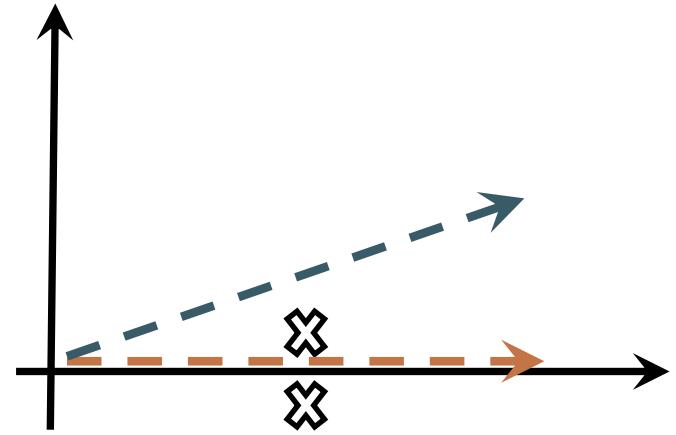
1. In the complex plane, the integral routine will be fixed between two singularity.

$$\int \frac{dx}{(x + i\epsilon)^2} \rightarrow \text{Convergence} \quad \int \frac{dx}{(x + i\epsilon)(x - i\epsilon)} \rightarrow \text{Divergence}$$

$$q_b = q_{on} - i\epsilon'$$

2. For $\cos \theta$, it is a one order singularity, thus, it will be convergence except at the edge.

$$\cos \theta = -1 \text{ or } 1$$



preliminary

How about Box Singularity

$$\frac{dX(q)}{dq} \Big|_{q=q_{on}} - \frac{d\tilde{Y}(q)}{dq} \Big|_{q=q_{on}} < 0 \quad \Longleftrightarrow \quad \frac{\nu_1 - \nu_{4\parallel 1}}{\nu_{4\perp 1}} + \frac{\nu_1 - \nu_{3\parallel 1}}{\nu_{3\perp 1}} < 0$$

$$\begin{aligned} \frac{dX(q)}{dq} \Big|_{q=q_{on}} - \frac{d\tilde{Y}(q)}{dq} \Big|_{q=q_{on}} &= \frac{dX(q)}{dq} \Big|_{q=q_{on}} - \frac{d\tilde{Y}}{dY} \frac{dY(q)}{dq} \Big|_{q=q_{on}} \\ &= \frac{k_2^0}{k_2 q_{on}} \left(\frac{q_{on}}{\omega_1} - \frac{k_2}{k_2^0} \cos \theta \right) + \frac{\sin \theta}{q_{on} \sin(\theta_1 \mp \theta)} \left(\frac{q_{on}}{\omega_1} \frac{k_1^0}{k_1} - \cos(\theta_1 \mp \theta) \right) \\ &= \frac{\sin \theta}{q_{on}} \left(\frac{k_2^0}{k_2 \sin \theta} \left(\frac{q_{on}}{\omega_1} - \frac{k_2 \cos \theta}{k_2^0} \right) + \frac{k_1^0}{k_1 \sin(\theta_1 \mp \theta)} \left(\frac{q_{on}}{\omega_1} - \frac{k_1 \cos(\theta_1 \mp \theta)}{k_1^0} \right) \right) \\ &= \frac{\sin \theta}{q_{on}} \left(\frac{\omega_1 + \omega_4}{\omega_4 |\nu_{4\perp 1}|} \left(\frac{q_{on}}{\omega_1} - \frac{k_2 \cos \theta}{\omega_1 + \omega_4} \right) + \frac{\omega_1 + \omega_3}{\omega_3 |\nu_{3\perp 1}|} \left(\frac{q_{on}}{\omega_1} - \frac{k_1 \cos(\theta_1 \mp \theta)}{\omega_1 + \omega_3} \right) \right) \\ &= \frac{\sin \theta}{q_{on}} \left(\frac{1}{|\nu_{4\perp 1}|} \left(\frac{q_{on}}{\omega_1} - \frac{k_2 \cos \theta - q_{on}}{\omega_4} \right) + \frac{1}{|\nu_{3\perp 1}|} \left(\frac{q_{on}}{\omega_1} - \frac{k_1 \cos(\theta_1 \mp \theta) - q_{on}}{\omega_3} \right) \right) \\ &= \frac{\sin \theta}{q_{on}} \left(\frac{\nu_1 - \nu_{4\parallel 1}}{|\nu_{4\perp 1}|} + \frac{\nu_1 - \nu_{3\parallel 1}}{|\nu_{3\perp 1}|} \right) \end{aligned}$$

