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Positivity bounds and their implications for SMEFT

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in memory of
my collaborator Cen Zhang

Are all EFTs allowed?

EFTs are widely used in physics: particle physics, gravity/cosmology…

$$\mathscr{L}_{\text{EFT}} = \sum_i \Lambda^4 \, c_i \, \mathcal{O}_i \left(\frac{\text{boson}}{\Lambda}, \frac{\text{fermion}}{\Lambda^{3/2}}, \frac{\partial}{\Lambda} \right)$$

Λ: EFT cutoff $\quad c_i$: Wilson coefficients

Is every set of Wilson coefficients $\{c_i\}$ allowed? No!

UV completion satisfies: Lorentz invariance, causality/analyticity, unitarity, crossing symmetry, …

↓ bootstrap

Positivity bounds on Wilson coefficients

Toy example: $P(X)$

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

2 to 2 scattering amplitude: $A(s, t=0) = \cdots + \frac{2\lambda s^2}{\Lambda^4} + \cdots$

Forward positivity bound: $\lambda > 0$

theories with $\lambda < 0$ do not have a standard UV completion

$$\mathscr{L}_{\mathrm{DBI}} \sim -\sqrt{1+(\partial\phi)^2}$$

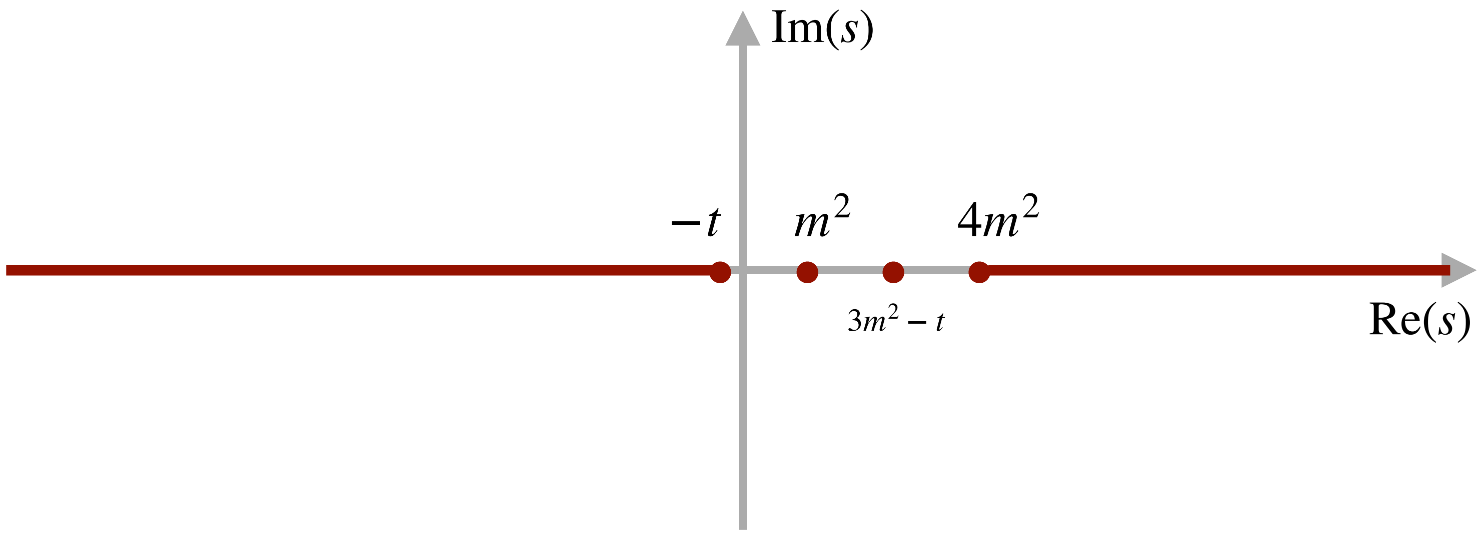
$$\mathscr{L}_{\overline{\mathrm{DBI}}} \sim -\sqrt{1-(\partial\phi)^2}$$

Similar to swampland idea

But positivity bounds take more conservative approach

Ingredients needed to prove positivity bounds

Causality/Analyticity: $A(s, t)$ as analytic function



s, t, u: Mandelstam variables

rigorously proven in 60'
Martin, …

Crossing symmetry: $A(s, t) = A(u, t)$ (one scalar)

Unitarity/Optical theorem: $\mathrm{Im}[A(s, t = 0)] \propto \sigma(s) > 0$

Frossart bound: as $s \to \infty, |A(s,0)| < C s \ln^2 s$

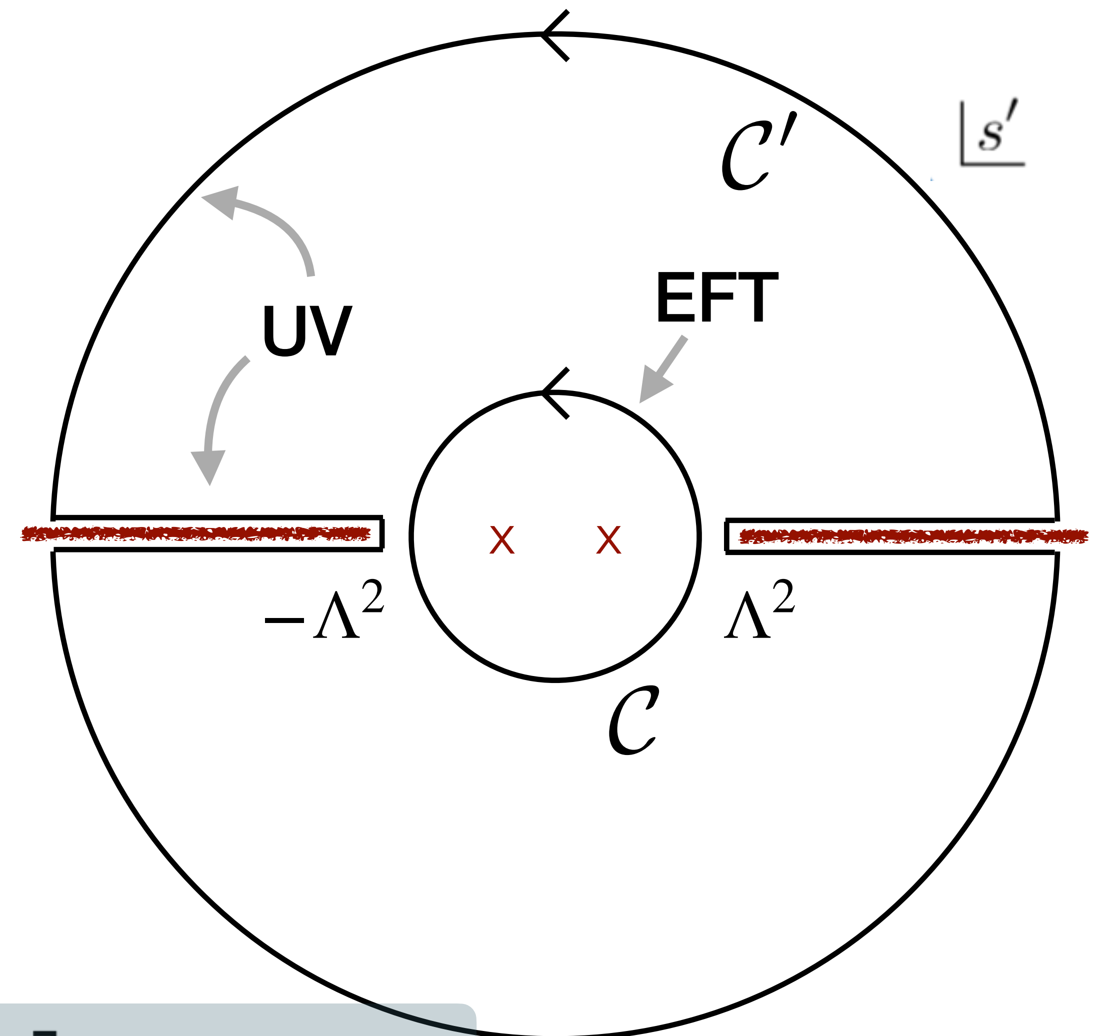
Froissart, 1961

Fixed t dispersion relation

Analyticity in complex s plane (fixed t)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s}$$

Cauchy's integral formula



Fixed t dispersion relation

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu-s} + \frac{u^2}{\mu-u}\right] \mathrm{Im}\, A(\mu,t)$$

$\mu > \Lambda^2$

EFT amplitude

IR ~ UV connection

UV full amplitude

Forward positivity bounds

Expand both sides of dispersion relation $A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t)$

Match the s^2 term on both sides $A(s,t) = c_{2,0}s^2 + \cdots$

$$c_{2,0} = \int \frac{2\,\mathrm{d}\mu}{\pi\mu^3} \mathrm{Im}\, A(\mu, 0)$$

Optical theorem: $\mathrm{Im}[A(s, t=0)] \propto \sigma(s) > 0$

$$c_{2,0} \sim \left.\frac{\partial^2 A(s,t)}{\partial s^2}\right|_{s,t=0} > 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006 + earlier works

this is just leading order, sufficient for phenomenology

Our fascinating universe

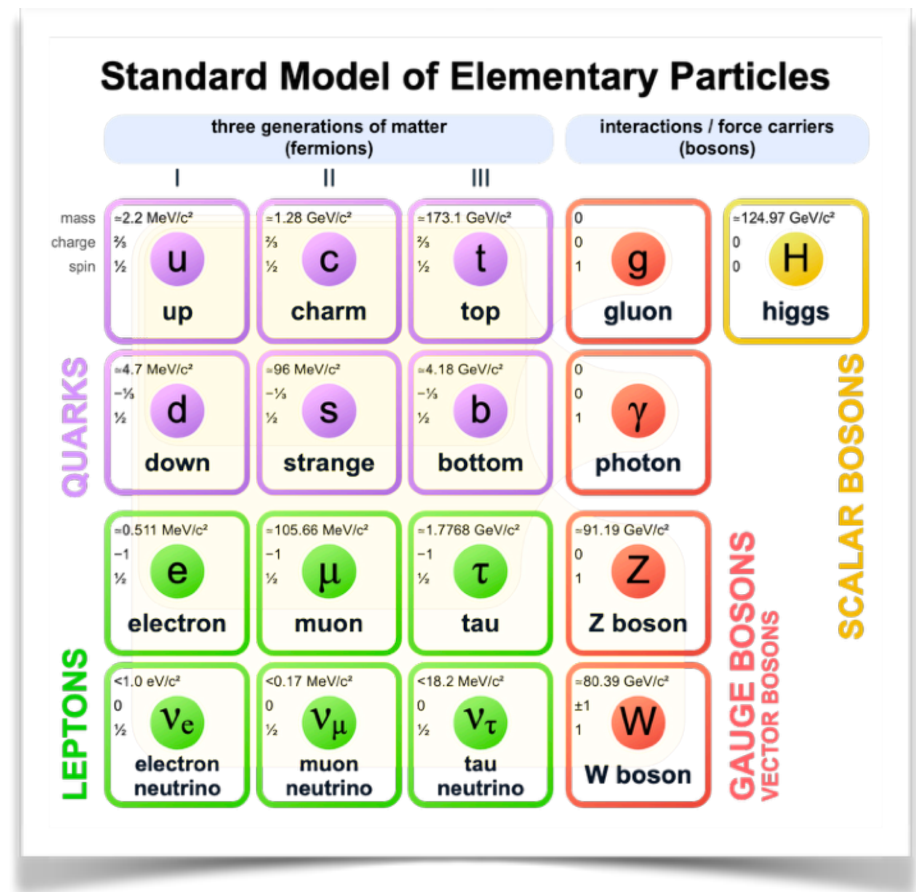
Universe is more complex than just one identical scalar!

SM Effective Field Theory (SMEFT)

$$\mathscr{L}_{\mathrm{SMEFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_j \frac{c_j^{(6)} O_j^{(6)}}{\Lambda^2} + \sum_i \frac{c_i^{(8)} O_i^{(8)}}{\Lambda^4} + \cdots$$

- SM particle contents and global symmetries
- SM gauge group structure

- Parametrize new physics
- Popular current approach



if consider up to dim-8, or order s^2

still huge parameter space!

How much can positivity bounds reduce the parameter space?

Generalized elastic positivity bounds

previous positivity bounds valid for identical particle scattering

Elastic scattering: particle i + particle j $\rightarrow$ particle i + particle j

$$M^{ijij} = c_{2,0}^{ijij} > 0$$

proof still goes through

Generalized elastic scattering: $a + b \rightarrow a + b$

massless case

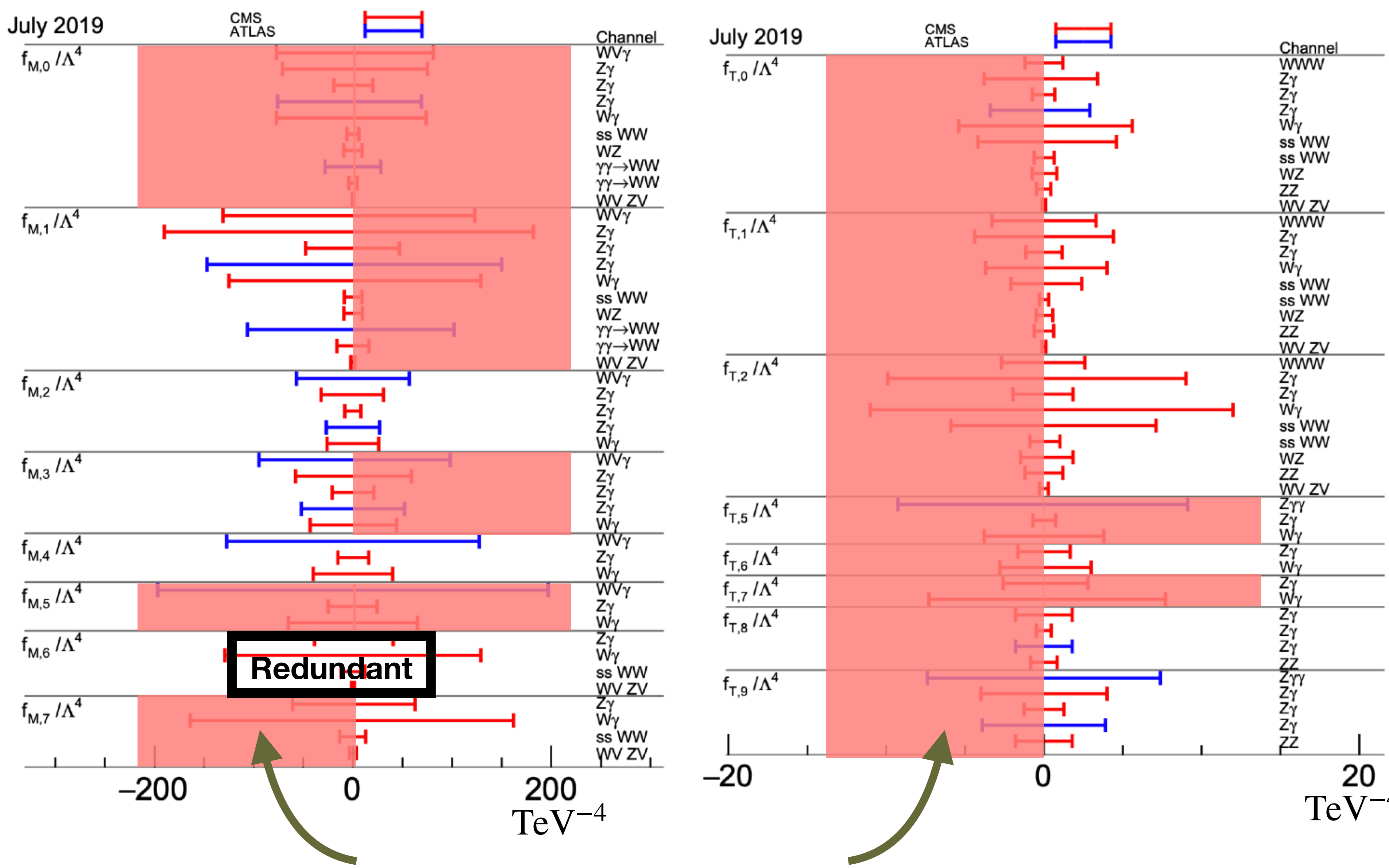
superposed states $|a\rangle = \sum_i u_i |i\rangle, \quad |b\rangle = \sum_j v_j |j\rangle$

u_i, v_i: arbitrary constants

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c_{2,0}^{ijkl} > 0$$

Positivity vs experimental bounds (1D bounds)

anomalous Quartic Qauge Couplings in VBS

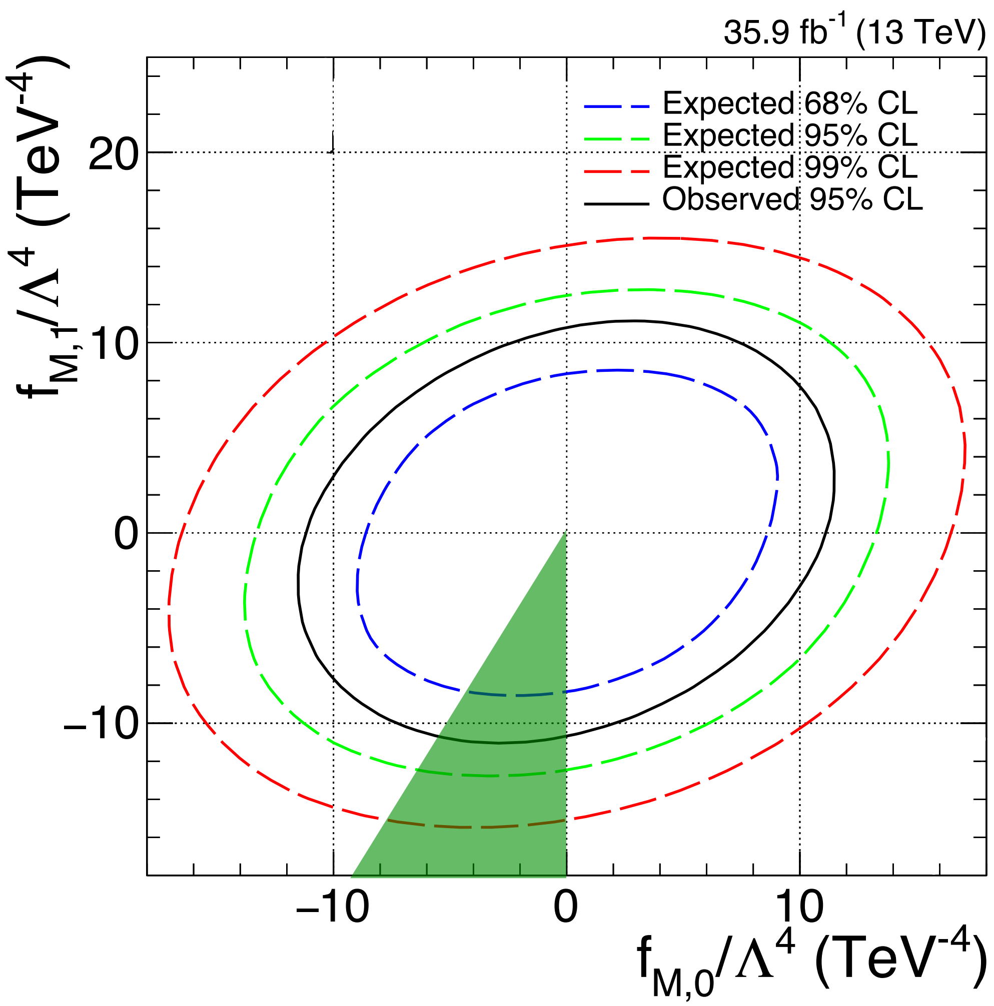


No UV completion/Inconsistent

Cen Zhang & **SYZ**,1808.00010

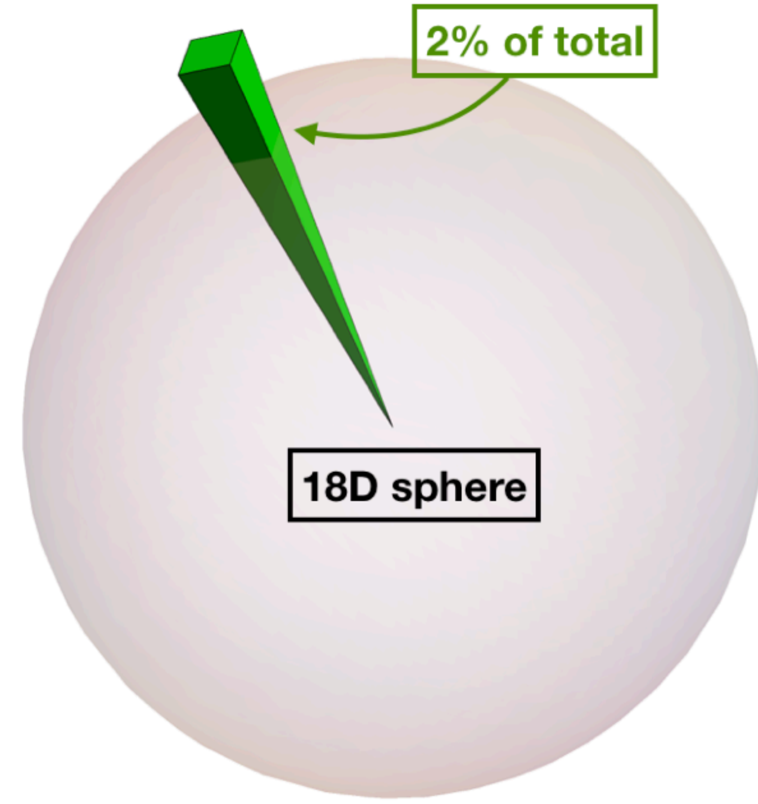
Positivity vs experimental bounds

O_{M0} and O_{M1}



Cen Zhang & **SYZ**,1808.00010

Space of 18 Wilson coeff's for aQGCs



Only **~2%** of the total aQGC parameter space admit an analytic UV completion!

Stronger positivity bounds?

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c_{2,0}^{ijkl} > 0$$

Is it possible such that

$$M^T = \sum_{ijkl} T_{ijkl} M^{ijkl} > 0, \text{ and } \{T_{ijkl}\} \supset \{u_i v_j u_k^* v_l^*\} ?$$

Yes, T_{ijkl} is more than $u_i v_j u_k^* v_l^*$!

Example: W-boson scatterings in SMEFT

$$F_{T,2} \geq 0, \quad 4F_{T,1} + F_{T,2} \geq 0$$
$$F_{T,2} + 8F_{T,10} \geq 0, \quad 8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$

scatterings of entangled states $T_{ijkl} \sim \Sigma_n \lambda_n U_{ij}^n U_{kl}^n$

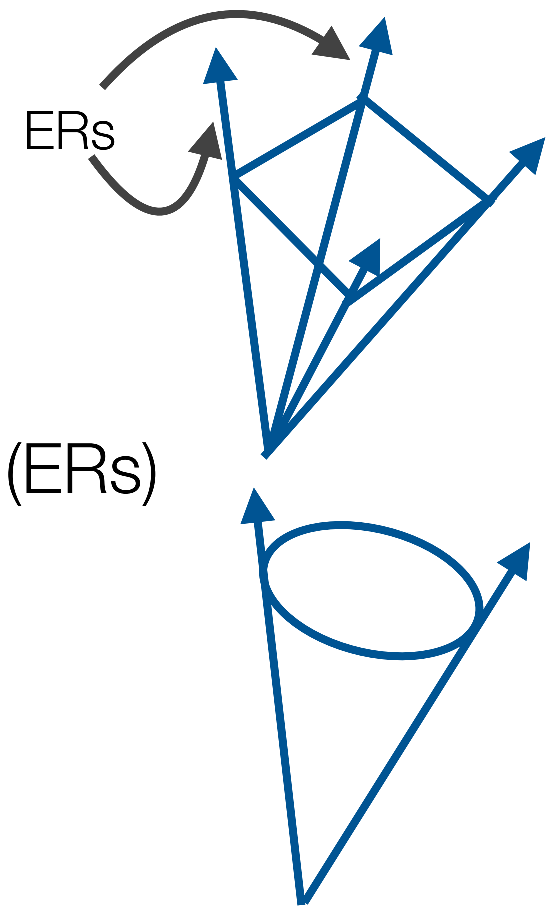
Cen Zhang & **SYZ**, PRL, 2005.03047

Best bounds from ERs of $\mathscr{T}$ cone

$$T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^+ \cap \overrightarrow{\mathbf{S}} \quad \begin{cases} \mathscr{T}^+ \equiv \left\{ T_{ijkl} \mid T_{ij,kl} \succeq 0 \right\} \\ \overrightarrow{\mathbf{S}} \equiv \left\{ T_{ijkl} \mid T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk} \right\} \end{cases}$$

T_{ijkl} forms spectrahedron $\mathscr{T}$ Li, Xu, Yang, Cen Zhang & **SYZ**, PRL, 2101.01191

(spectrahedron) = (convex cone of PSD matrices) ∩ affine-linear space



To find best bounds, find all ERs of $\mathscr{T}$

all elements of $\mathscr{T}$: $T_{ijkl} = \Sigma_p \alpha_p T^{(p)}_{ijkl}, \ \ \alpha_p > 0$

p enumerates all extreme rays (ERs)

Best positivity bounds:

$$\sum_{ijkl} T^{(p)}_{ijkl} M^{ijkl} > 0$$

Convex cone $\mathscr{C}$ of amplitudes

$$\mathscr{C} \equiv \{M^{ijkl}\} = \text{cone}\left(\{m^{i(j}m^{|k|l)}\}\right)$$

$$m^{ij} \sim M^{ij\to X}$$

X: intermediate state

For m^{ij} to be extremal, it can not be split to two amplitudes

$$m^{ij}_{(\text{ER})} \sim M^{ij\to X_{\text{irrep}}} \sim C^{r,\alpha}_{i,j}$$

CG coefficient

Get $\mathscr{C}$ cone by symmetries of EFT

$$\mathscr{C} = \text{cone}\left(\{P^{i(j|k|l)}_r\}\right)$$

$$P^{ijkl}_r \equiv \sum_\alpha C^{r,\alpha}_{i,j}\left(C^{r,\alpha}_{k,l}\right)^*$$

group projector

Cen Zhang & **SYZ**, PRL, 2005.03047

The inverse problem

For weakly coupled UV completion

Cen Zhang & **SYZ**, PRL, 2005.03047

ER ⟷ UV particle

Example: Higgs $\mathscr{C}$ cone in SMEFT

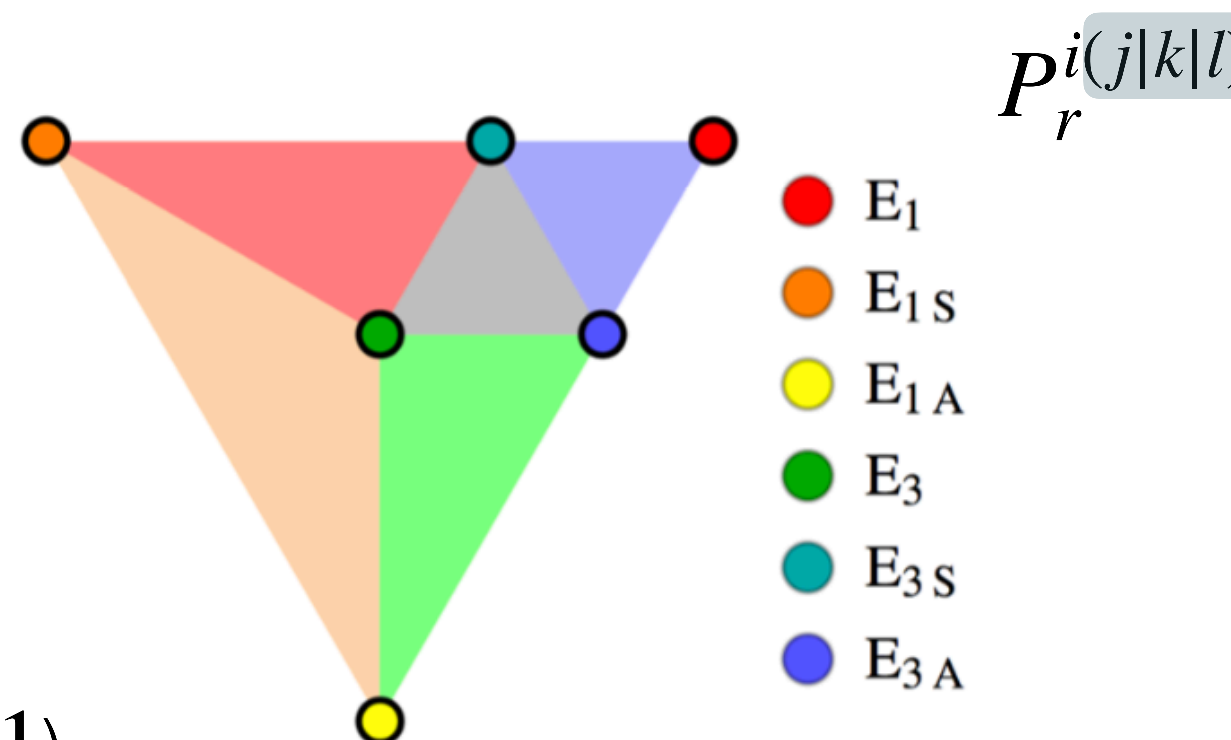
Wilson coeff's fall in blue region

↓

E_1 must exit

↓

new UV state ($SU(2)_L$ singlet, $Y = 1$)



ERs of $\mathscr{C}$ (or dim-8 operators) are important to inverse-engineer UV physics!

Semi-definite program (SDP)

spectrahedron is parameter space of a semi-definite program

Use SDP to check M^{ijkl} is in $\mathscr{C}$ cone

minimize $\sum_{ijkl} T_{ijkl} M^{ijkl}$

subject to $T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^{+} \cap \overrightarrow{\mathbf{S}}$

$\min(T \cdot M) > 0$, then M^{ijkl} is within positivity bounds

Compared to elastic approach ($uvuvM > 0$)

- stronger bounds
- more efficient (polynomial complexity)

Can also randomly sample and iterate to find ERs of $\mathscr{T}$

Li, Xu, Yang, Cen Zhang & **SYZ**, PRL, 2101.01191

Applications of convex cone approach

- Transversal Vector Boson Scattering

  10D parameter space: 0.681%

  Yamashita, Cen Zhang & **SYZ**, 2009.04490

  work in progress towards full VBS/aQGC

  Liu, Zhang & **SYZ**, 22XX.XXXXX

$$
\begin{aligned}
O_{T,0} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}] \\
O_{T,2} &= \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}] \\
O_{T,5} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta} \\
O_{T,7} &= \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha} \\
O_{T,8} &= \hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
O_{T,1} &= \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}] \\
O_{T,10} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}] \\
O_{T,6} &= \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\hat{B}_{\mu\beta}\hat{B}^{\alpha\nu} \\
O_{T,11} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta} \\
O_{T,9} &= \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta}\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}
\end{aligned}
$$

- 4-gluon SMEFT operators

  7D parameter space: 1.6628%

  obtained bounds both in $\mathscr{C}$ and $\mathscr{T}$ cones

  Li, Xu, Yang, Cen Zhang & **SYZ**, PRL, 2101.01191

$Q_{G^4}^{(1)}$	$(G^A_{\mu\nu}G^{A\mu\nu})(G^B_{\rho\sigma}G^{B\rho\sigma})$
$Q_{G^4}^{(2)}$	$(G^A_{\mu\nu}\widetilde{G}^{A\mu\nu})(G^B_{\rho\sigma}\widetilde{G}^{B\rho\sigma})$
$Q_{G^4}^{(3)}$	$(G^A_{\mu\nu}G^{B\mu\nu})(G^A_{\rho\sigma}G^{B\rho\sigma})$
$Q_{G^4}^{(4)}$	$(G^A_{\mu\nu}\widetilde{G}^{B\mu\nu})(G^A_{\rho\sigma}\widetilde{G}^{B\rho\sigma})$
$Q_{G^4}^{(7)}$	$d^{ABE}d^{CDE}(G^A_{\mu\nu}G^{B\mu\nu})(G^C_{\rho\sigma}G^{D\rho\sigma})$
$Q_{G^4}^{(8)}$	$d^{ABE}d^{CDE}(G^A_{\mu\nu}\widetilde{G}^{B\mu\nu})(G^C_{\rho\sigma}\widetilde{G}^{D\rho\sigma})$
Q_G	$f^{ABC}G^{A\nu}_{\mu}G^{B\rho}_{\nu}G^{C\mu}_{\rho}$

What about other coefficients?

forward bounds

non-forward bounds

$$A(s,t) \sim c_{2,0}s^2 + c_{2,1}s^2t + c_{2,2}s^2t^2 + \cdots$$

Again, use su symmetric dispersion relation

new gradient:
st crossing symmetry

Triple crossing bounds:

All $c_{i,j}$ with $i > 2,\ j \geq 0$ have **two-sided** bounds

Wilson coefficients are $O(1)$!

used to be a folklore; now is a theorem

Tolley, Wang & **SYZ**, 2011.02400

Caron-Huot & Duong, 2011.02957

Further developments

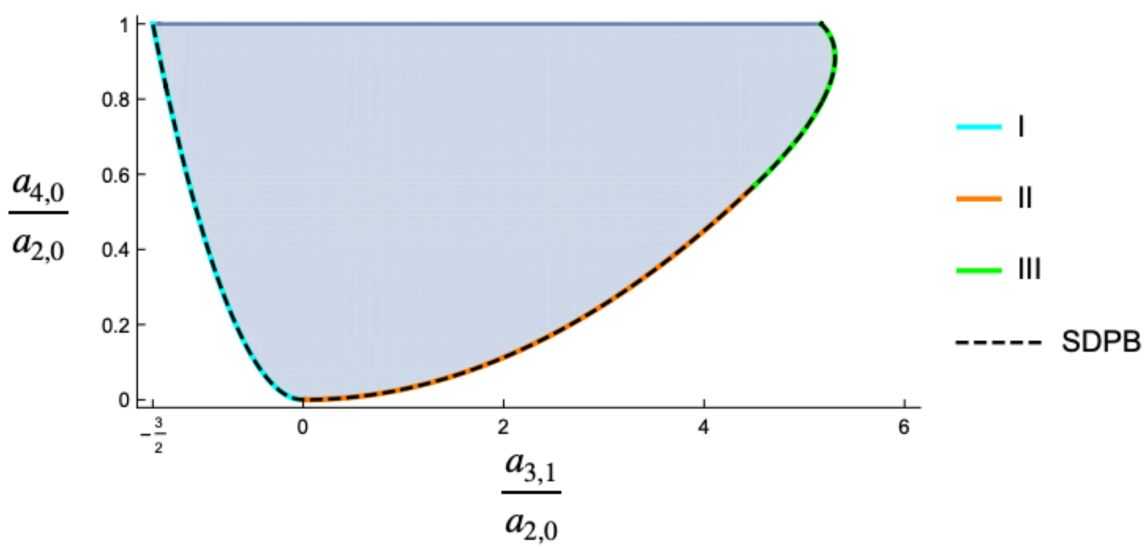
Fully crossing symmetric dispersion relation

Sinha & Zahed, 2012.04877

Semi-analytical approach

reduce to bi-variate moment problem
(GL rotations + triple-crossing slices)

Chiang, Huang, Li, Rodina & Weng, 2105.02862



Bounds from fixed impact parameter

can deal with spin-2 t-pole

Caron-Huot, Mazac, Rastelli, Simmons-Duffin, 2102.08951

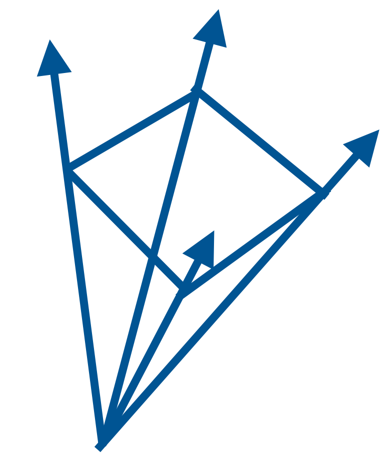
Triple crossing bounds for multi-fields

still, all the coefficients are $O(1)$

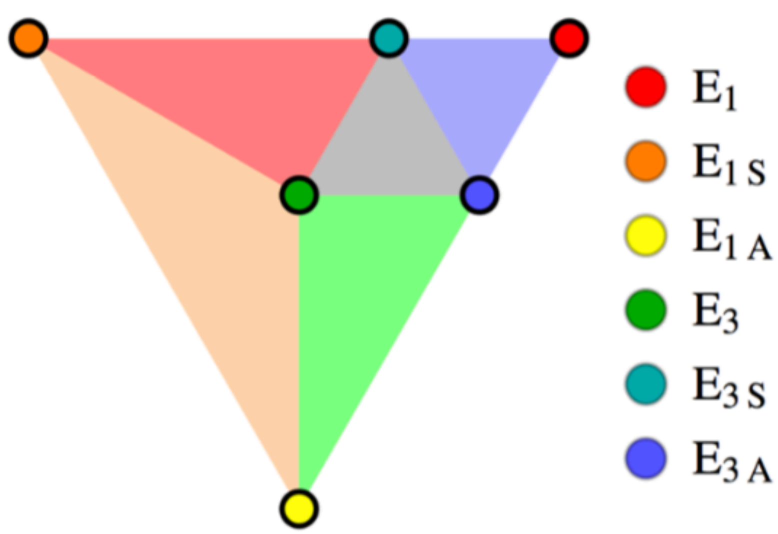
Du, Cen Zhang & **SYZ**, 2111.01169

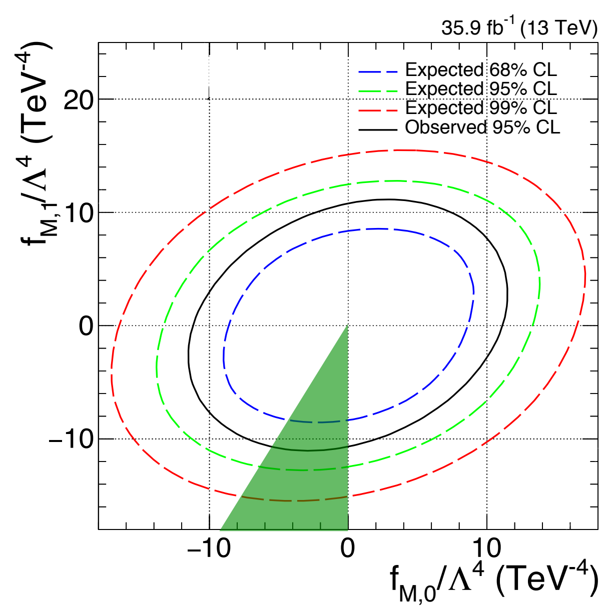
Summary

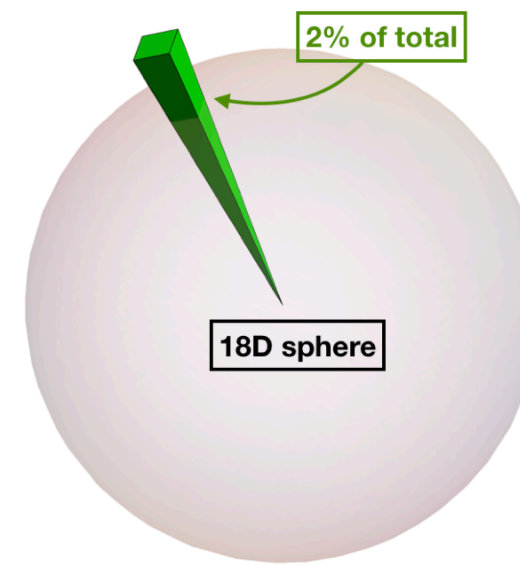
- Positivity bounds are **robust** — from axioms of QFT
- s^2 positivity bounds for **multi-fields** form a convex **cone**.



- **dim-8** operators are important to **infer UV theory**
- Positivity bounds **severely constrain SMEFT**'s parameter space







- Wilson coeff's are bounded from both sides: $c_i \sim O(1)$

Thank you!