The Subthreshold $\frac{1}{2}^-$ Nucleon Resonance from Roy-Steiner Equation Analyses

Xiong-Hui Cao (曹雄辉)

In collaboration with **Qu-Zhi Li (李衢智)** and **Han-Qing Zheng (郑汉青)**

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School of Physics, Peking University

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Outline

A brief historical review

- In 1971, Roy developed an exact integral equation based on axiomatic field theory [Roy, PLB (1971)]. Subsequently, Basdevant et al. and Pennington et al. realized the importance of Roy eq. and applied it to $\pi\pi$ data [Basdevant, et al., Nuovo Cim., PLB (1972), NPB (1973), Pennington and Protopopescu PRD (1973)]
- In 1973, Hite and Steiner analyzed previous dispersive approaches and proposed a new Roy-like integral equations (Roy-Steiner eqs.), which can be applied to unequal mass scatterings such as $\pi N \to \pi N$, etc. [Hite and Steiner, Nuovo Cim. (1973)]

Shasanka Mohan Roy

Frank Steiner

★ **With the development of Chiral perturbation theory (χPT), Roy (-Steiner) eqs. have been revived**

- Leutwyler et al. reanalyzed $\pi\pi$ Roy eq. using new data with chiral constrains [Colangelo, et al., NPB (2001), Ananthanarayan, et. al., Phys. Rept. (2001)] and demonstrate the existence of $\sigma/f_0(500)$–the lowest-lying resonance in QCD [Caprini, et al., PRL (2006)]
- Moussallam et al. analyzed πK low energy partial waves (PWs) using RS eq. [Buettiker, et al., EPJC (2004)] and found $\kappa/K_0^*(700)$ [Descotes-Genon, et al., EPJC (2006)]
- Hoferichter et al. given a comprehensive RS eq. analyses of πN scattering [Ditsche, et al., JHEP (2012), Hoferichter, et al., Phys. Rept. (2016)], and applied it to extract πN σ term, $\sigma_{\pi N} = (59.1 \pm 3.5)$ MeV [Hoferichter, et al., PRL 115, 092301 (2015)] and χPT low energy constants [Hoferichter, et al., PRL 115, 192301 (2015)]
- Other processes: $\gamma\pi \to \pi\pi$ [Hannah, NPB (2001)], $\gamma\gamma \to \pi\pi$ [Hoferichter, et al., EPJC (2011)] and $\gamma^*\gamma^* \to \pi\pi$ [Hoferichter and Stoffer, JHEP (2019)], etc.;
  Theoretical improvements: (once-sub DR) GKPY eqs.; high energy, $\pi\pi$ [Moussallam, EPJC (2011), Garcia-Martin, et al., PRL (2011), Caprini, et al., EPJC (2011)] and πK [Pelaez and Rodas, PRL (2020)]$\cdots$

Roy equations

Roy eqs. $=$ Analyticity (Causality) $+$ Crossing symmetry $+$ Unitarity $+$ Froissart-Martin bound

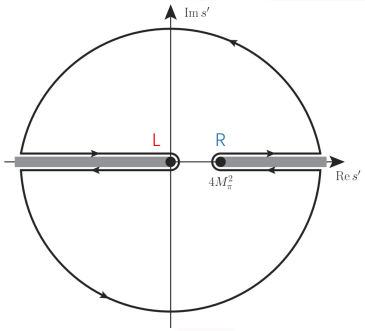
- Fixed-t (twice sub.) DR for $\pi\pi$ scattering:

$$T(s,t,u) = \alpha(t) + s\beta(t) + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}s' \frac{\mathrm{Im}_s\, T(s',t,u')}{s'^2(s'-s)} + \frac{s^2}{\pi}\int_{-\infty}^{-t} \mathrm{d}s' \frac{\mathrm{Im}_u\, T(s',t,u')}{s'^2(s'-s)}$$

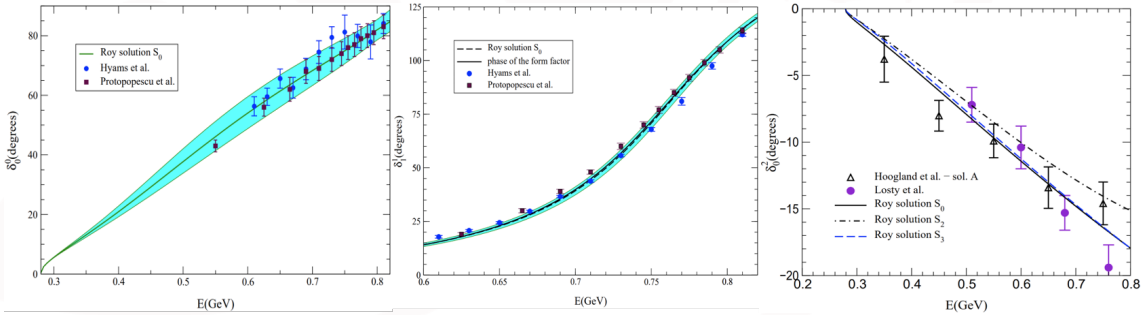
- Using crossing symmetry, (a) the contribution of the left-hand cuts are represented by the right-hand cuts; (b) $\alpha(t), \beta(t)$ are expressed by S wave scattering lengths a_0^0, a_0^2
- Partial wave expansion $\Longrightarrow$ Roy equations,

$$\mathrm{Re}\, t_J^I(s) = k_J^I(s) + \sum_{I'}\sum_{J'} \mathcal{P}\int_{4m_\pi^2}^{\infty} \mathrm{d}s' \underbrace{K_{JJ'}^{II'}(s',s)}_{\frac{1}{\pi}\frac{\delta_{JJ'}\delta_{II'}}{s'-s-i\epsilon}+\bar{K}_{JJ'}^{II'}(s,s')} \mathrm{Im}\, t_{J'}^{I'}(s')$$

- Phase shifts : $t_J^I(s) = \frac{e^{2i\delta_J^I(s)}-1}{2i\rho_{\pi\pi}(s)}$
- Scattering lengths: "free" parameters in $k_J^I(s)$



[Colangelo, et al., NPB (2001), Ananthanarayan, et. al., Phys. Rept. (2001)]



Roy-Steiner equations

- Hite and Steiner proposed a specific, hyperbolic DR, different from traditional fixed-t and fixed-angle DRs.
  It is equivalent to a fixed-$(s-a)(u-a)$ DR.
  Three advantages:
  √ No further kinematic singularity
  √ Curve $b=(s-a)(u-a)$ passes through **the physical regions of all crossing channel**
  √ It does not touch **the double spectral region**
- Roy-like coupled integral equations (RS eqs.) for πN scattering,

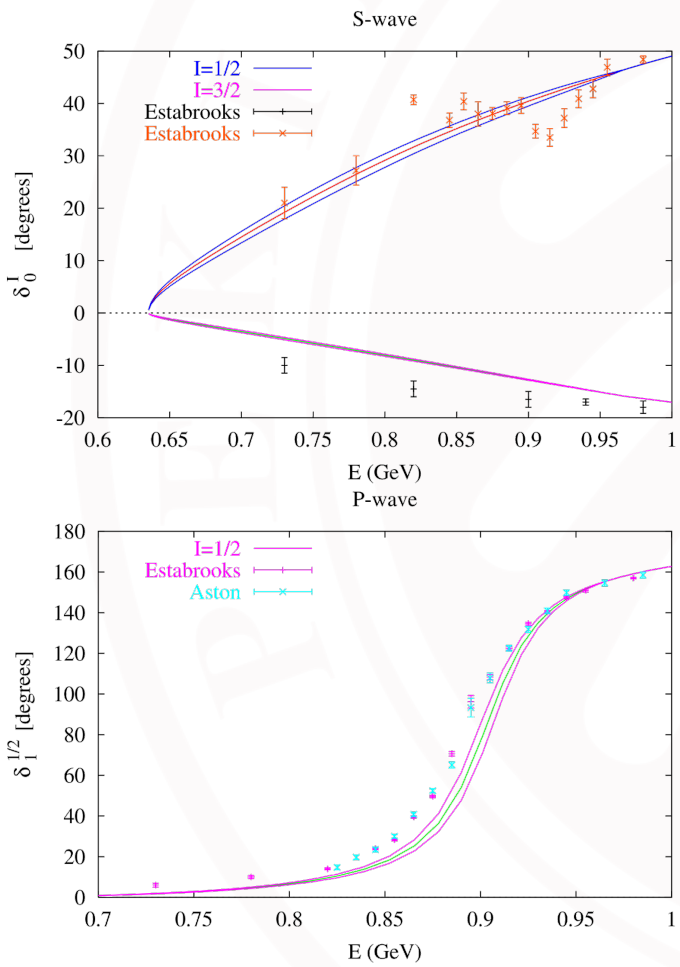
$$f_{l+}^{I}(W) = N_{l+}^{I}(W) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \sum_J \left\{ G_{lJ}(W,t')\,\mathrm{Im}\, f_+^J(t') + H_{lJ}(W,t')\,\mathrm{Im}\, f_-^J(t') \right\}$$
$$+ \frac{1}{\pi}\int_{m_\pi+m_N}^{\infty} \mathrm{d}W' \sum_{l'}^{\infty} \left\{ K_{ll}^{I}(W,W')\,\mathrm{Im}\, f_{l'+}^{I}(W') + K_{ll'}^{I}(W,-W')\,\mathrm{Im}\, f_{(l'+1)-}^{I}(W') \right\}$$

- The t-channel amplitudes $f_\pm^J$ also satisfies a RS equation.
  The difference is that the t-channel problem ($\pi\pi \to N\bar{N}$) can recast as a **Muskhelishvili-Omnès** (MO) problem!

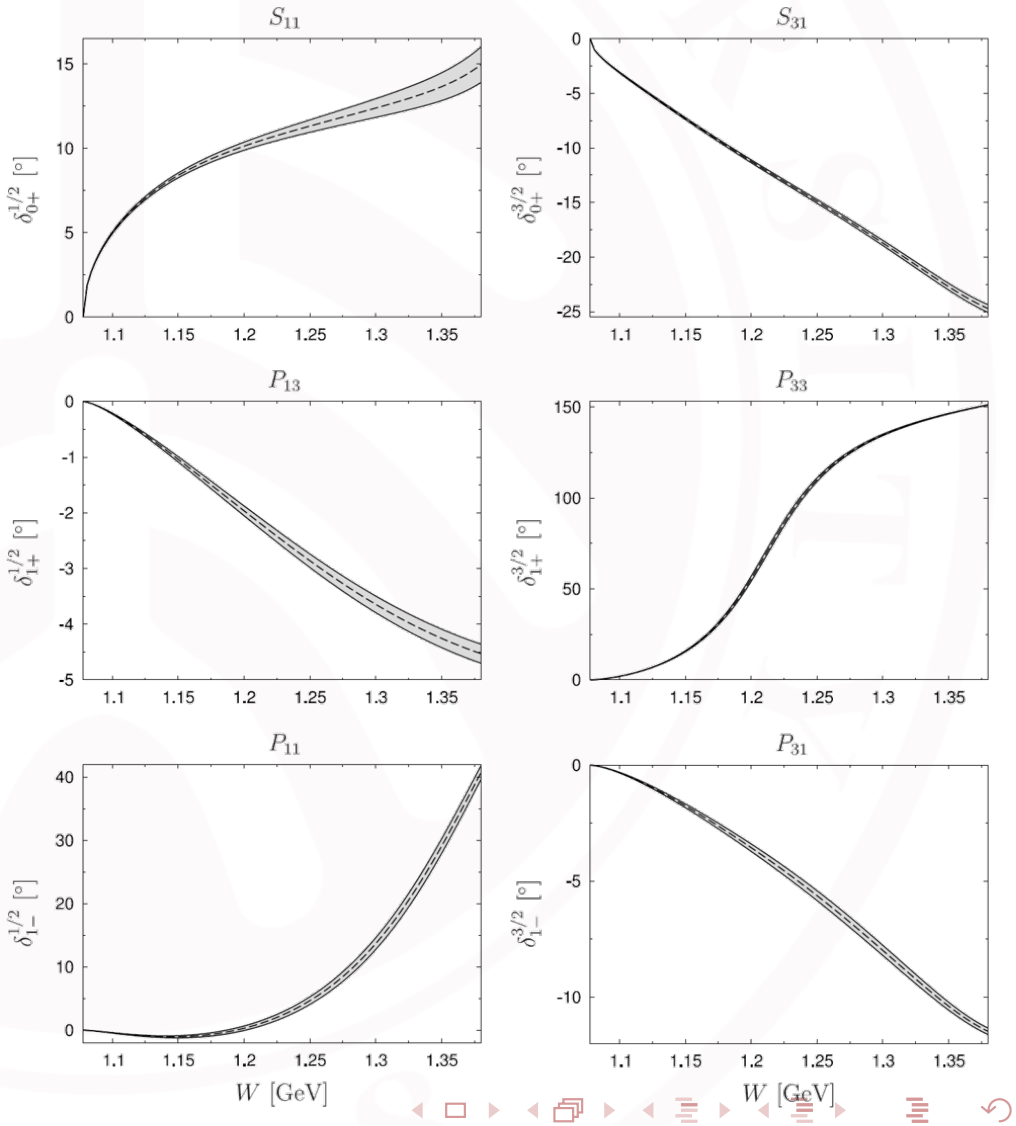
$$\text{RS equations} \overset{\text{MO solution}}{\Longrightarrow} \text{Roy-like equations}$$

RS analyses for πK and πN scatterings

πK scattering Buettiker et al. [2004]:

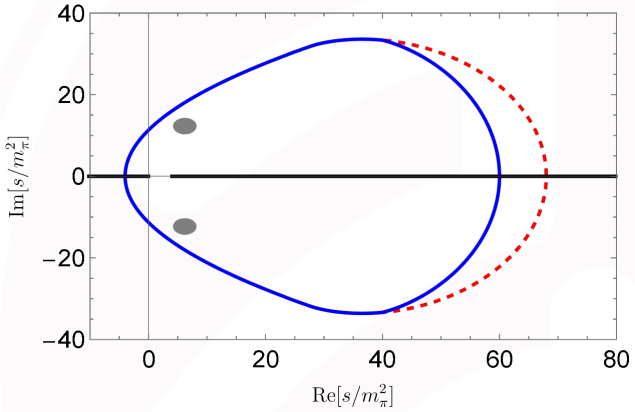


πN scattering Hoferichter et al. [2016]:



Validity Domain

Lehmann-Martin ellipse + **Mandelstam analyticity** (**analyticity from axiomatic field theory**) $\Longrightarrow$ Validity domain of $\pi\pi$ scattering

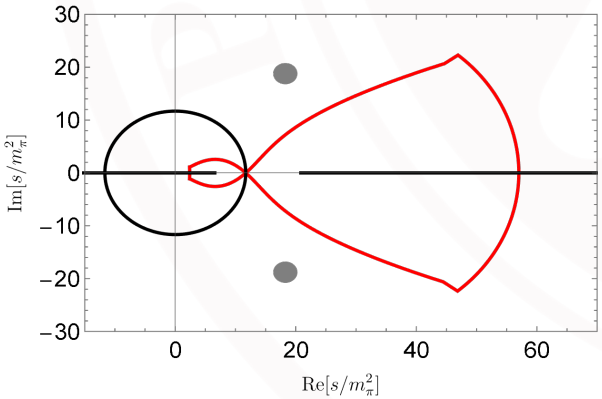


$$m_\sigma = 441^{+16}_{-8}\ \mathrm{MeV}$$

$$\Gamma_\sigma = 544^{+18}_{-25}\ \mathrm{MeV}$$

Caprini et al. [2006]

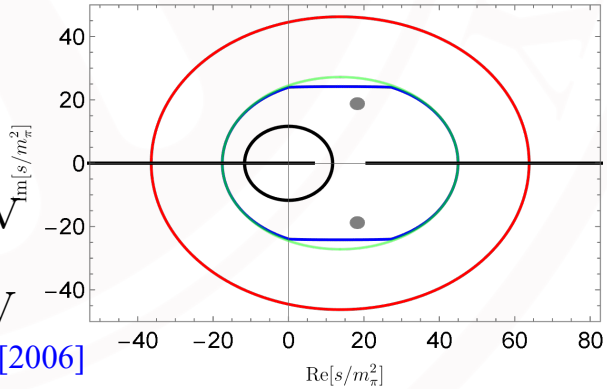
πK scattering $\Longrightarrow$ Extend the validity domain $\Rightarrow$ From fixed-t DR to hyperbolic DR!



$$m_\kappa = 658 \pm 13\ \mathrm{MeV}$$

$$\Gamma_\kappa = 557 \pm 24\ \mathrm{MeV}$$

Descotes-Genon and Moussallam [2006]



πN scattering

- Theoretical discussions:
  - S_{11} channel ($L_{2I\,2J}$ convention) $N^*(1535)$:
    √ lies above the P wave first resonance $N^*(1440)$ √ large couple channel effects with πN and ηN
  - P_{11} channel $N^*(1440)$ (Roper resonance):
    √ low mass, large decay width, coupling to σN channel √ two-pole structure
  - $\cdots$
- Reaction: $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$
- Isospin structure: $T^{ba}(s,t) = \delta^{ba} T^+(s,t) + \frac{1}{2}\left[\tau^b, \tau^a\right] T^-(s,t)$
- Lorentz structure ($I = \pm$): (where $D^I(s,t) = A^I(s,t) + \frac{s-u}{4m_N} B^I(s,t)$)

$$T^I(s,t) = \bar{u}\left(p'\right)\left\{A^I(s,t) + \frac{\not{q}' + \not{q}}{2} B^I(s,t)\right\} u(p) = \bar{u}\left(p'\right)\left\{D^I(s,t) - \frac{[\not{q}', \not{q}]}{4m_N} B^I(s,t)\right\} u(p)$$

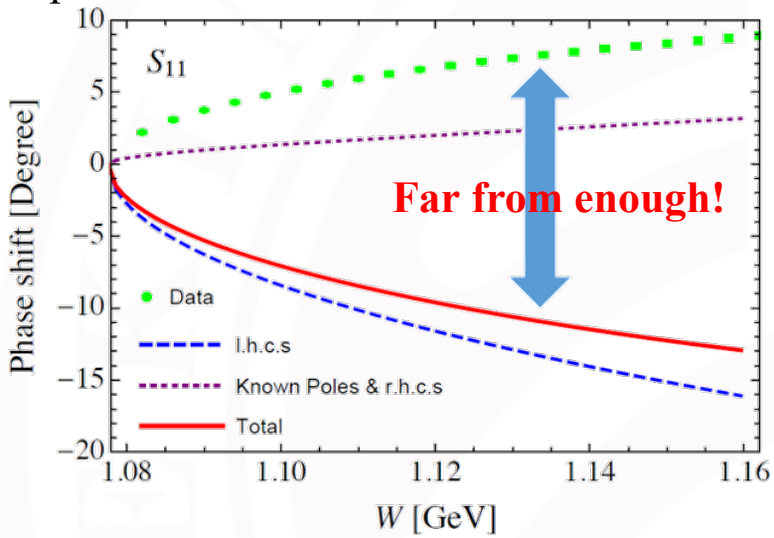
- s-channel isospin relation:
  $X^{1/2} = X^+ + 2X^-\,, \quad X^{3/2} = X^+ - X^-\,, \quad X \in \{T, A, B, D, f_{l\pm}, N_{l\pm}, K_{ll'} \cdots\}$
- PW projection ($\pm \rightleftharpoons J = |l \pm 1/2|$): (where $X_l^I(s) = \int_{-1}^{1} \mathrm{d}z_s P_l(z_s)\, X^I(s,t)\Big|_{t=-2q^2(1-z_s)}$, $X \in \{A, B\}$)

$$f_{l\pm}^I(W) = \frac{1}{16\pi W}\left\{(E+m_N)\left[A_l^I(s) + (W-m_N)\, B_l^I(s)\right] + (E-m_N)\left[-A_{l\pm1}^I(s) + (W+m_N)\, B_{l\pm1}^I(s)\right]\right\}$$

$\hookrightarrow$ MacDowell symmetry relation: $f_{l+}^I(W) = -f_{(l+1)-}^I(-W)$

The subthreshold resonance

- Experimental S_{11} Phase shift vs. Known contributions

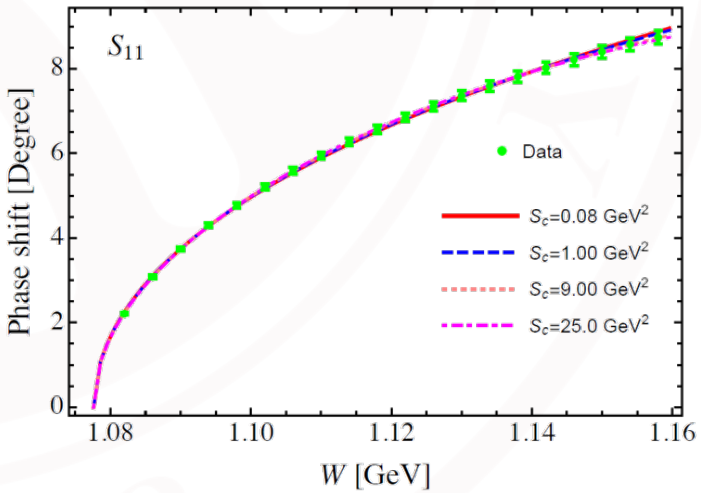


- bound states $\longrightarrow$ negative phase shift
- virtual states (usually hidden) $\longrightarrow$ positive phase shift
- resonances $\longrightarrow$ positive phase shift
- left hand cuts $\longrightarrow$ (empirically) negative phase shift (proved in quantum mechanical potenial scatterings) [T. Regge, Nuovo Cim. (1958)]

- A "crazy" resonance is essential to compensate the diacrepancy [Y.-F. Wang, et al. EPJC (2018), CPC (2019)]
- Hidden pole below πN threshold:

$$\sqrt{s} = (895 \pm 81) - i(164 \pm 23)\ \text{MeV}$$

- The long-distance resonance can be confirmed by the rigorous RS equation method (like $\sigma/f_0(500)$ and $\kappa/K_0^*(700)$)



Lehmann-Martin ellipse

Jost-Lehmann-Dyson representation

A causal Green function satisfies a necessary and sufficient representation:

$$F(q) = \int \mathrm{d}^4 u \int_0^\infty \mathrm{d}\kappa^2 \epsilon\left(q_0 - u_0\right) \delta\left[(q-u)^2 - \kappa^2\right] \Phi\left(u, \kappa^2\right)$$

$\Phi(u, \kappa^2)$ is arbitrary but differs from 0 only in some domains Jost and Lehmann [1957] Dyson [1958].

- $F_R(q) = \frac{i}{2\pi} \int \mathrm{d}q_0' \frac{1}{q_0 - q_0' + i\epsilon} F\left(q'\right)\Big|_{q' = (q_0', q)} \xRightarrow{\text{"retarded" LSZ}} -iT = \frac{1}{2\pi} \int \mathrm{d}^4 u \, \mathrm{d}\kappa^2 \frac{\Phi\left(u, \kappa^2, p, k\right)}{\left((k'-p')/2 - u\right)^2 - \kappa^2}$
- PW unitarity ($\mathrm{Im}\ T_\ell = \frac{2q}{\sqrt{s}} |T_\ell|^2$) + Analyticity of Legendre function ($(T_\ell)^{1/\ell} < \frac{1}{z_0 + \sqrt{z_0^2 - 1}}$)

Lehmann-Martin ellipse

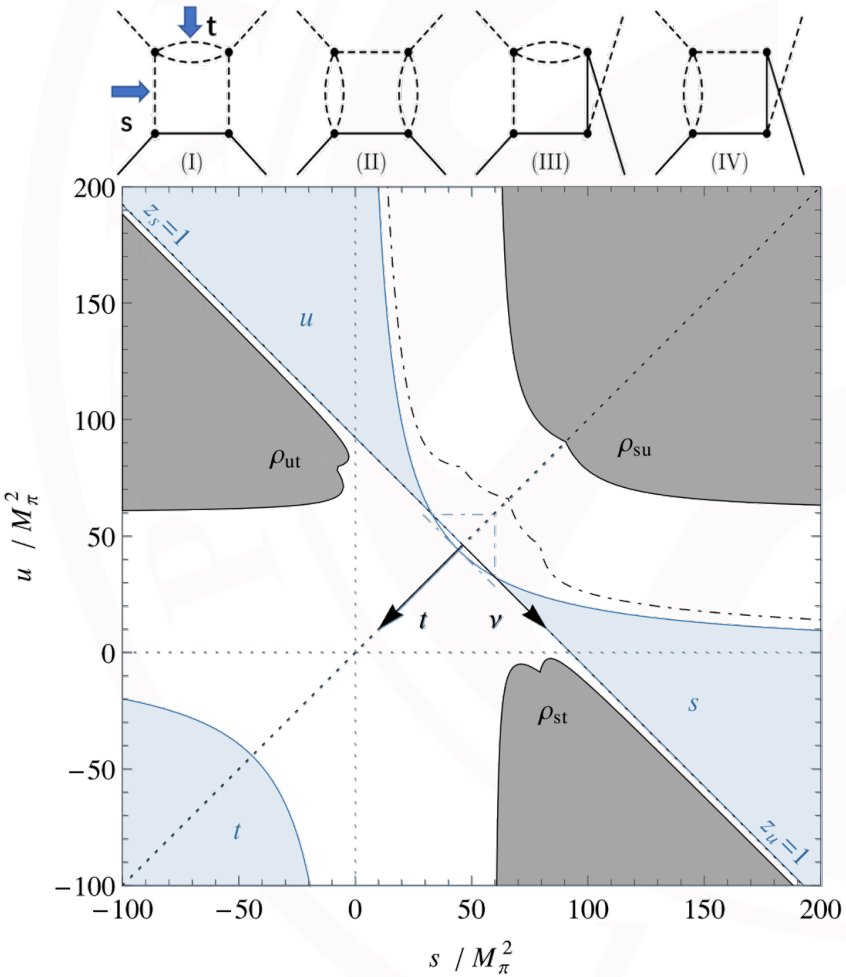
Any DR (or $\mathrm{Im}\ T_\ell$) can be shown to be valid in a finite region of t. The range of validity is rely on Lehmann-Martin ellipse Lehmann [1958].

★ Foci: $z = \pm 1$; Semi-major axis: $z_{\max} = 1 + \frac{2s}{\lambda_s} \mathcal{T}(s)$

Mandelstam Analyticity

Mandelstam Analyticity

The scattering amplitudes satisfy Mandelstam double spectral representation Mandelstam [1958, 1959].



- Mandelstam double spectral representation:

$$T(s,t) = \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}u' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint \mathrm{d}t'\mathrm{d}u' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}t' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

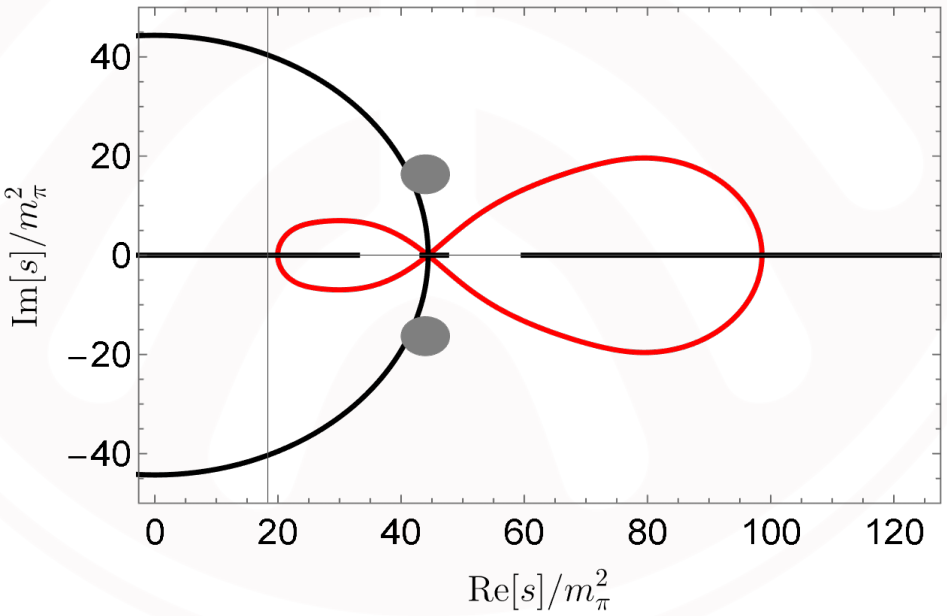
- This concept can be justified in perturbation theory
- A rigorous proof from axiomatic field theory is absent.

Fixed-t representation

- The foci of Lehmann-Martin ellipse is located at $z = \cos\theta_s = \pm 1$
- $T_{st}(s) = \min\{T_{\rm I}(s), T_{\rm II}(s)\}$, $T_{su}(s) = \max\{T_{\rm III}(t), T_{\rm IV}(t)\}$
- Focus on T_{st}: $T(\theta) = \min_{s' \geq s_+} T(s', \theta)$

$$T\left(s', \theta\right) = \frac{T_{st}\left(s'\right)\left(\lambda_{s'} + s' T_{st}\left(s'\right)\right)}{\lambda_{s'} \cos^2\frac{\theta}{2} + s' T_{st}\left(s'\right)}$$

- PW projection $\int_{-\lambda_s/s}^{0} \cdots \mathrm{d}t \Rightarrow$ solving $\lambda_s + sT(\theta)\exp(i\theta) = 0$



Fixed-$(s-a)(u-a)$ representation

- The hyperbolic DR is restricted in a hyperbola $(s-a)(u-a)=(s'-a)(u'-a)=b$, e.g.,

$$A^+(s,t)=\frac{1}{\pi}\int_{s_+}^{\infty}\mathrm{d}s'\left(\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{1}{s'-a}\right)\mathrm{Im}_s A^+\left(s',t'\right)+\frac{1}{\pi}\int_{t_\pi}^{\infty}\mathrm{d}t'\frac{\mathrm{Im}_t A^+\left(s',t'\right)}{t'-t}$$

- Two ellipses: one is the ellipse of $z_{s'}$, and the other is $z_{t'}$. From $s',t'\Longrightarrow b\equiv B\exp(i\theta)$

$$\frac{\left(1+\frac{2s'}{\lambda_{s'}}\left(\Sigma-s'-a-\frac{B_s\left(s',\theta\right)\cos\theta}{s'-a}\right)\right)^2}{A_{s'}^2}+\frac{\left(\frac{2s'}{\lambda_{s'}}\left(\frac{B_s\left(s',\theta\right)\sin\theta}{s'-a}\right)\right)^2}{A_{s'}^2-1}=1\,,$$

$$\frac{\left(\frac{\left(t'-\Sigma+2a\right)^2-4B_t\left(t',\theta\right)\cos\theta}{(t'-t_\pi)(t'-t_N)}-\frac{1}{2}\right)^2}{A_{t'}^2-\frac{1}{2}}+\frac{\left(\frac{4B_t\left(t',\theta\right)\sin\theta}{(t'-t_\pi)(t'-t_N)}\right)^2}{A_{t'}\sqrt{A_{t'}^2-1}}=1\,,$$

where $\Sigma=2(m_\pi^2+m_N^2)$, $t_N=4m_N^2$, $A_{s'}=1+\frac{2s'\,T_{st}(s')}{\lambda_{s'}}$ $(s'>s_+)$ and $A_{t'}^2=\frac{16m_N^2N_{st}(t')}{(t'-t_\pi)(t'-t_N)}$ $(t'>t_N)$, $A_{t'}^2=1-\frac{16m_N^2N_{st}(t')}{(t'-t_\pi)(t-t_N)}$ $(t_N>t'>t_\pi)$ Ditsche et al. [2012].

- **The discriminant of the above quadratic equations should be nonnegative** $\Longrightarrow$

$$-2.59m_\pi^2<a<4m_\pi^2$$

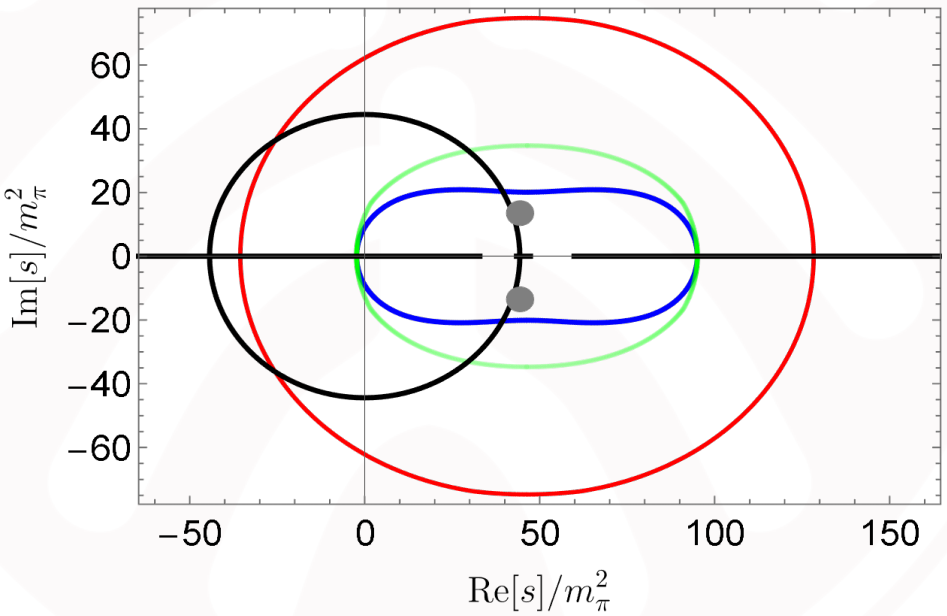
- In [Hoferichter, et al., Phys. Rept. (2016)], it is chosen that $a=-29.3m_\pi^2<-2.59m_\pi^2$, thus the solution of PWs cannot be extended to the complex s plane.

Fixed-$(s-a)(u-a)$ representation

- $B_s(\theta) = \min_{m_+^2 \le s'} B_s(s', \theta)$ and $B_t(\theta) = \min_{t_\pi \le t'} B_t(t', \theta)$
- PW projection

$$\int_{(s-a)((m_\pi^2-m_N^2)^2/s-a)}^{(s-a)(\Sigma-s-a)} \cdots \mathrm{d}b \Rightarrow \begin{cases} (s-a)(\Sigma-s-a) - B_{s/t}(\theta)\exp(i\theta) = 0 \\ (s-a)\left(\left(m_N^2-m_\pi^2\right)^2/s-a\right) - B_{s/t}(\theta)\exp(i\theta) = 0 \end{cases}$$

★ Validity domain of the fixed-b RS representation $(a=0)$



★ s' integral, ρ_{su} boundary; t' integral, ρ_{st} boundary; s' integral, ρ_{st} boundary

- The largest validity value of $\sqrt{s}$ is $W_\mathrm{m} = 1.36$ GeV, slightly smaller than 1.38 GeV in [Hoferichter, et al., Phys. Rept. (2016)]

Existence and Uniqueness

- (πN) RS equations are useful only at low eneriges and we only solve S and P waves up to matching point $W_{\rm m} = 1.36$ GeV
- D and F waves and the intermediate energy contributions of $\ell \leq 4$ PWs $\Longleftarrow$ GWU/SAID (up to $W = 2.5$ GeV); Other high PW ($\ell > 4$) driving terms can be estimated by Regge theory $\rightsquigarrow$ Negligible
- t-channel PWs $f_{\pm}^{J}$ (matching point $\sqrt{t_{\rm m}} = 2m_N$):
  S wave $\longmapsto$ Couple channel ($\pi\pi$, $K\bar{K}$) MO formalism, phase shifts from Garcia-Martin et al. [2011]; Pelaez and Rodas [2018];
  P and D waves: $\longmapsto$ Single channel MO formalism, phase shifts from Garcia-Martin et al. [2011]; Hoferichter et al. [2016]; High PWs $\rightsquigarrow$ Negligible

⊛ **Multiplicity** of the coupled integral equations, $m = \sum_i m_i, \quad m_i = \begin{cases} \left\lfloor \frac{2\delta_i(s_{\rm m})}{\pi} \right\rfloor & \text{if } \delta_i(s_{\rm m}) > 0 \\ -1 & \text{if } \delta_i(s_{\rm m}) < 0 \end{cases},$
Matching point $W_{\rm m} = 1.36$ GeV $\Longrightarrow m = -2$ [Wanders, EPJC (2000)]

⊛ Six "no cusp" constraints for phase shifts at the matching points

⊛ Two S wave scattering lengths were fixed precisely by chiral symmetry and the pionic atom spectrum Baru et al. [2011],

$$m_\pi\, a_{0+}^{1/2} = (169.8 \pm 2.0) \times 10^{-3}\,, \quad m_\pi\, a_{0+}^{3/2} = (-86.3 \pm 1.8) \times 10^{-3}$$

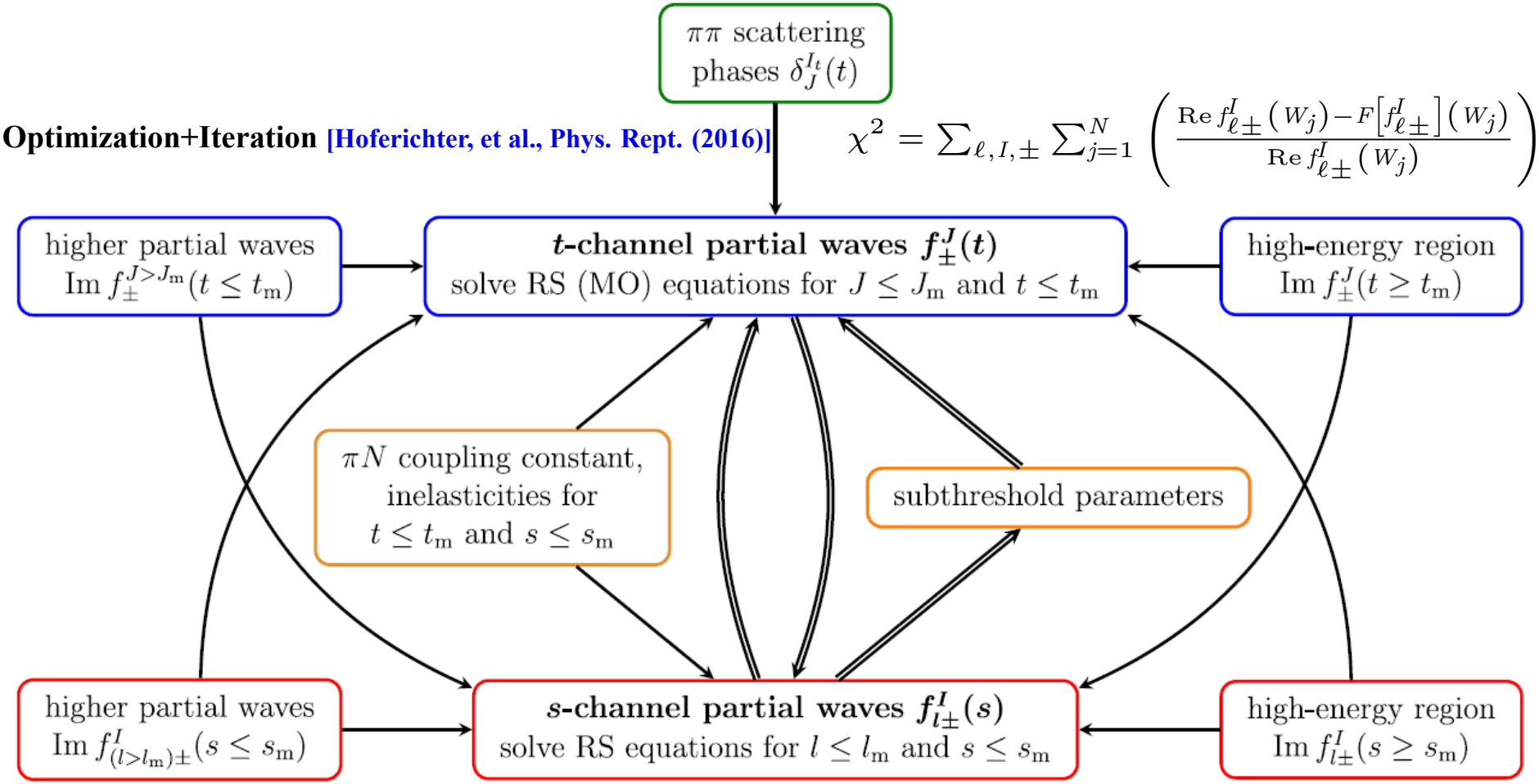
⊛ $|m| + 6 + 2 = 10$ comstraints, implies a unique solution exists **if and only if** the system also has 10 free parameters

⊛ Subthreshold subtractions ($s = u = m_\pi^2 + m_N^2$, $t = 0$): 10 subthreshold expansion parameters

Fitting stratagy

Optimization+Iteration [Hoferichter, et al., Phys. Rept. (2016)]

$$\chi^2 = \sum_{\ell,I,\pm} \sum_{j=1}^{N} \left(\frac{\mathrm{Re}\, f^I_{\ell\pm}(W_j) - F\left[f^I_{\ell\pm}\right](W_j)}{\mathrm{Re}\, f^I_{\ell\pm}(W_j)} \right)^2$$



$\pi\pi \to N\bar{N}$ and MO problem

Two-channel MO formalism with **finite matching point**, S wave

- The $\pi\pi \to N\bar{N}$ and $K\bar{K} \to N\bar{N}$ S-waves $f_+^0(t)$ and $h_+^0(t)$ fulfill the unitarity relation:

$$\mathrm{Im}\,\mathbf{f}(t) = T^*(t)\Sigma(t)\mathbf{f}(t), \quad \mathbf{f}(t) = \begin{pmatrix} f_+^0(t) \\ \frac{2}{\sqrt{3}} h_+^0(t) \end{pmatrix}$$

- RS equations provide a DR: $\mathbf{f}(t) = \mathbf{\Delta}(t) + (\mathbf{a} + \mathbf{b}t)(t - t_N) + \frac{t^2(t-t_N)}{\pi} \int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\,\mathbf{f}(t')}{t'^2(t'-t_N)(t'-t)}$ where

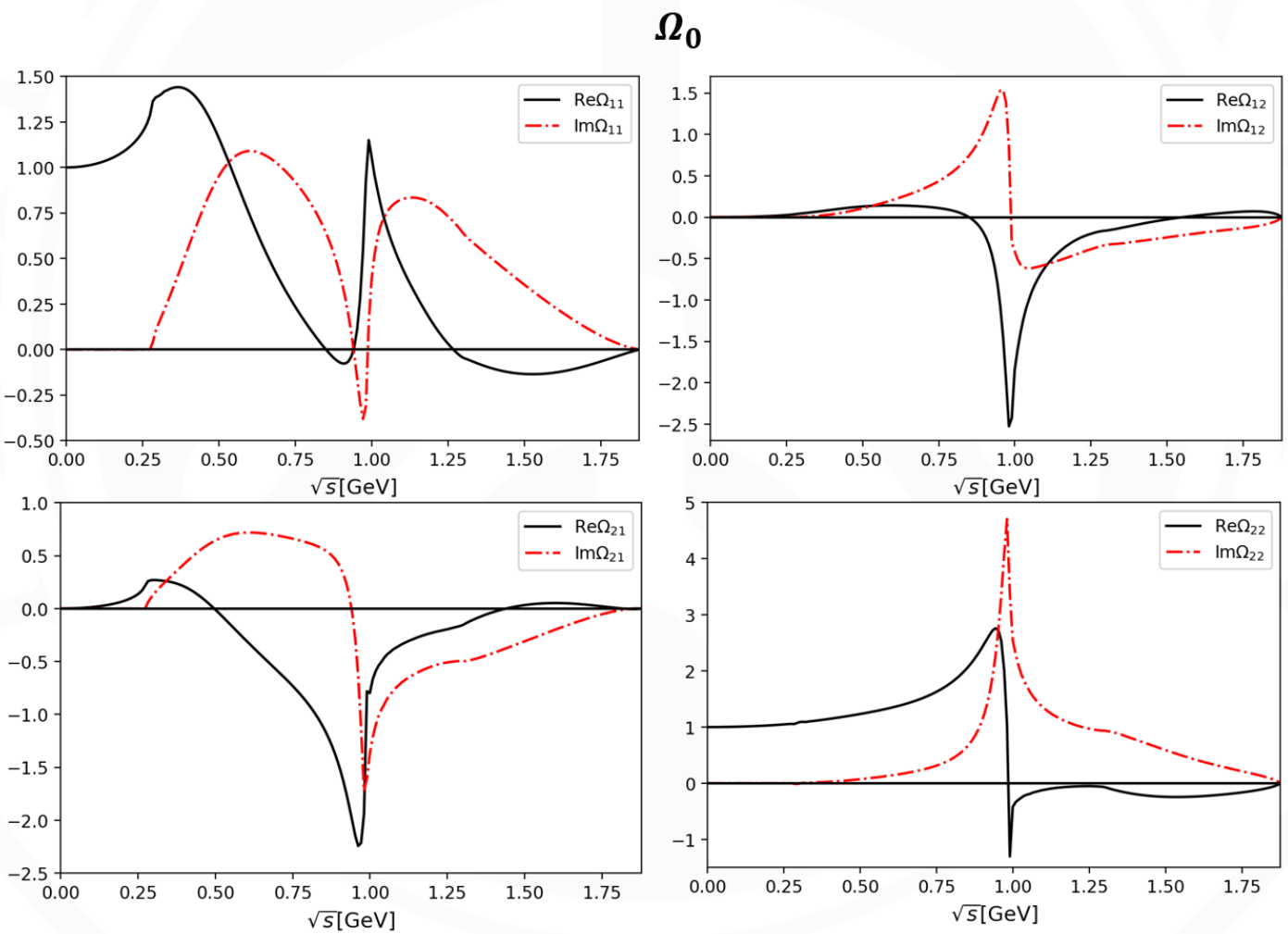
$$\Delta_1(t) = \tilde{N}_+^0(t) + \frac{1}{\pi}\int_{W_+}^{\infty} \mathrm{d}W' \sum_{l=0}^{\infty} \left\{ \tilde{G}_{0l}(t, W')\,\mathrm{Im}\,f_{l+}^I(W') + \tilde{G}_{0l}(t, -W')\,\mathrm{Im}\,f_{(l+1)-}^I(W') \right\}$$
$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \sum_{J'=J+2}^{\infty} \frac{1 + (-1)^{J+J'}}{2} \left\{ \tilde{K}_{J'}^1(t, t')\,\mathrm{Im}\,f_+^{J'}(t') + \tilde{K}_{0J'}^2(t, t')\,\mathrm{Im}\,f_-^{J'}(t') \right\}$$

- The major challenge in the RS application concerns the generalization of MO formalism to the case of a finite matching point [Hoferichter, et al., JHEP (2012)]
- The final solution:

$$\mathbf{f}(t) = \mathbf{\Delta}(t) + (t - t_N)\,\Omega(t)(\mathbb{1} - t\dot{\Omega}(0))\mathbf{a} + t\,(t - t_N)\,\Omega(t)\mathbf{b} - \frac{t^2\,(t - t_N)}{\pi}\Omega(t) \int_{t_\pi}^{t_m} \mathrm{d}t' \frac{\mathrm{Im}\,\Omega^{-1}(t')\,\mathbf{\Delta}(t')}{t'^2\,(t' - t_N)\,(t' - t)}$$

$\pi\pi \to N\bar{N}$ and MO problem

- Solutions to MO problem with **finite matching point**, S wave [Hoferichter, et al., J. High Energy Phys. 1206 (2012) 063]
- Discretization method + SVD [Moussallam, EPJC (2000), D.-L. Yao, et al., EPJC (2018)]



$\pi\pi \to N\bar{N}$ and MO problem

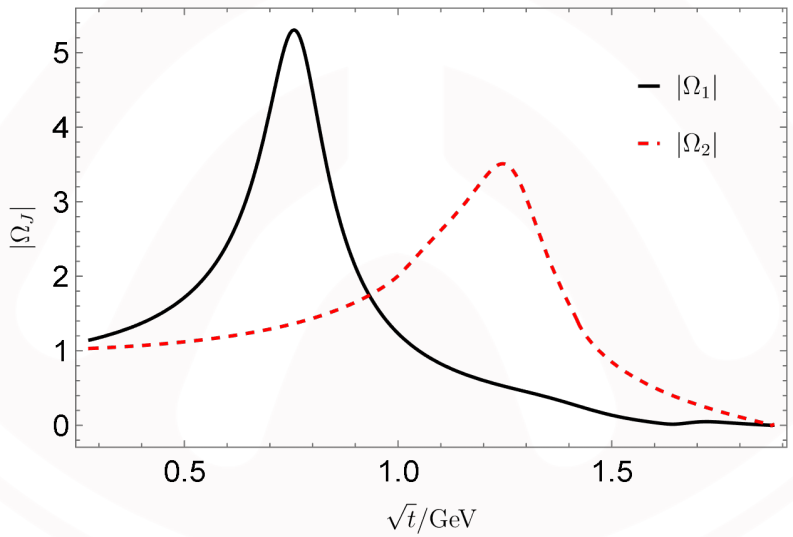
Single-channel MO formalism with finite matching point

- For example, $f_-^{1,2}(t)$ waves:

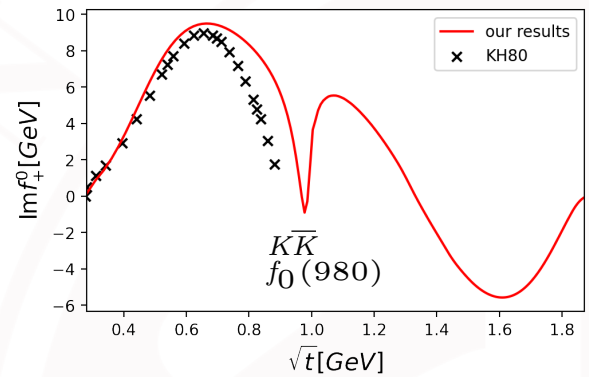
$$f_-^1(t) = \Delta_-^1(t) + \frac{\sqrt{2}}{12\pi}\left\{\left(b_{00}^- - \frac{g^2}{2m_N^2}\right)\left(1 - t\dot{\Omega}_1(0)\right) + b_{01}^- t\right\}\Omega_1(t) + \frac{t^2\Omega_1(t)}{\pi}\int_{t_\pi}^{t_m} \mathrm{d}t' \frac{\Delta_-^1(t')\sin\delta_1(t')}{t'^2(t'-t)|\Omega_1(t')|}$$

$$f_-^2(t) = \Delta_-^2(t) + \frac{\sqrt{6}}{60\pi m_N} b_{00}^+ \Omega_2(t) + \frac{t\Omega_2(t)}{\pi}\int_{t_\pi}^{t_m} \mathrm{d}t' \frac{\Delta_-^2(t')\sin\delta_2(t')}{t'(t'-t)|\Omega_2(t')|}$$

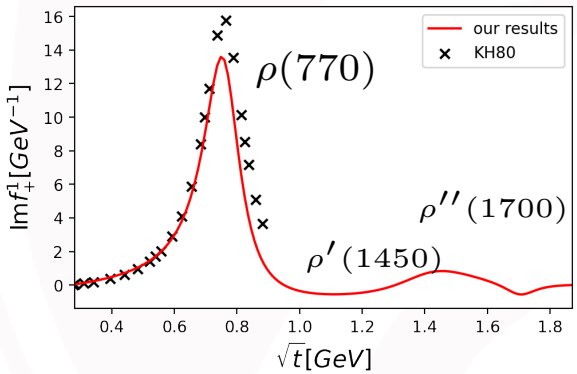
- Solutions to MO problem with finite matching point, P and D wave

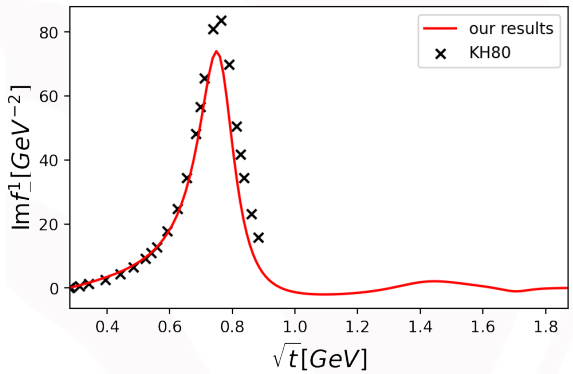


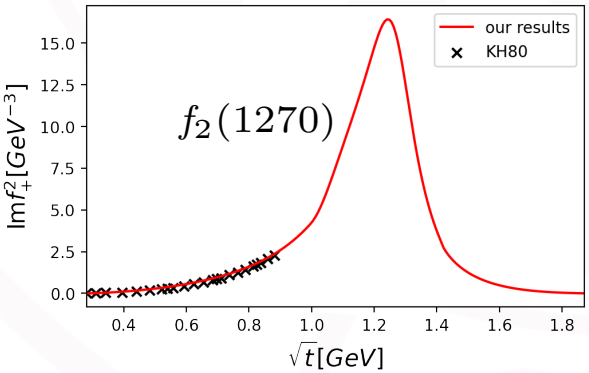
t-channel: $\pi\pi \to N\bar{N}$

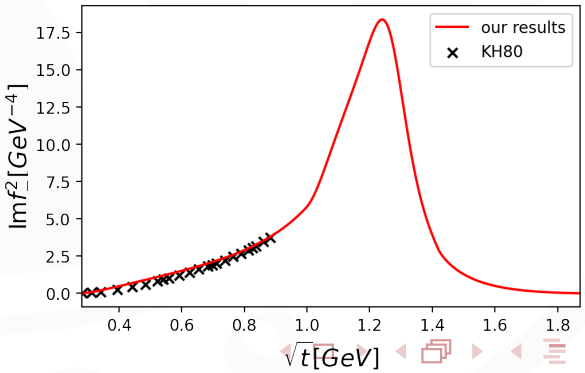


PW analyses from Karlsruhe-Helsinki (1980):
Höhler, Pion Nucleon Scattering.
Part 2: Methods and Results of Phenomenological Analyses
(Springer, 1983)









s-channel: $\pi N \to \pi N$

★ **Phase shifts:**

s_R — Threshold

To be solved by RS equations

s_m — Matching point

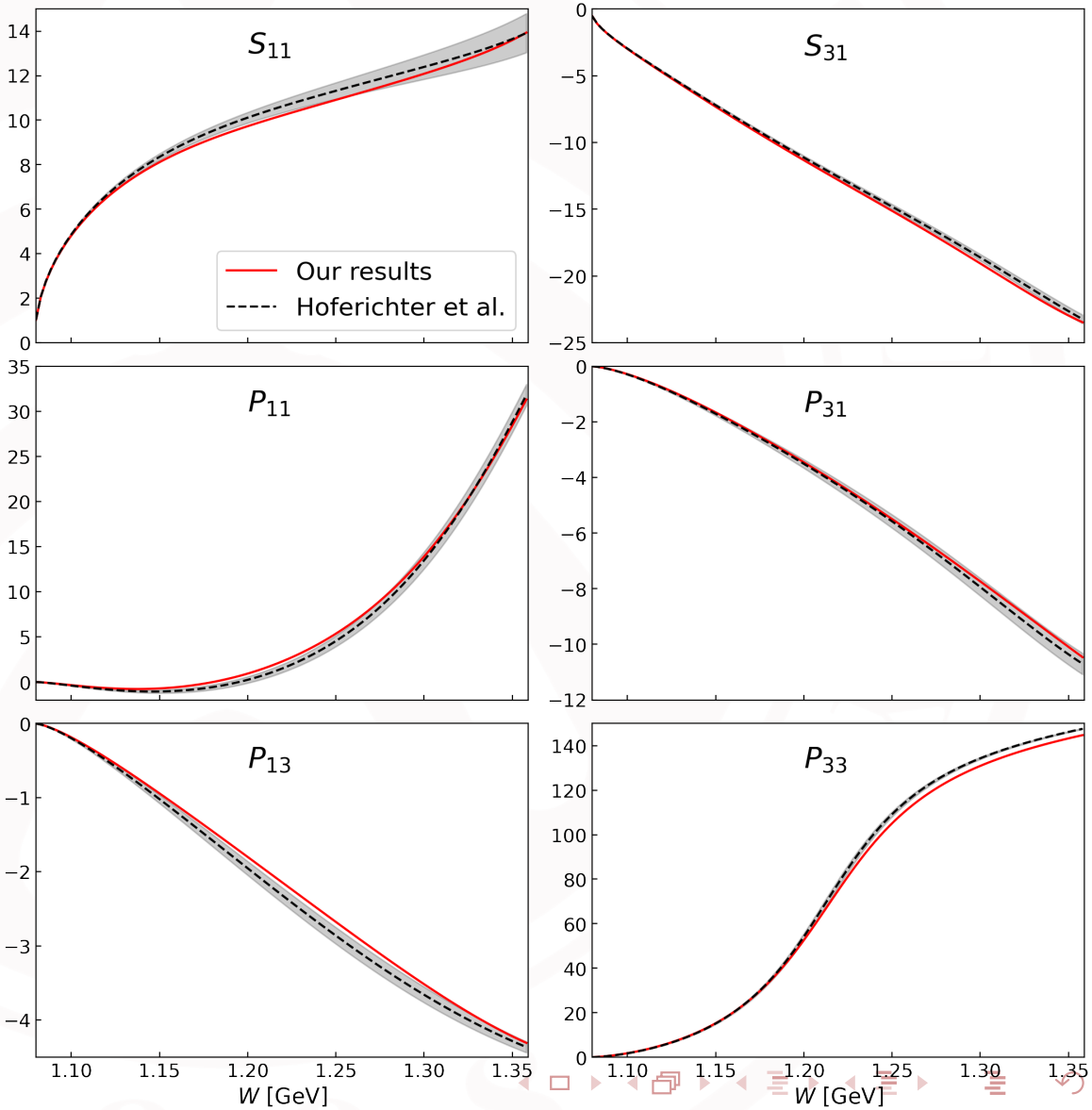
Driving terms: S and P waves (s_m, s_C)
D and F waves (s_R, s_C)

PW analyses from George Washington Univ.
WI08 Solution [Workman, et al. PRC (2012)]

s_c

Driving terms from Regge theory
[F. Huang, et al., EPJA (2010)]

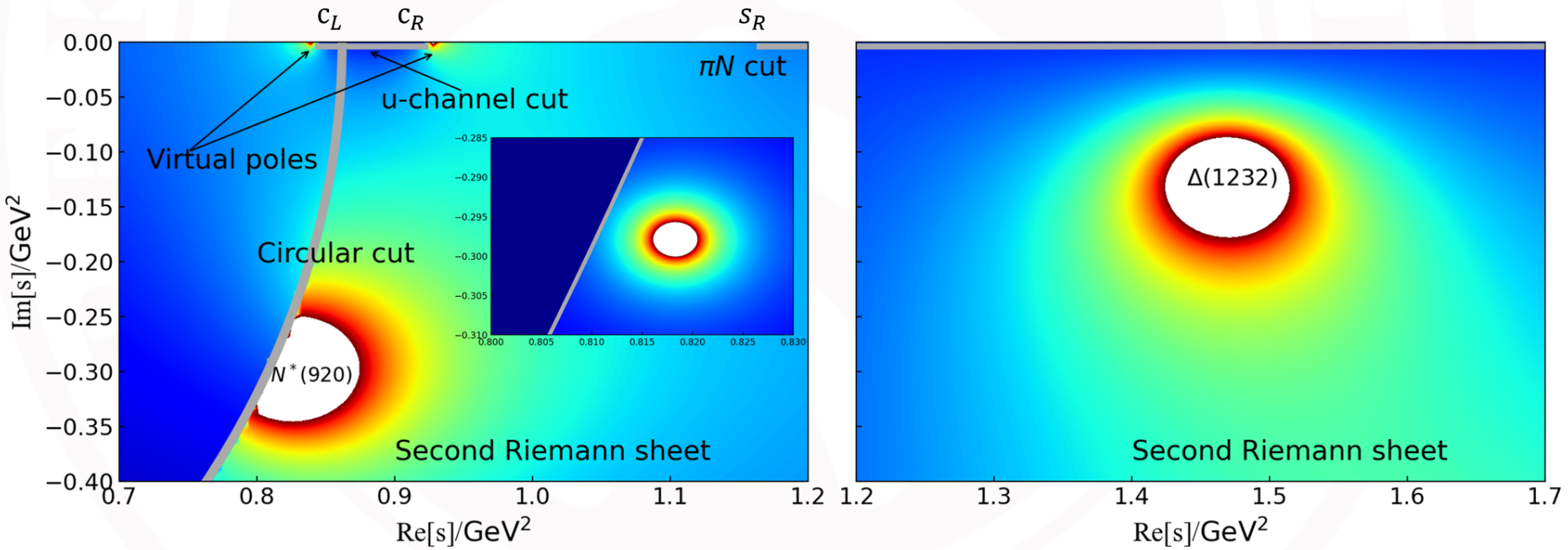
∞



The lowest-lying $\frac{1}{2}^-$ resonance

- S_{11} S matrix: $S_{0+}^{1/2}(s) = 1 - \frac{\sqrt{(s_+ - s)(s - s_-)}}{\sqrt{s}} f_{0+}^{1/2}(\sqrt{s})$
- Analytic continuation: $S_{0+}^{1/2\mathrm{II}} = 1/S_{0+}^{1/2}$
- $\sqrt{s_{N^*(920)}} = (919 \pm 4) - i(162 \pm 7)\mathrm{MeV}$ and $\sqrt{s_{\Delta(1232)}} = (1213 \pm 2) - i(50 \pm 2)\mathrm{MeV}$ all inside the validity domain. $\Delta_{33}(1232)$ PDG average value: $(1210 \pm 1) - i(50 \pm 1)$ MeV

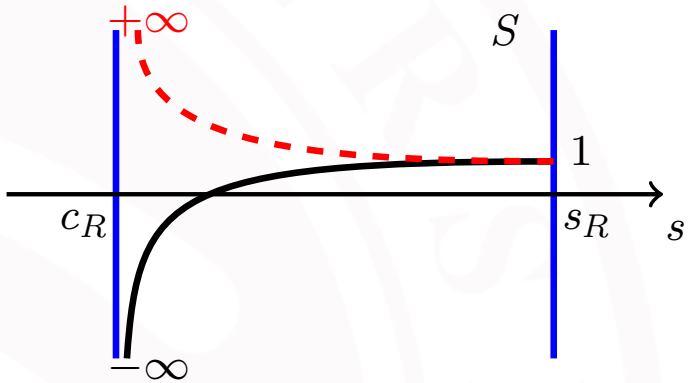
★



- Two virtual poles, how to understand it?

Virtual pole mechanism

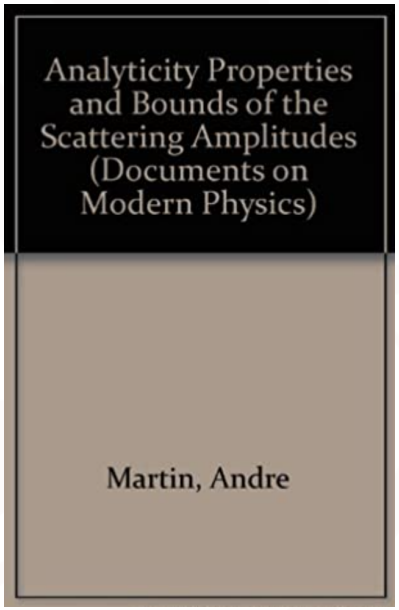
- Such general phenomenon was firstly discussed in [Blankenbecler, Goldberger, MacDowell, and Treiman, Phys. Rev. (1961)] (rediscovered in $\pi\pi$ scatterings [Z.-Y Zhou, et al., JHEP (2005)], in πN scatterings [Q.-Z. Li and H.-Q. Zheng, CTP (2022)])
- These virtual poles hardly affect physical observables such as phase shifts (if no bound state), in the scheme of the PKU decomposition



- An extra CDD pole is needed in P_{11} channel [Gasparyan and Lutz, NPA (2010)]

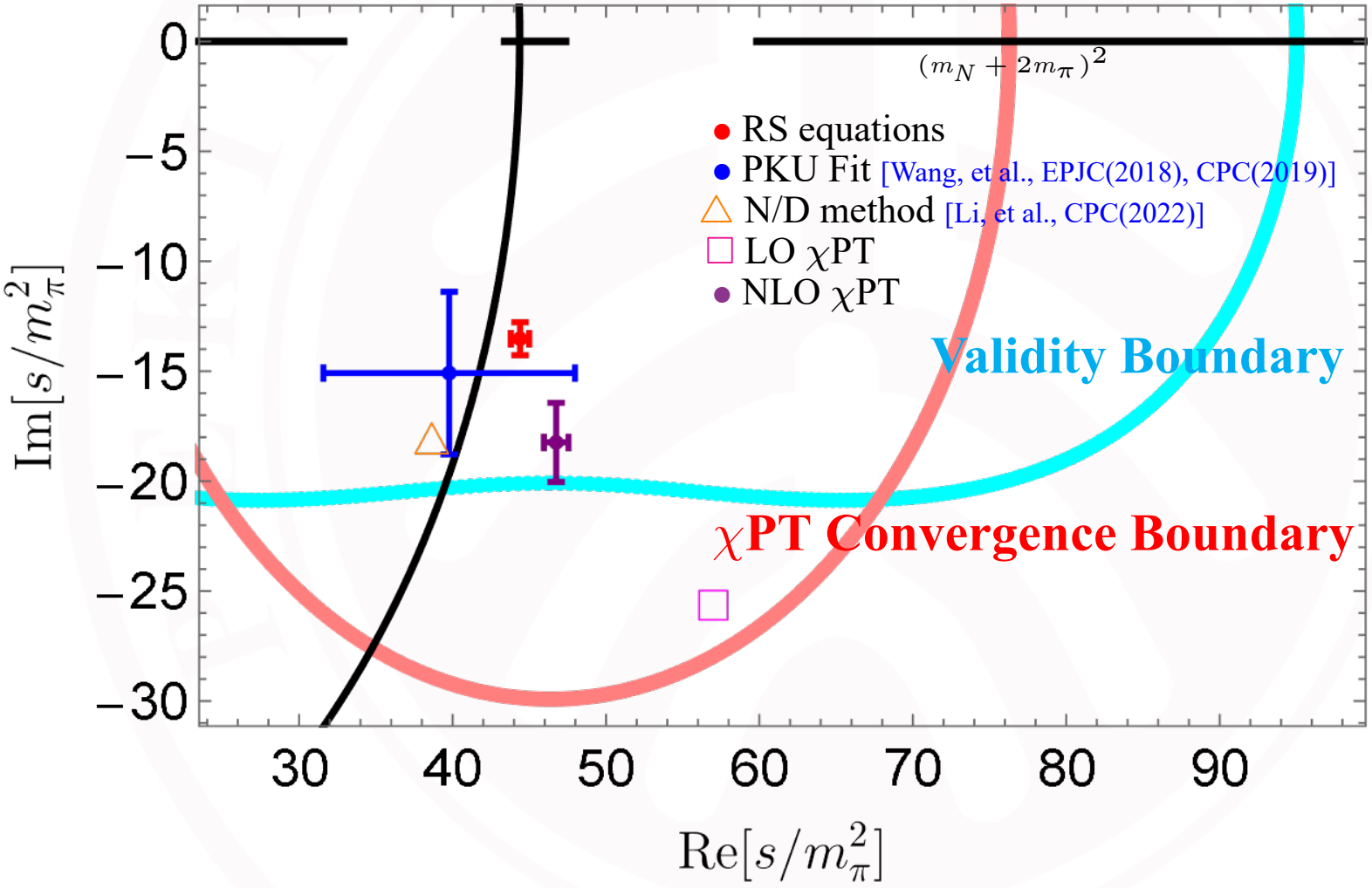
★ Virtual poles: RS equations vs. Leading order χPT

	S_{11}	S_{31}	P_{11}	P_{31}	P_{13}	P_{33}
RS equations Virtual states: $\sqrt{s_{\text{pole}}}$ [MeV]	961 916		976	917	961	908
χPT LO Virtual states: $\sqrt{s_{\text{pole}}}$ [MeV]	961 918		983	919	963	907



- ⊛ **André Martin (1970)**: $s = 0$ is an accumulation of singularities on the second sheet for $\pi\pi$ scattering amplitudes.
  ⊛ **Q.-Z. Li and H.-Q. Zheng (2022)**: $s = c_{L,R}$ are accumulation of singularities on the second sheet for πN case.

Comparison

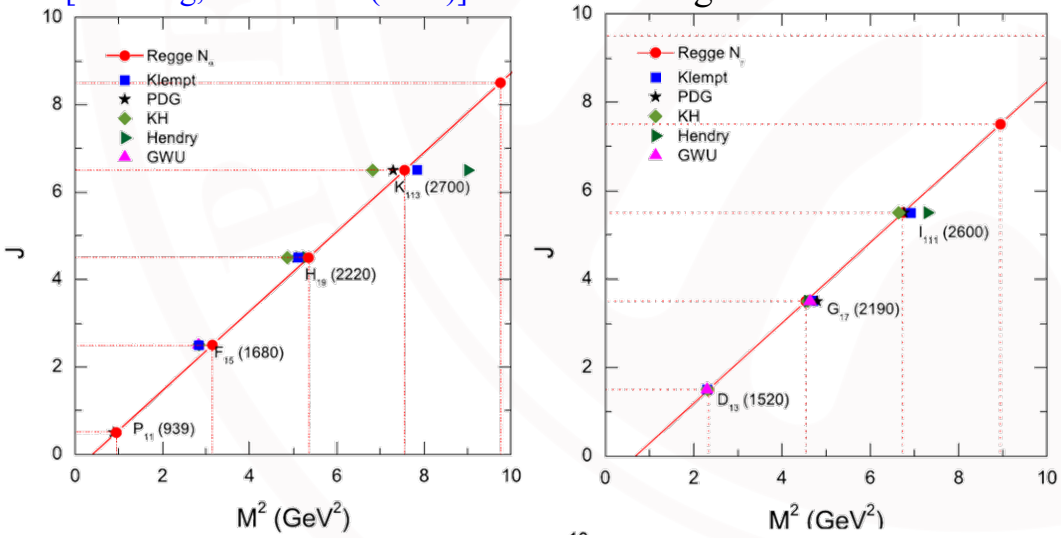

$(m_N + 2m_\pi)^2$
RS equations
PKU Fit [Wang, et al., EPJC(2018), CPC(2019)]
N/D method [Li, et al., CPC(2022)]
LO χPT
NLO χPT
Validity Boundary
χPT Convergence Boundary
$\mathrm{Im}[s/m_\pi^2]$
$\mathrm{Re}[s/m_\pi^2]$
0
−5
−10
−15
−20
−25
−30
30
40
50
60
70
80
90

Regge trajectories

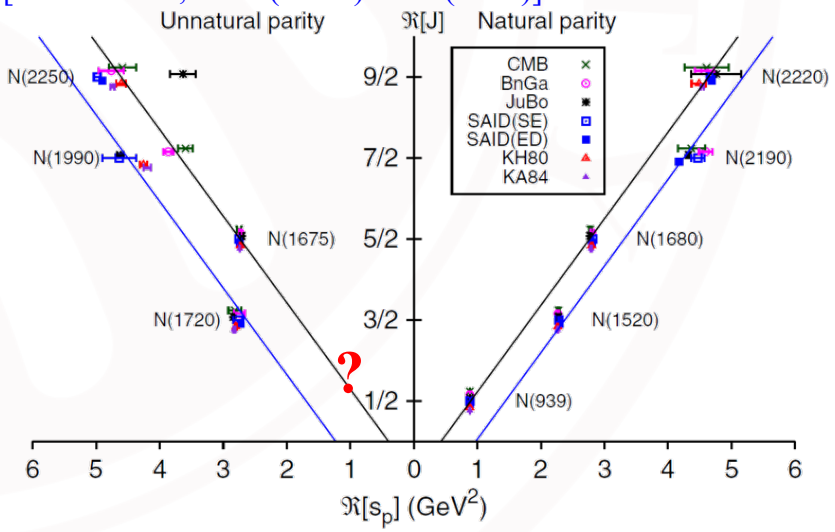
- Regge theory: analytic continuation in the angular momentum plane
- Trajectories are (approximately) straight lines with a similar slope parameter ($1\ \mathrm{GeV}^{-2}$)
- **MacDowell symmetry**: $\alpha^{\pm}(\sqrt{s}) \simeq \alpha_0 + \alpha' s$

Trajectory	Isospin	Parity	Signature $(-1)^{J-1/2}$	Parity partner
N_α	$1/2$	$+$	$+$	N_β
N_γ	$1/2$	$-$	$-$	N_δ
Δ_δ	$3/2$	$+$	$-$	Δ_γ
Δ_β	$3/2$	$-$	$+$	Δ_α

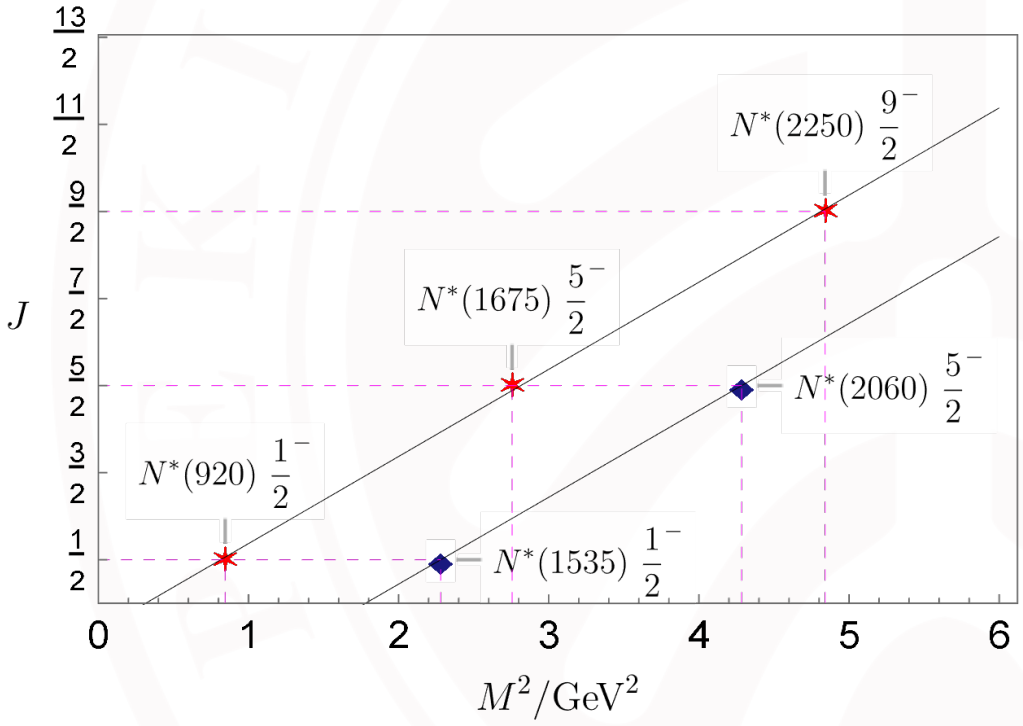
[F. Huang, et al. EPJC (2010)] from scattering data



[Silva-Castro, et al. (JPAC) PRD (2019)] from resonance data



$N^*(920)$ and Regge trajectories



- $N^*(1675)$ and $N^*(2250)$ are PDG four-stars resonances
- Long missing lowest-lying $\frac{1}{2}^-$ state: $N^*(920)$
- The mass of $N^*(920)$ is even lower than nucleon!
- It has large width similar to broad resonances $\sigma/f_0(500)$ and $\kappa/K_0^*(700)$

Summary and Outlook

- The unity of rigorous dispersive techniques and experimental data is powerful to investigate the pole properties of low-lying resonances in QCD
  - Widely-used unitarization methods such as K matrix, Padé approximation (IAM), etc., are poor in light meson & baryon studies
  - (Covariant) Chiral pertubation theory obeys perturbative unitarity and analytic properties (correct LHC structure), but is hard to predict broad resonances such $\sigma/f_0(500)$, etc.
  - Rigorous dispersive approaches, N/D (+comformal mapping), PKU factorization, Roy-Steiner equations, etc. are necessary when touching the mysterious area in the complex plane
- $N^*(920)$ as the lowest-lying ($J^P = \frac{1}{2}^-$) nucleon resonance in QCD
  - $N^*(920)$: $\sqrt{S_{N^*(920)}} = (\mathbf{919} \pm \mathbf{4}) - \mathbf{i}(\mathbf{162} \pm \mathbf{7})$ MeV $\hookrightarrow$**Model independent!**
    - Why it is physically relevant meanwhile it is far from the physical region?
    - How to define rigorously the distinction between **an S matrix pole** and **a physical resonance**?
  - A possible universal phenomenon for the appearance of such a (S wave) broad structure $\sigma/f_0(500)$, $\kappa/K_0^*(700)$, $N^*(920)\cdots$? [Z.-Y. Zhou and Z.-G. Xiao, EPJC (2021)]
- Future applications
  - $NN \to NN \& N\bar{N} \to N\bar{N}$ and $\gamma^{(*)}\gamma^{(*)} \to N\bar{N} \& \gamma^{(*)}N \to \gamma^{(*)}N$ in RS equations analyses, etc.
  - **Lattice data + Roy and Roy-Steiner eqs.** for $\pi\pi$, πK and πN ... (CXH, Q.-Z. Li and Z.-H. Guo, in preparation)
  - ...

Thank you very much for your attention!

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