When Nekrasov partition function meets orientifold 5-plane in the thermodynamic limit

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Contents

Mirror Symmetry on Calabi-Yau Geometry

- Mirror symmetry is a string duality relating the A-model on a Calabi-Yau manifold X (defined in terms of the symplectic structure on X) to the B-model on another Calabi-Yau manifold X' (which is the mirror of X defined in terms of the complex structure on X').
- Mirror symmetry relates A-model topological string theory on X to B-model topological string theory on X'.
- A-side: Gromov-Witten invariants (numbers)$\longleftrightarrow$ B-side: period integrals (numbers).
- Genus zero: Givental's Mirror formula and Lian-Liu-Yau's Mirror Principle.
- Higher genus: Quantization.

- Question: Is there a global view beyond correspondence of numbers ? Answer: Generating functions !
- Mirror Symmetry: Nekrasov partition functions $\longleftrightarrow$ Seiberg-Witten prepotentials. c.f. [Nekrasov, Okounkov '03] [Nakajima, Yoshioka '03,'05][Braveman, Etingof '05]
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In particular, for toric(noncompact) Calabi-Yau 3-fold, we have

- Topological A-model: The (local) Gromov-Witten invariants for all genus, all degree can be computed by using "topological vertex formalism".
  - Physics[M.Aganagic, A.Klemm, M.Marino, C.Vafa '03];
  - Mathematics[J.Li, C.M.Liu, K.Liu, J.Zhou '04]
- Topological B-model: The genus-zero B-model topological strings on a Calabi-Yau 3-fold can be described in terms of variation of Hodge structures. In addition, the genus zero Gromov-Witten invariants can be also computed from Seiberg-Witten geometry, which is systematically obtained for **toric** Calabi-Yau 3-fold.

Motivation

- The resolutions of **A-type** singularity can be always described by **toric** geometry.
- But the resolution of **D-type** singularity is **NOT toric**.
- Physicists claim that "**Orientifold-plane**" (O-plane) is useful to describe D-type singularity.
- Orientifold is the generalization of orbifold. Orientifold action acts not only to the target space like orbifold, but also act on the string world sheet(Riemann surface Σ) in such a way to change the orientation.

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(p, q)5-brane with O5 plane

- Mathematical physicists conjectured that there is a one-to-one correspondence between "**5-brane web diagram**" (5+1 dimensional object to which the strings can be attached) and "**toric diagram**". [N.C.Leung, C.Vafa '97]
- In physics, it is natural to introduce **O5-plane** (5+1 dimensional orientifold-plane) to the 5-brane web diagram.

Question:

Is there geometry corresponding to "5-brane web with O5-plane"?

Suppose a "5-brane web diagram with O5-plane" is given.

Questions

- How to construct the corresponding geometry for the given diagram?
- How to compute the corresponding Gromov-Witten invariants when the diagram is given?

- No systematic answer is known for the first question (although it is expected to be related to the resolution of D-type singularity.). There are conjectures only for limited examples.
- As for the second question, a systematic method is proposed by generalizing the topological vertex formalism (without answering the first question). [S-S.Kim, F.Yagi '17]

Topological Vertex without O5 plane

According to topological vertex formalism in physics, topological string partition function can be computed systematically based on (p, q) 5-brane web.

1. Assign different Young diagram $\lambda, \mu, \nu, \ldots$ to different edges in the web diagram.
2. Introduce the edge factor $(-Q)^{|\lambda|}\left[(-1)^{|\lambda|}g^{\frac{1}{2}(||\lambda^t||^2-||\lambda||^2)}\right]^n$ to each edge and vertex factor $C_{\lambda\mu\nu}$ to each vertex;

Then the topological string partition function can be computed by multiplying these factors and summing over all possible Young diagrams as

$$Z = \sum_{\lambda,\mu,\nu\cdots} \prod (\text{Edge factor}) \cdot \prod (\text{Vertex factor}). \tag{1}$$

Topological Vertex with O5 plane

For a (p, q) 5-brane web with an O5-plane, the following two new rules are introduced for the topological vertex computations:

- Assign *identical* Young diagram Y to both the $(p, -1)$ 5-brane and the $(-p, -1)$ 5-brane as in Fig.(a)($p = 0 \,\text{or}\, 1$).
- Introduce the new type of edge factor,

$$(+Q_1 Q_2)^{|Y|} \left[(-1)^{|Y|} g^{\frac{1}{2}(||Y^t||^2 - ||Y||^2)}\right]^{\mathfrak{n}} \tag{2}$$

  for the configuration including the edges corresponding to $(p, -1)$ and $(-p, -1)$ 5-branes, where

$$\mathfrak{n} = (p_2, -q_2) \wedge (p_1, q_1) + 1 = p_1 q_2 + p_2 q_1 + 1. \tag{3}$$

Equipped with these two new rules, the topological string partition function can be computed even if the 5-brane web diagram includes O5-planes.

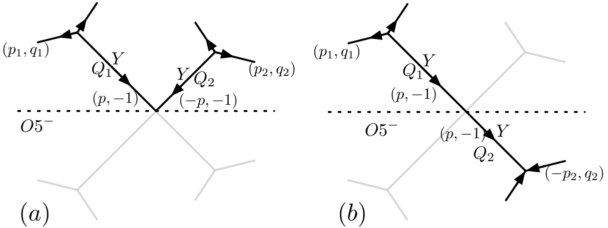

Figure: (a) New rule for topological vertex formalism including an O5-plane. (b) Interpretation of the rule in terms of the reflected image by an O5-plane.

Finally, we obtain topological vertex formula with O5 plane:

$$Z^{O5} = \sum_{\lambda,\mu,\nu\cdots} \prod (\text{Edge factor}) \cdot \prod (\text{Vertex factor}) \cdot \prod (\text{O5} - \text{plane factor}).$$

The global picture of Mirror Symmetry

5-brane web (with O5-plane)

Boundary condition [Hayashi-kim-lee-yagi,17']

Topological vertex formalism with O5-plane [kim-yagi,17']

Seiberg-Witten curve

Nekrasov partition function

(Seiberg-Witten prepotential)

(Topological string partition function)

$F \longleftrightarrow Z$

Thermodynamic limit

$$F = -\lim_{\hbar \to 0} \hbar^2 \log Z \qquad (\epsilon_1 = -\epsilon_2 = \hbar)$$

Step1: Rewrite Nekrasov partition function in terms of profile function

- Sum over Young diagram $\sum_{\lambda} \rightarrow$ Sum over profile function $\sum_{f_\lambda(x)}$

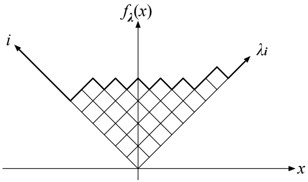

Figure: Young diagram λ and its corresponding profile function $f_{\boldsymbol{\lambda}}(x)$.

Step2: Take thermodynamic Limit $\epsilon_1 = -\epsilon_2 = \hbar \to 0$

- Sum over profile function $\to$ "path integral" over profile function
- $\sum_{f_\lambda(x)} \to \int \mathcal{D}f$
- $$Z = \int \mathcal{D}f e^{-\frac{1}{\hbar^2}(\epsilon(f)+\xi constraints)}$$
  $$= e^{-\frac{1}{\hbar^2}\epsilon(f_*)+\mathcal{O}(\hbar^{-1})}$$

Step3: Introduce resolvent

- $$R(z) := \int_{-\infty}^{\infty} \frac{f''_*(x)}{1 - e^{z-x}} dx$$
- Analytic continuation shows that the resolvent is defined on a Riemann surface (two sheet).

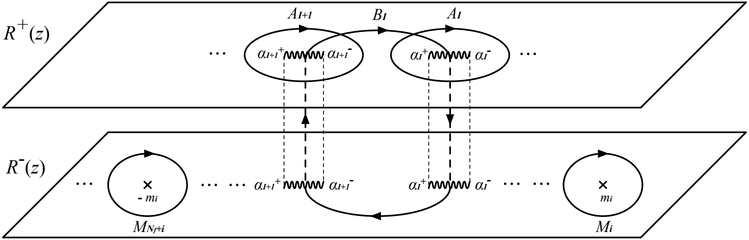

Figure: A_I, B_I, and M_i-cycles depicted on the two copies of complex plane with coordinate z. $R^-(z)$ is defined on the upper complex plane

Step4: Reproduce Seiberg-Witten solution

- By finding a proper function of the resolvent,

$$t(w = e^{-z}) = t(R(z)),$$

the constraints can be identified as the Seiberg-Witten curve

$$t^2 + P(w)t + Q(w) = 0.$$

- Boundary conditions can be reproduced:

$$P(1)^2 - 4Q(1) = 0, \quad P(-1)^2 - 4Q(-1) = 0$$

Step5: Reproduce Seiberg-Witten solution by evaluating integrals over cycles

- $\lambda^{\pm}_{SW} = \frac{\beta z}{4\pi i}\left(R^{\pm}(z) - \sum_{i=1}^{N_f} \frac{2e^{-\beta(m_i+z)}}{1-e^{-\beta(m_i+z)}} - 4(N+2)\right) dz,$
- $$\oint_{A_I} \lambda_{SW} = a_I, \quad (I = 1, 2, \cdots, N, N+1),$$
$$\oint_{B_I} \lambda_{SW} = \frac{1}{2\pi i\beta}\left(\frac{\partial F}{\partial a_I} - \frac{\partial F}{\partial a_{I+1}}\right) \quad (I = 1, 2, \cdots, N-1),$$
$$\oint_{B_N} \lambda_{SW} = \frac{1}{2\pi i\beta}\frac{\partial F}{\partial a_N},$$
$$\sum_{I=1}^{N+1} a_I = -m_0 + \frac{1}{2}\sum_{j=1}^{N_f} m_j. \qquad (4)$$

Summary

The proof mainly consists of the following two ingredients:

- We study 5d $N = 1$ $Sp(N)$ gauge theory with N_f flavors by using 5-brane web with O5-plane.
- Seiberg-Witten curve can be obtained from 5-brane web with O5-plane. Especially, we discuss the boundary condition originated from O5-plane.
- We compute Nekrasov partition function by using Topological vertex formalism with O5-plane.
- We take thermodynamic limit and reproduced the Seiberg-Witten curve (prepotential).

Thank you very much !