Perturbative QCD Theory Review

Yang Zhang 张扬
University of Science and Technology of China

CLHCP 2021
Nov. 25, 2021

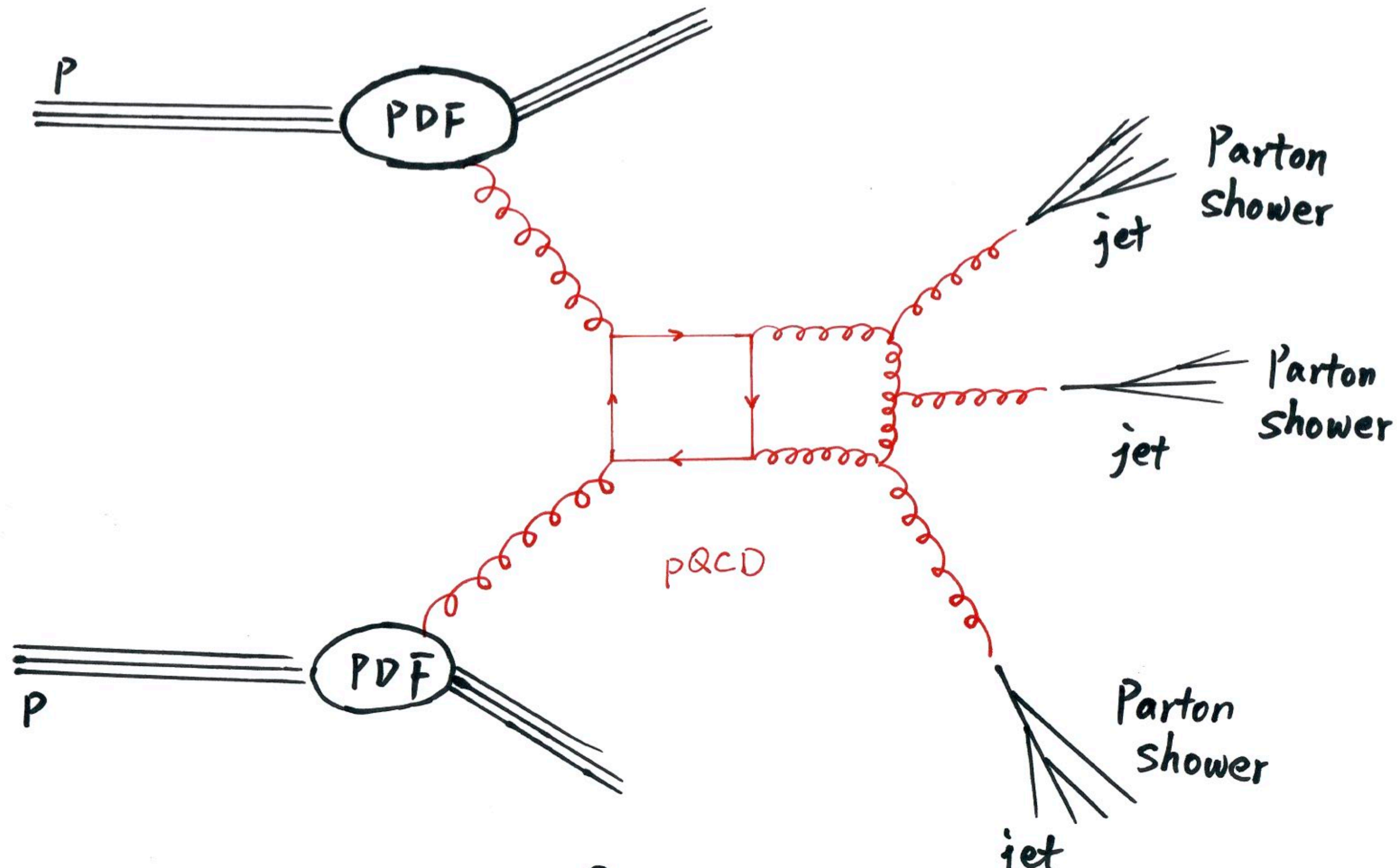
Outline

- pQCD status overview
- new techniques for the computations in pQCD

  Loop integrand reconstruction
  Feynman integral reduction
  Feynman integral evaluation

- NNLO 2 -> 3 scattering
- Outlook

The Era of Precision Physics



$$d\sigma(pp \to X) \sim \sum_{i,j} \int dx_1\, dx_2\, f_i(x_1, Q^2)\, f_j(x_2, Q^2)\; d\hat{\sigma}(ij \to X)$$

pQCD: perturbative expansion

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

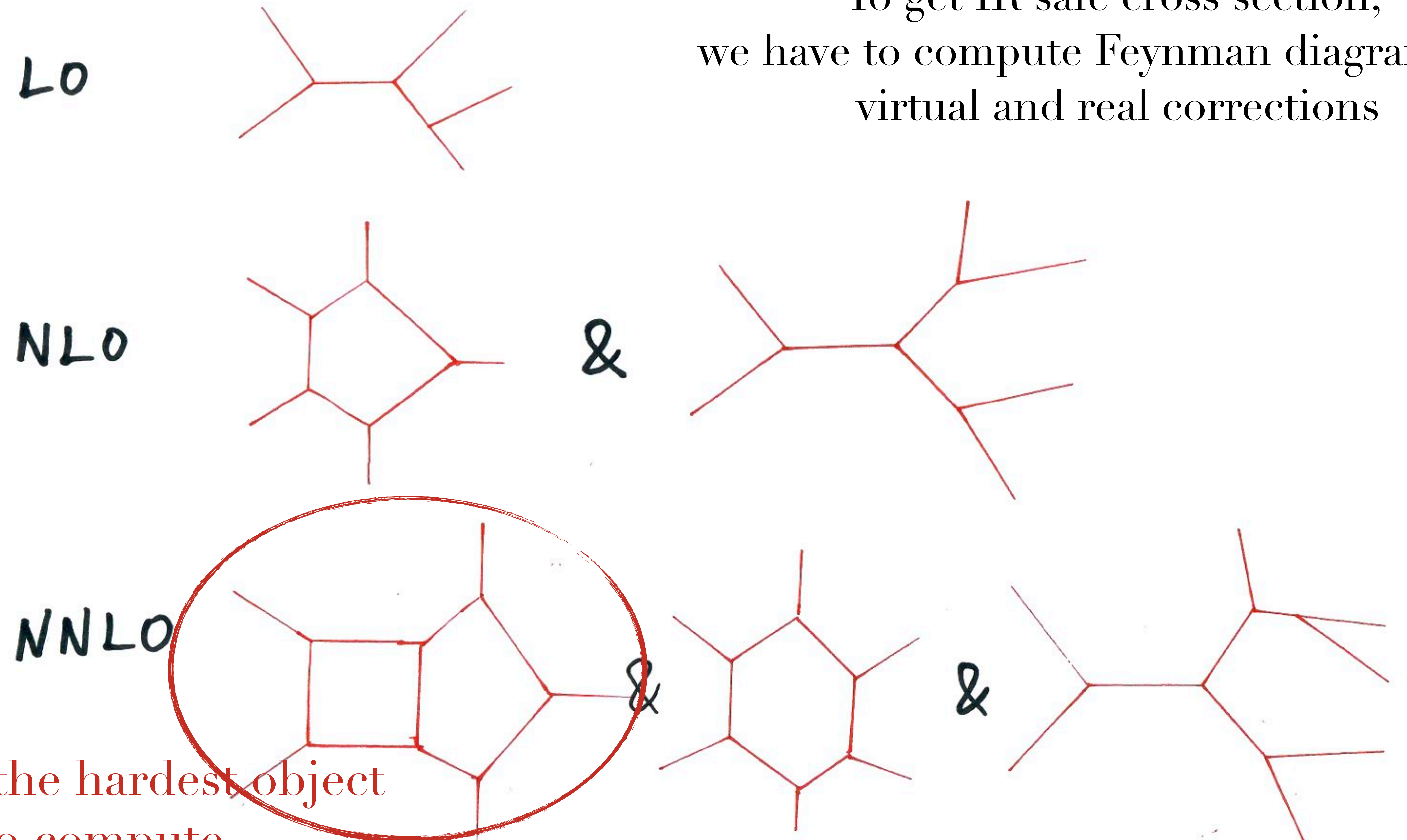
α_s^m α_s^{m+1} α_s^{m+2}

10~30% 1~10%

Frequently, to achieve the precision goal for a theoretical prediction, Next-to-Next-to-Leading-Order cross section is needed.

Precision frontier: 2 → 3 Scattering

To get IR safe cross section,
we have to compute Feynman diagrams for
virtual and real corrections



Usually the hardest object
to compute

pQCD: challenges

Complicated color and tensor structure from Feynman Rules

Complicated kinematics: a multi-scale problem

For two-loop and the higher, mathematical function for loop integrals

pQCD: recent progress

Analytic 2-loop 2 -> 3 massless master integrals

Planar Gehrmann, Henn, Lo Presti 2015

non-Planar Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019

2-loop 2 -> 3 massless scattering amplitudes

Leading color

pp -> 3 jets, Abreu, Cordero, Page, Sotnikov 2021
pp -> 3 photons, Abreu, Page, Pascual, Sotnikov 2020
pp -> diphoton + one jet, Agarwal, Buccioni, von Manteuffel, Tancredi 2021

Full color

pp -> diphoton + one jet, Agarwal, Buccioni, von Manteuffel, Tancredi 2021
Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Marzucca, Moodie, Peraro, Krys, Zoia, 2021

pQCD: recent progress

Analytic 2-loop 2 -> 3 with ONE MASS master integrals

Planar

Papadopoulos, Tommasini, Wever 2019
Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

non-Planar

(partial) Abreu, Ita, Page, Tschernow 2021

2-loop 2 -> 3 with ONE MASS scattering amplitudes

Leading color

pp -> W q qbar, Badger, Hartanto, Zoia 2021
pp -> H q qbar, Badger, Hartanto, Krys, Zoia 2021

pQCD: recent progress

2-loop 2 -> 2 with MASSIVE propagators scattering amplitudes

e p -> mu mu, Bonciani, Broggio, Di Vita, Ferroglia, Mandal Mastrolia,
Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla, Tramontano 2021

Leading color

gg -> t tbar, Badger, Chaubey, Hartanto, Marzucca 2021

pQCD computation: workflow

Scattering process at certain order

↓ Feynman rules/ unitarity

Feynman integrand

↓ color decomposition and tensor reduction

Scalar Integrals

↓ IBP reduction

Master Integrals

↓ Integral evaluation

Bare Amplitude

↓ infrared subtraction and RG

Differential cross section

Then phase space integral
to get total section

Integrand generation

Feynman Rules	**Qgraf** + FORM/Mathematica
Unitarity	Numeric Unitarity **Caravel**: Abreu, Dormans, Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov 2020

For complicated cases, the mandelstam variables + mass should be set to rational numbers or finite field numbers

Color decomposition

decompose the amplitude to a Lie algebra "trace" basis, by color ordered Feynman rules

Planar diagram = Leading color order
non-Planar diagram = sub-Leading color order

Tensor reduction

Rewrite the tensor from Feynman rules as scalar products

If necessary, pull out the polarisation vectors by
projection operators (Boels, Jin, Luo 2018)
adaptive integral (Mastrolia, Peraro, Primo 2016)

IBP reduction

Scalar Feynman integrals -> master integrals

NNLO	10^4~10^5	10~10^2

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0$$

Chetyrkin, Tkachov 1981

IBP reduction: Standard Algorithm

IBP reduction via Linear algebra (Laporta 2000)

Softwares: FIRE, Kira, Reduze2

Sometime this is the most CPU-time-consuming step of a pQCD computation

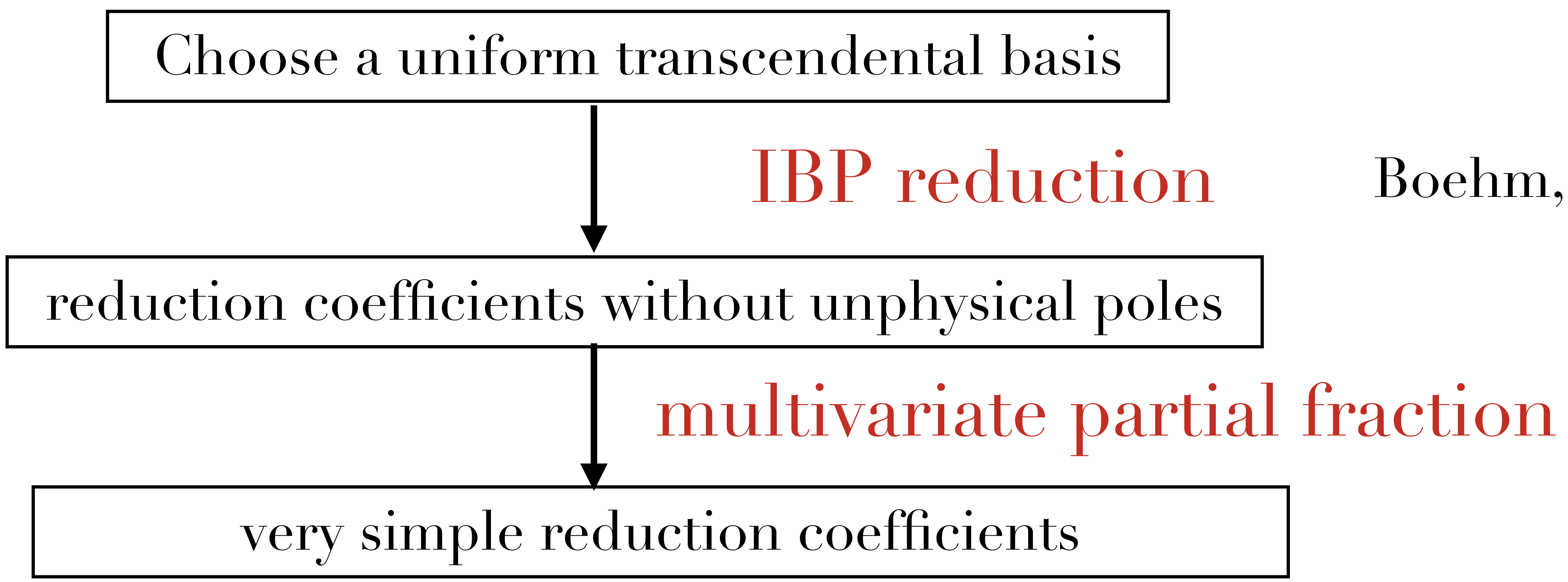
IBP reduction: new Algorithms

Algebraic geometry based methods	Boehm, Georgoudis, Larsen, Schulze, YZ 2017 Gluza, Kajda, Kosower 2010
Auxiliary mass flow method	Auxiliary mass expansion, Liu and Ma, 2019
Intersection theory	Mastrolia, Mizera 2018

IBP reduction:

Optimize the master integral basis/ Simplify the reduction result

Different master integral basis choice dramatically changes the reduction results



Boehm, Wittman, Wu, Xu, YZ, 2020

This method can shrink the size of reduction result by two orders of magnitudes in IBP for NNLO computations

Integral evaluation

Technically, this is the most hardcore part of a loop computation

Analytic

Canonical differential equation, Henn, 2013
HypInt, Panzer, 2015
Dimensional Recursion, Lee, 2014

mainstream analytic method

Numeric

Sector Decomposition
Numeric Differential equation
Auxiliary mass flow method   Liu and Ma, 2019

Integral evaluation: Canonical differential equation

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

$$= \epsilon\left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l\right)\tilde{I}(\bar{x},\epsilon)$$

Proportional to ϵ

Symbol letters

Constant rational number matrix

the equation can be solved perturbatively in ϵ
the solution is an iterative integrals in symbol letters

Henn 2013

Integrals with uniform transcendental (UT) weights satisfy canonical differential equation

Not every integral family has a UT basis. If there is no UT basis, then one has to compute elliptic/hyperelliptic iterative integrals.

Integral evaluation: Canonical differential equation

$$\tilde{I} = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$

Feynman integrals are the iterated integration of rational functions ⇒ polylogarithm functions

$$\tilde{I}(x) = \underbrace{P\exp\left(\epsilon \int_{\mathcal{C}} dA\right)}\tilde{I}(x_0)$$

path-ordered

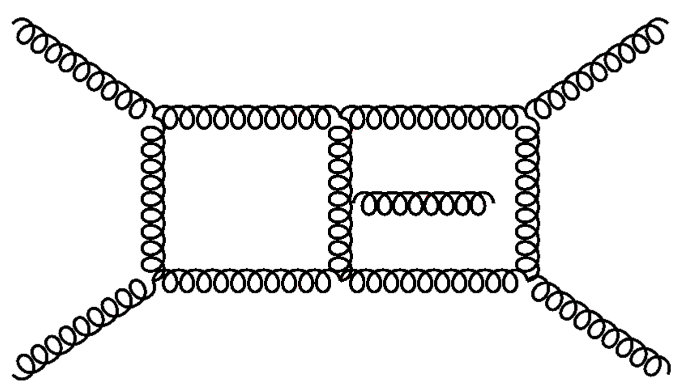
Analogy of Dyson Series

Goncharov Polylogarithm (GPL)

$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\left(\log z\right)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t)$$

implemented in GiNaC

Examples: 2-loop 5-point nonplanar integrals



108 MIs

$$d\tilde{I}(s_{ij};\epsilon) = \epsilon \left(\sum_{k=1}^{31} a_k d \log W_k(s_{ij}) \right) \tilde{I}(s_{ij};\epsilon)$$

31 (108,108) matrices with rational number entries, fitted numerically, 291 kB

symbol letters

$W_1 = v_1\,,\quad W_6 = v_3 + v_4\,,\quad W_{11} = v_1 - v_4\,,\quad W_{16} = v_1 + v_2 - v_4\,,$

$W_2 = v_2\,,\quad W_7 = v_4 + v_5\,,\quad W_{12} = v_2 - v_5\,,\quad W_{17} = v_2 + v_3 - v_5\,,$

$W_3 = v_3\,,\quad W_8 = v_5 + v_1\,,\quad W_{13} = v_3 - v_1\,,\quad W_{18} = v_3 + v_4 - v_1\,,$

$W_4 = v_4\,,\quad W_9 = v_1 + v_2\,,\quad W_{14} = v_4 - v_2\,,\quad W_{19} = v_4 + v_5 - v_2\,,$

$W_5 = v_5\,,\quad W_{10} = v_2 + v_3\,,\quad W_{15} = v_5 - v_3\,,\quad W_{20} = v_5 + v_1 - v_3\,,$

$$v_1 = s_{12},\ v_2 = s_{23},\ v_3 = s_{34},\ v_4 = s_{45},\ v_5 = s_{15}$$

$W_{21} = v_3 + v_4 - v_1 - v_2\,,$

$W_{22} = v_4 + v_5 - v_2 - v_3\,,$

$W_{23} = v_5 + v_1 - v_3 - v_4\,,$

$W_{24} = v_1 + v_2 - v_4 - v_5\,,$

$W_{25} = v_2 + v_3 - v_5 - v_1\,,$

only for nonplanar

$$W_{26} = \frac{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,$$

$$W_{27} = \frac{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,$$

$$W_{28} = \frac{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,$$

$$W_{29} = \frac{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,$$

$$W_{30} = \frac{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,$$

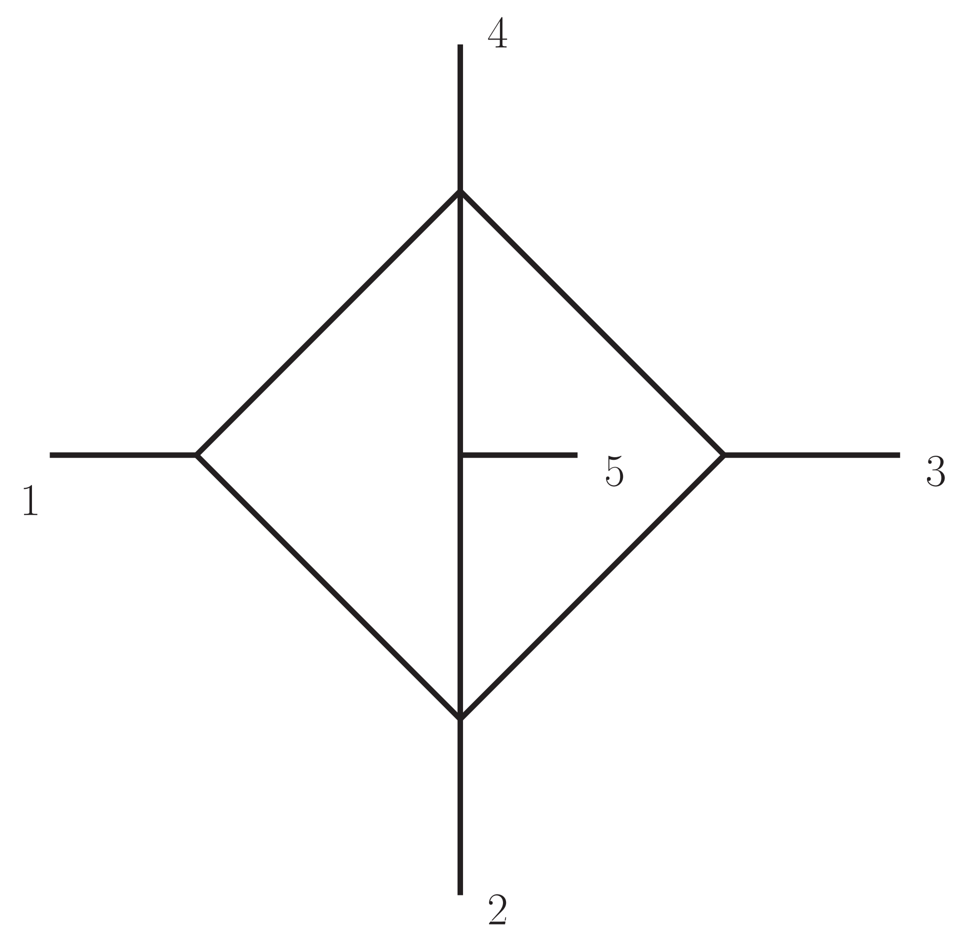
odd letters

$$W_{31} = \sqrt{\Delta}\,.$$

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019

Examples: 2-loop 5-point nonplanar integrals

A tiny piece



$$I = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)\,.$$

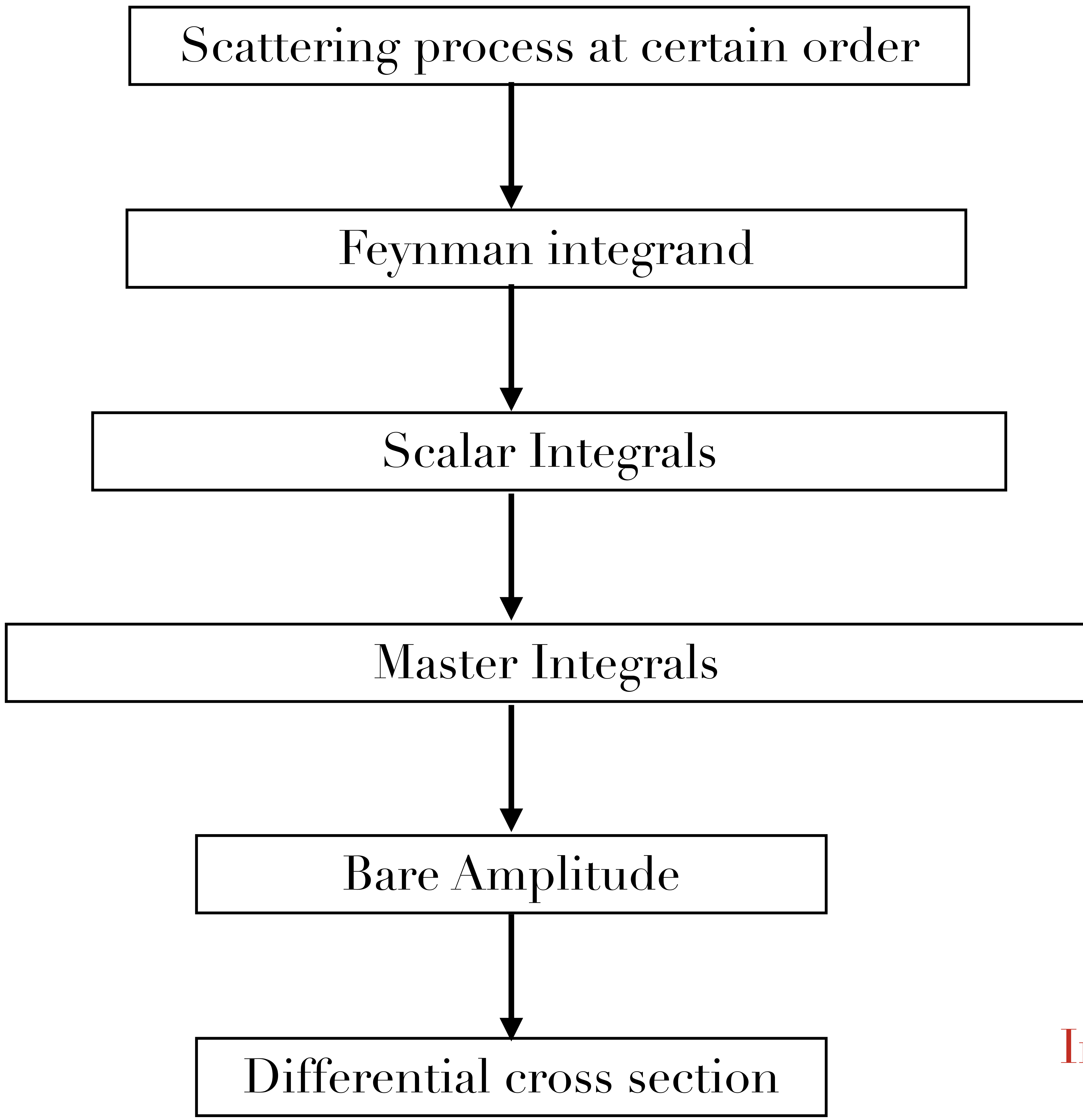
Leading part:

$$\begin{aligned} f^{(2)} = & -3\Big[\mathrm{Li}_2\Big(\frac{1}{W_{27}}\Big) - \mathrm{Li}_2\big(W_{27}\big) + \mathrm{Li}_2\Big(\frac{1}{W_{28}}\Big) - \mathrm{Li}_2\Big(\frac{1}{W_{27}W_{28}}\Big) \\ & - \mathrm{Li}_2\big(W_{28}\big) + \mathrm{Li}_2\big(W_{27}W_{28}\big)\Big]\,. \end{aligned}$$

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019

The analytic result generate high precision numerics
and is much much faster than sector decomposition with GPU

Fast arithmetics: Finite field



Computation
with finite field Z/p
numerics
for Mandelstam/Mass

Repeat many times

Interpolate the finite field result to a
rational number phase point
or the fully analytic result

Summary

- pQCD: recently, a revolution in NNLO multiple scale computations
- necessary techniques: finite field computation/differential equations
- progress in the analytic computation of massive propagator diagrams
- waiting for the phenomenology applications

Thanks!