Inclusive B decays

Ref. ① "Heavy Quark Physics", chapter 6, Manohar & Wise

② 1501.00314, Paulo Gambino ($\bar{B} \to X_c \ell^- \bar{\nu}$)

③ 2007.04191, Huber, Hurth, Jenkins, Lunghi, Qin, Vos ($\bar{B} \to X_s \ell^+ \ell^-$)

1. What is inclusive?

Exclusive — all f-state particles detected.

e.g. $\bar{B}^0 \to K^- \pi^+$, $e^+ e^- \to \pi^+ \pi^- \gamma$

Inclusive — some (all) particles unmeasured

e.g. $e^+ e^- \to X$, $X \ni \mu^+ \mu^-, \pi^+ \pi^- \pi^0, \cdots$

$\bar{B} \to X_s \ell^+ \ell^-$, $X_s \ni K^-, K^- \pi^+, K^*$

2. Why inclusive?

(1) Complementary to exclusive

- different theo. frameworks
- different exp. measurents

$\Big\}$ ⇒ strong cross check

e.g. V_{ub} & V_{cb}

e.g. B anomalies P_5', R_{K^*}, $B \to X_s \ell^+ \ell^-$

(2) lifetime

$$\tau(B) = 1/\Gamma(B) = 1/\Gamma(B \to X)$$

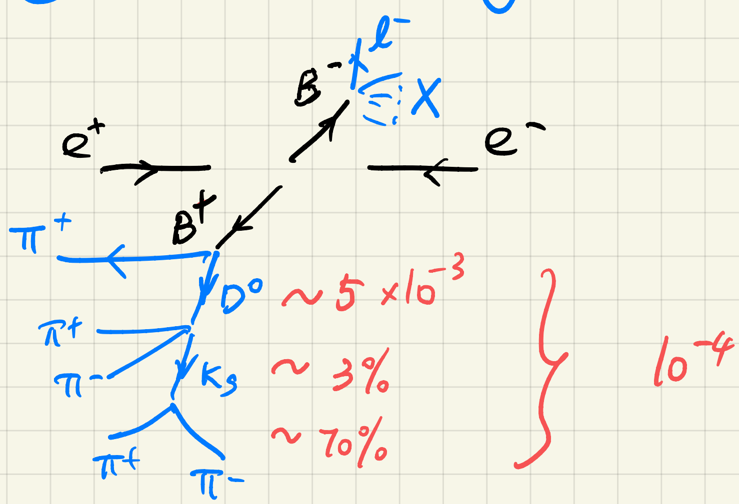
3. How to measure inclusive?

(1) Quasi-inclusive.

sum of exclusive
(all)

(2) Really inclusive

- leave "X" unmeasured
- fully reconstruct the tag side.



☆ Inclusive measurements more difficult.

Belle II $\sim 10^{11}$ B mesons

4. How to calculate inclusive?

4.1 Heavy quark effective theory

B meson = b quark + "mud" (quark + gluon)

$P_B = m_B v$ $\qquad$ $P_b = \underset{\sim}{m_b} v + k$ $\qquad$ soft

$k \sim \Lambda_{QCD}$

Integrating out the constant big P, only Λ_{QCD} left.

Define $Q_v(x) \equiv \underbrace{e^{im_Q v\cdot x}}Q(x)$

$$h_v(x) \equiv \underbrace{\frac{1+\cancel{v}}{2}}_{\hookrightarrow \cancel{p}+m} Q_v(x) = e^{im_Q v\cdot x}\frac{1+\cancel{v}}{2}Q(x)$$

$$H_v(x) = \underbrace{\frac{1-\cancel{v}}{2}}_{\hookrightarrow \cancel{p}-m} Q_v(x) = e^{im_Q v\cdot x}\frac{1-\cancel{v}}{2}Q(x)$$

$Q(x) = e^{-im_Q v\cdot x}(h_v(x)+H_v(x))$. $\quad \cancel{v}h_v(x) = h_v(x)$

$\cancel{v}H_v(x) = -H_v(x)$

Expand the $\mathcal{L}$ by $\frac{\Lambda_{QCD}}{m_Q}$

- Leading power

$$\mathcal{L} = \bar{Q}(i\cancel{D} - m_Q)Q = \boxed{(\bar{h}_v + \cancel{\bar{H}_v})e^{im_Q v\cdot x}(i\cancel{D}-m_Q)\cdot e^{-im_Q v\cdot x}(h_v+\cancel{H_v})}$$

$$= \bar{h}_v i\cancel{D} h_v + \cdots$$

$$= \frac{1}{2}\bar{h}_v\cancel{v}i\cancel{D}h_v + \frac{1}{2}\bar{h}_v i\cancel{D}\cancel{v}h_v$$

$$\mathcal{L} = \bar{h}_v\, iv\cdot D\, h_v + \cdots$$

mass? HQ flavor symmetry

spin? HQ spin symmetry

- Subleading power

$$\mathcal{L} = \bar{h}_v\, iv\cdot D\, h_v - \bar{H}_v(iv\cdot D + 2m_Q)H_v$$
$$+ \bar{h}_v i\cancel{D} H_v + \bar{H}_v i\cancel{D} h_v$$

$$\Rightarrow (iv\cdot D + 2m_Q)H_v = i\cancel{D}_\perp h_v$$

$$\Rightarrow \quad \mathcal{L} = \bar{h}_v \, i v\cdot \underbrace{D}_{\Lambda_{QCD}} h_v - \frac{1}{2m_Q} \bar{h}_v \underbrace{\not{D}_\perp \not{D}_\perp}_{\Lambda_{QCD}^2} h_v + \cdots$$

$$\not{D}_\perp \not{D}_\perp = \left(\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} + \tfrac{1}{2}[\gamma_\mu, \gamma_\nu] \right) D_\perp^\mu D_\perp^\nu$$

$$= D_\perp^2 + \frac{g_s}{2} \sigma_{\mu\nu} G^{\mu\nu}$$

$$\Rightarrow \mathcal{L} = \bar{h}_v \, i v\cdot D h_v - \bar{h}_v \frac{D_\perp^2}{2m_Q} h_v - g_s \bar{h}_v \frac{\sigma_{\mu\nu} G^{\mu\nu}}{4m_Q} h_v$$

$\sigma_{\mu\nu} G^{\mu\nu}$ → spin-dependent

★ Keep in mind: in HQET, $\sim \Lambda_{QCD}$ are left.

all physical quantities expanded $\left(\frac{\Lambda_{QCD}}{m_Q}\right)^n$.

4.2 Operator Product Expansion.

$$G_{12}(x, 0; y_1, \cdots, y_n) = \langle T\{ \underbrace{O_1(x) O_2(0)}_{\text{interaction vertices.}} \underbrace{\phi(y_1) \cdots \phi(y_n)}_{\text{external states}} \} \rangle$$

If x is small,

$$T\{O_1(x) O_2(0)\} \longrightarrow \sum \underbrace{C_{12}^{(n)}(x)}_{\text{Wilson coefficients.}} \underbrace{O_n^{(d)}(0)}_{\text{local operator basis}}$$

(d) → dimension

$$\Rightarrow \quad G_{12}(x, 0; y_1, \cdots, y_n) = \sum_n C_{12}^{(n)}(x) \langle T\{ O_n^{(d)}(0) \phi(y_1) \cdots \phi(y_n) \} \rangle$$

★ $O_n^{(d)}(0)$ have global symmetries of $T\{\cdots\}$

★ $C_{12}^{(n)}(x)$ universal to different processes.

(simple)

• A familiar example — beta decay.

$$T\left\{ \frac{g}{\sqrt{2}} \bar{u}(x) \gamma^\mu P_L d(x) W_\mu(x) \otimes \frac{g}{\sqrt{2}} \bar{e}(0) \gamma^\nu P_L \nu(0) W_\nu(0) \right\}$$

when x is small, guess

$\bar{u}(0) \gamma^\mu P_L d(0) \bar{e}(0) \gamma_\mu P_L \nu(0)$

$+ \partial$ terms.

Use $d(p_d) \to u(p_u) + e(p_e) + \bar{\nu}(p_\nu)$

$$\int d^4x \langle u e \bar{\nu} | T\{\cdots\} | d \rangle$$

$$= -\frac{ig^2}{2} \bar{u}(p_u) \gamma^\mu P_L u(p_d) \frac{g_{\mu\nu}}{l^2 - m_W^2} \bar{u}(p_e) \gamma^\nu P_L v(p_\nu)$$

$l = p_d - p_u$

$$l^2 \ll m_W^2 \Rightarrow \frac{1}{l^2 - m_W^2} = -\frac{1}{m_W^2}\left(1 + \frac{l^2}{m_W^2} + \cdots\right)$$

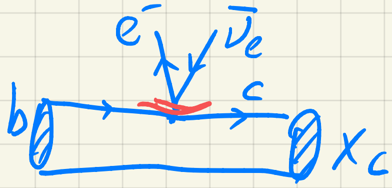
① leading power $\quad O_0 = \bar{u}(0) \gamma^\mu P_L d(0) \bar{e}(0) \gamma_\mu P_L \nu(0)$

$$C_0 = \frac{ig^2}{2m_W^2} \delta^4(x) = i \frac{4G_F}{\sqrt{2}} \delta^4(x)$$

② subleading $\quad O_1 = \bar{u}(0) \gamma^\mu P_L (\overrightarrow{\partial} - \overleftarrow{\partial})^2 d(0) \bar{e}(0) \gamma_\mu P_L \nu(0)$

$$C_1 \propto \frac{1}{m_W^4}$$

4.3 $\bar{B} \to X_c e^- \bar{\nu}_e$



$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} \bar{c} \gamma^\mu P_L b \, \bar{e} \gamma_\mu P_L \nu_e$$

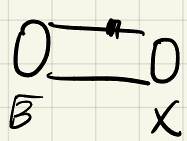
$$\frac{d^3\Gamma}{dq^2\, dE_e\, dE_\nu} = \frac{1}{4} \underbrace{\sum_{X_c}}_{\uparrow \; \sum_X \int \frac{d^3\vec{P}_X}{(2\pi)^3 2E_X}} \sum_{\text{spin}} \frac{|\langle X e \bar{\nu} | H_W | \bar{B} \rangle|^2}{2m_B} \, \delta^{(4)}[P_B - \underbrace{(P_e + P_\nu)}_{\parallel \; q} - P_X]$$

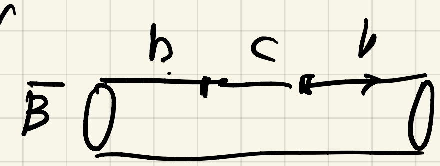
$$= \frac{1}{(2\pi)^3} \, 2 G_F^2 |V_{cb}|^2 \underbrace{W_{\alpha\beta}}_{\text{hadronic piece}} \underbrace{L^{\alpha\beta}}_{\text{leptonic piece}}$$

$$L^{\alpha\beta} = \mathrm{Tr}[\not{P}_e \, \gamma^\beta P_L \not{P}_\nu \, \gamma^\alpha P_L]$$

$$W^{\alpha\beta} = \sum_{X_c} (2\pi)^3 \delta^{(4)}(P_B - q - P_X) \frac{1}{2m_B} \underbrace{\langle \bar{B} | J_L^{+\alpha} | X_c \rangle \langle X_c | J_L^\beta | \bar{B} \rangle}$$

— Optical theorem

$\sum_X \Big|$ 

$\Big|^2 \propto \mathrm{Im}\Big($ 

$\Big)$

$$T^{\alpha\beta} \equiv -i \int d^4x \, e^{-iq\cdot x} \frac{\langle \bar{B} | T\{ J_L^{+\alpha}(x) J_L^\beta(0) \} | \bar{B} \rangle}{2m_B}$$

$$J_L^\beta = \bar{c} \gamma^\beta P_L b$$

$$\Rightarrow -\frac{1}{\pi} \mathrm{Im}\, T_{\alpha\beta} = \underbrace{\sum_{X_c} (2\pi)^3 \delta^{(4)}(P_B - q - P_X) \frac{\langle \bar{B} | J_{L\alpha}^+ | X_c \rangle \langle X_c | J_{L\beta} | \bar{B} \rangle}{2m_B}}_{W_{\alpha\beta}}$$

$$\bar{B} + e + \bar{\nu}_e \to X_{\bar{c}bb} \rightarrow \quad + \sum_{X_{\bar{c}bb}} (2\pi)^3 \delta^{(4)}(P_B + q - P_X) \frac{\langle \bar{B} | J_{L\beta} | X_{\bar{c}bb} \rangle \langle X_{\bar{c}bb} | J_{L\alpha}^+ | \bar{B} \rangle}{2m_B}$$

Next task: OPE for $T[J^{\dagger}_{L\alpha}(x) J_{L\beta}(0)]$

* basis of local operators
* perturbatively calculate the WCs

$\bar{b}(x)\gamma_\alpha P_L c(x)\,\bar{c}(0)\gamma_\beta P_L b(0)$

$\bar{b}\,b(0)$? $\underbrace{\bar{b}\,\partial^\mu b}_{\sim m_b}$? $\rightarrow$ $\underbrace{\bar{h}_v \partial^\mu h_v}_{\sim \Lambda_{QCD}}$

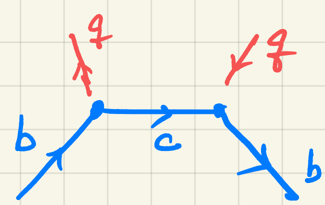
4.3.1 Operator match

$t_{\alpha\beta} \equiv -i\int d^4x\, e^{-iq\cdot x}\, T[J^{\dagger}_{L\alpha}(x) J_{L\beta}(0)] = ?$

Consider

$$\langle b(m_b v+k)|\, t_{\alpha\beta}\,|b(m_b v+k)\rangle \qquad (k\sim\Lambda_{QCD})$$

$$\overset{LO}{=} \frac{1}{(m_b v - q + k)^2 - m_c^2 + i\epsilon}\,\bar{u}\gamma_\alpha P_L (m_b \not{v} - \not{q} + \not{k})\times\gamma_\beta P_L u$$



(1) Leading power ($k\to 0$)

$$\langle b|t_{\alpha\beta}|b\rangle = \frac{1}{\underbrace{(m_b v - q)^2 - m_c^2 + i\epsilon}_{=\Delta_0}}\,\bar{u}\gamma_\alpha\gamma_\lambda\gamma_\beta P_L u\cdot(m_b v - q)^\lambda$$

$$= \frac{1}{\Delta_0}\bar{u}\left(g_{\alpha\lambda}\gamma_\beta + g_{\lambda\beta}\gamma_\alpha - g_{\alpha\beta}\gamma_\lambda - i\epsilon_{\alpha\beta\lambda\eta}\gamma^\eta\right)\times P_L u\,(m_b v - q)^\lambda$$

single γ's

One single operator

$$\boxed{\bar{b}\gamma^\mu P_L b} = \bar{h}_v\gamma^\mu P_L h_v + \cdots ?$$

Wilson coefficient $\rightarrow$ $\frac{1}{\Delta_0}\left(g_{\alpha\lambda}g_{\mu\beta} + g_{\lambda\beta}g_{\mu\alpha} - g_{\alpha\beta}g_{\mu\lambda} - i\epsilon_{\alpha\beta\lambda\mu}\right)\times(m_b v - q)^\lambda$

(2) subleading power.

$$\frac{k^\lambda}{(m_b v - q)^\lambda} \cdot LP + \frac{-2k\cdot(m_b v - q)}{\Delta_0} \cdot LP$$

$$\ni \frac{1}{\Delta_0} \bar{u}\left(k_\alpha \gamma_\beta + k_\beta \gamma_\alpha - g_{\alpha\beta} k^\lambda \gamma_\lambda - i \epsilon_{\alpha\beta\lambda\eta} k^\lambda \gamma^\eta \right) P_L u$$

$$\bar{u}\gamma_\lambda k_\tau P_L u \to \bar{b}\gamma_\lambda (iD_\tau - m_b v_\tau) P_L b$$

$$\xrightarrow{HQET} \bar{h}_v \gamma_\lambda iD_\tau P_L h_v + \cdots$$

$$\longrightarrow v_\lambda \bar{h}_v iD_\tau P_L h_v$$

$$\left(\frac{1+\not{v}}{2} \gamma_\lambda \frac{1+\not{v}}{2} = v_\lambda \frac{1+\not{v}}{2} \right.$$

(3) Subsubleading

3 sources $\begin{cases} k^2 \text{ terms} \\ k^1 \text{ terms} \\ b \to bg \text{ match} \end{cases}$

h_v, D_μ.

* k^2 terms.

$$\longrightarrow \bar{b}\gamma^\lambda (iD - m_b v)^{(\alpha} (iD - m_b v)^{\beta)} b$$

$$\longrightarrow v^\lambda \bar{h}_v iD^{(\alpha} iD^{\beta)} h_v + \cdots$$

$$a^{(\alpha} b^{\beta)} \equiv \frac{1}{2}(a^\alpha b^\beta + a^\beta b^\alpha$$

Other sources.

$$\bar{h}_v g \frac{G_{\alpha\beta}\sigma^{\alpha\lambda}}{2m_b} h_v, \quad \bar{h}_v g G^{\mu\nu} \gamma^\lambda \gamma_5 h_v, \cdots$$

A short summary.

* leading $\bar b \gamma b$ $(\bar h_v \gamma_\lambda h_v)$
* subleading, one additional iD_μ (Λ_{QCD}/m_b)
* subsubleading, two iD_μ, iD_ν
* beyond this. 3 iD's. $\bar b_v \Gamma_1 q \bar q \Gamma_2 b_v$

4.3.2 Matrix elements $\langle \bar B(v) | O(0) | \bar B(v) \rangle$

(1) leading power $\bar b \gamma_\lambda P_L b$

$$\langle \bar B(v) | \bar b \gamma_\lambda \gamma_5 b | \bar B(v) \rangle = 0$$

$$\langle \bar B(v) | \bar b \gamma_\lambda b | \bar B(v) \rangle = ?$$

The b-number charge $\hat Q_b \equiv \int d^3\vec x\, \bar b \gamma^0 b(x)$

$$\langle \bar B(p_1) | \hat Q_b | \bar B(p_2) \rangle = \langle \bar B(p_1) | \bar B(p_2) \rangle = 2E_B (2\pi)^3 \delta^{(3)}(\vec p_1 - \vec p_2)$$

$$= \int d^3\vec x\, \langle \bar B(p_1) | \bar b \gamma^0 b(x) | \bar B(p_2) \rangle$$

$$= \int d^3\vec x\, e^{-i(\vec p_1 - \vec p_2)\cdot \vec x} \langle \bar B(p_1) | \bar b \gamma^0 b(0) | \bar B(p_2) \rangle$$

$$\Rightarrow \langle \bar B(v) | \bar b \gamma^0 b(0) | \bar B(v) \rangle = 2m_B v^0$$

$$\Rightarrow \langle \bar B(v) | \bar b \gamma^\lambda b | \bar B(v) \rangle = 2m_B v^\lambda$$

(2) Subleading power (HQET)

to be determined.

$$\langle \bar B(v) | \bar h_v iD_\tau h_v | \bar B(v) \rangle = X v_\tau$$

$v^\tau \times$ $$\langle \bar B(v) | \bar h_v \underline{iv\cdot D h_v} | \bar B(v) \rangle = X$$

Recall $\mathcal{L}_{HQET} = \bar{h}_v \, i v \cdot D \, h_v + \cdots$

$$\xrightarrow{EOM} \quad \underline{i v \cdot D \, h_v = 0} \quad + \text{ power corrections}$$

$$\Rightarrow \quad X = 0 \quad + \quad \text{power corrections.}$$

(3) Subsubleading power (HQET)

Define
$$\begin{cases} 2\lambda_1 = -\langle \bar{B} | \bar{h}_v D_\perp^2 h_v | \bar{B} \rangle \\ -12\lambda_2 = \langle \bar{B} | \bar{h}_v \, g \, \sigma_{\alpha\beta} G^{\alpha\beta} h_v | \bar{B} \rangle \end{cases}$$

↳ determined by B, B^* mass difference

eg $\langle \bar{B}(v) | \bar{h}_v \, iD_{(\lambda} iD_{\tau)} h_v | \bar{B}(v) \rangle = Y (g_{\lambda\tau} - v_\lambda v_\tau)$

$i v \cdot D h_v = 0$

$g^{\lambda\tau}$

$$\Rightarrow \langle \bar{B}(v) | \bar{h}_v (iD)^2 h_v | \bar{B}(v) \rangle = 3Y$$

$$\Rightarrow Y = +\frac{2}{3}\lambda_1$$

Up to now, all WCs and MEs have been obtained $(\alpha_s^0, \frac{1}{m_b^2})$

— Final result

* Leading power

$$\frac{d^2\Gamma}{d\hat{q}^2 \, dy} = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} \, 12 (y - \hat{q}^2)(1 + \hat{q}^2 - \rho - y)$$

$$= \frac{d^2\Gamma(b \to c \ell^- \bar{\nu})}{d\hat{q}^2 \, dy}$$

$$\begin{cases} y = \frac{2E_e}{m_b} \\ \hat{q}^2 = \frac{q^2}{m_b^2} \\ \rho = \frac{m_c^2}{m_b^2} \end{cases}$$

$m_b \gg \Lambda_{QCD} \Leftrightarrow$ free b.

* Including power corrections

$$\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[1 + \frac{\lambda_1}{2m_b^2} + \frac{3\lambda_2}{2m_b^2} \left(2\rho \frac{d}{d\rho} - 3 \right) \right] f(\rho)$$

$$f(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho.$$