Analytic Methods for Feynman Integrals

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In memorial of

Cen Zhang

1984-2021

Outline

Analytic Feynman integral: an overview

Recent development in Uniformally transcendental (UT) integrals

To simplify analytic IBP reduction coefficients

Based on

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia

"*All master integrals for three-jet production at NNLO*", PhysRevLett. 123 (2019), no. 4 041603

Boehm, Wittmann, Xu, Wu and **YZ**

"*IBP reduction coefficients made simple* " JHEP 12 (2020) 054

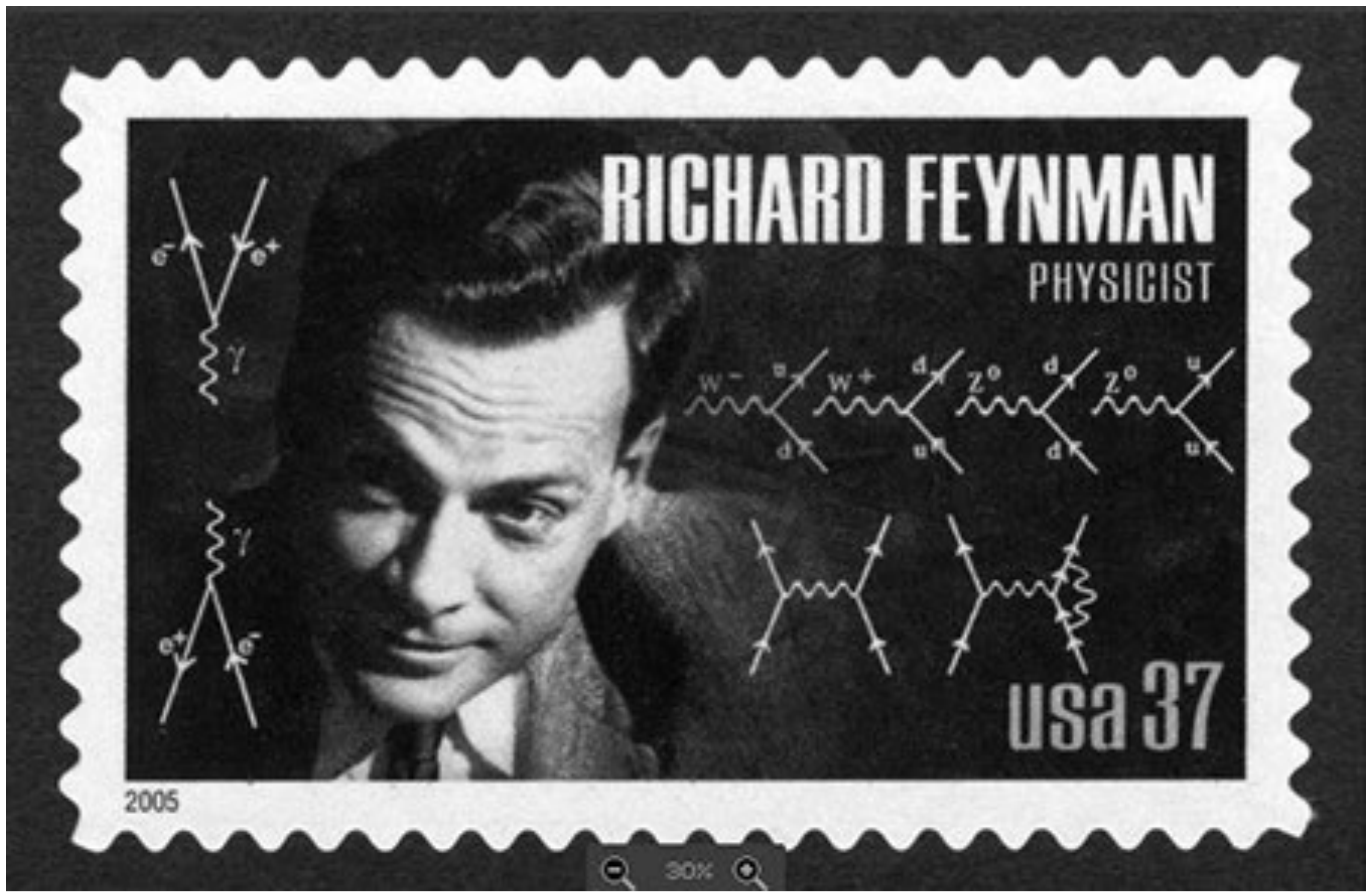
Dlapa, Li, **YZ**,

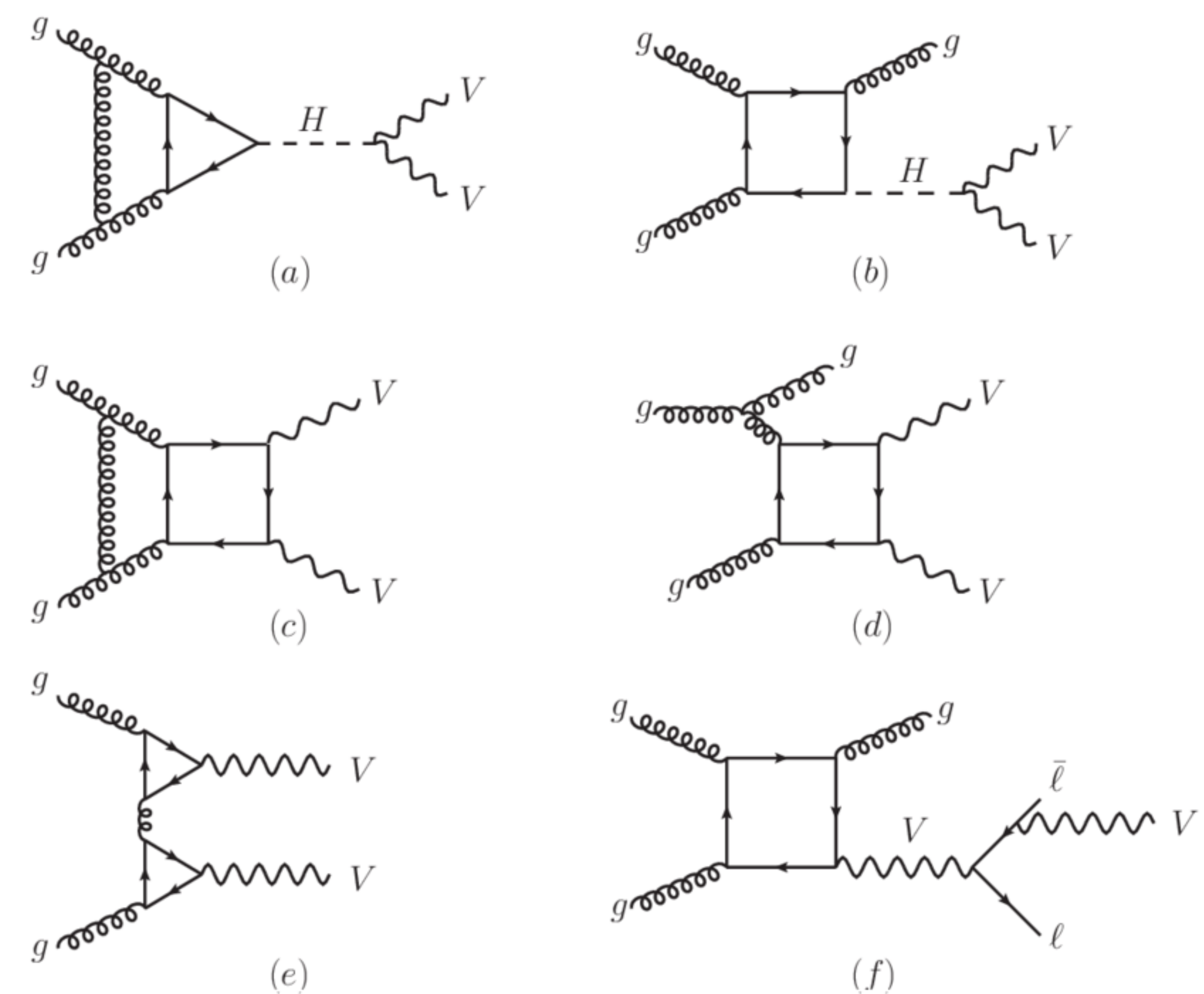
"Leading singularities in Baikov representation and Feynman integrals with uniform transcendental weight" 2103.04638

Bendle, Boehm, Heymann, Ma, Rahn, Wittman, Ristau, Wu, **YZ**

"Two-loop five-point integration-by-parts relations in a usable form" 2104.06866

Feynman integrals

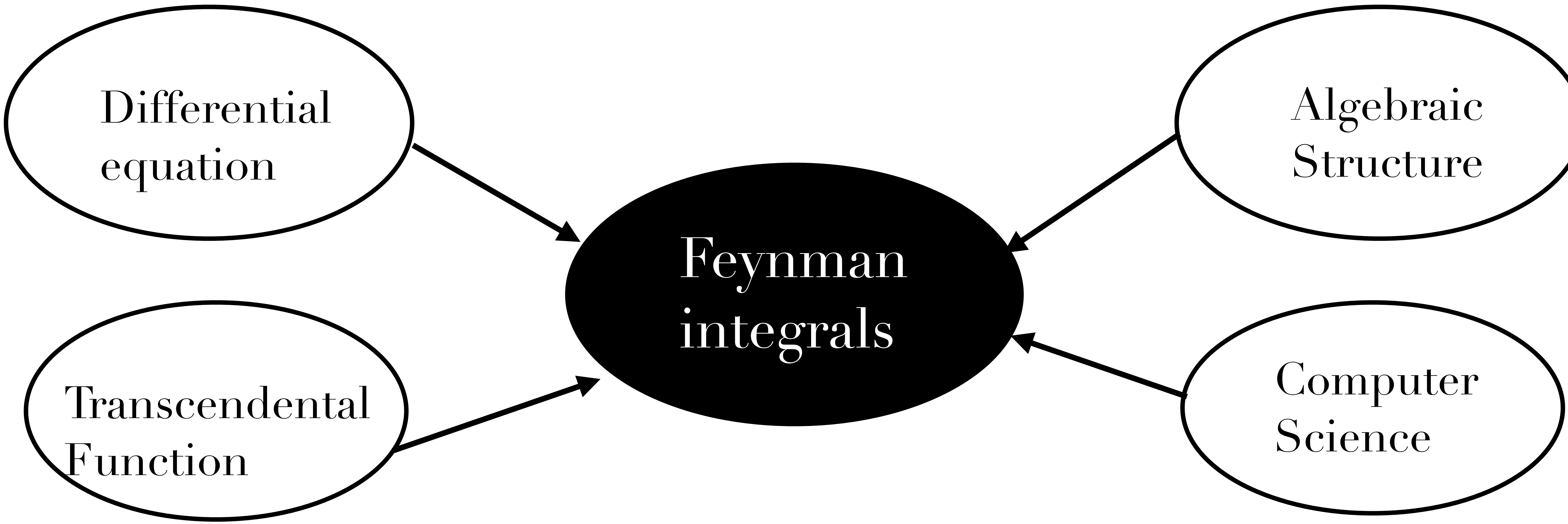




Basis computational tool in quantum field theory

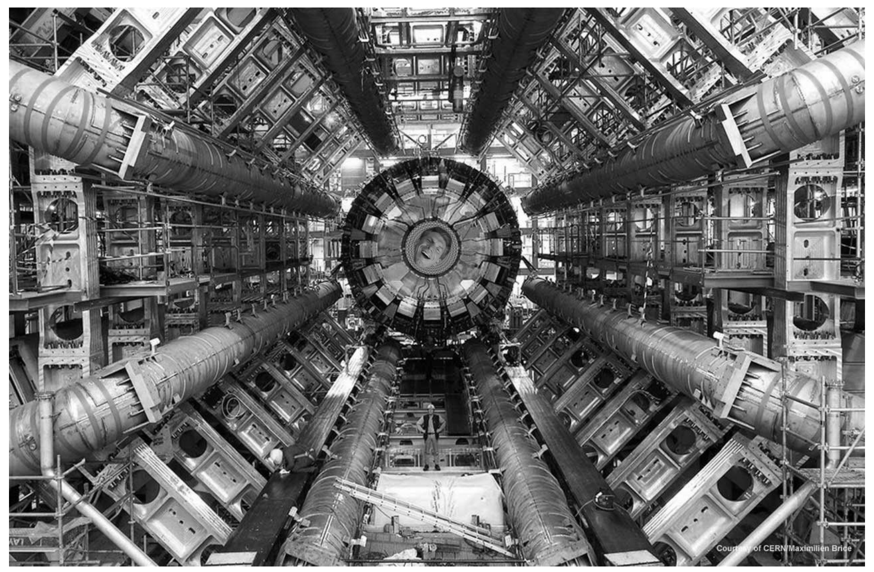
Feynman integrals, new developments

It is still a basic tool in quantum field theory;
Crucially for precision high-energy physics, formal theories,

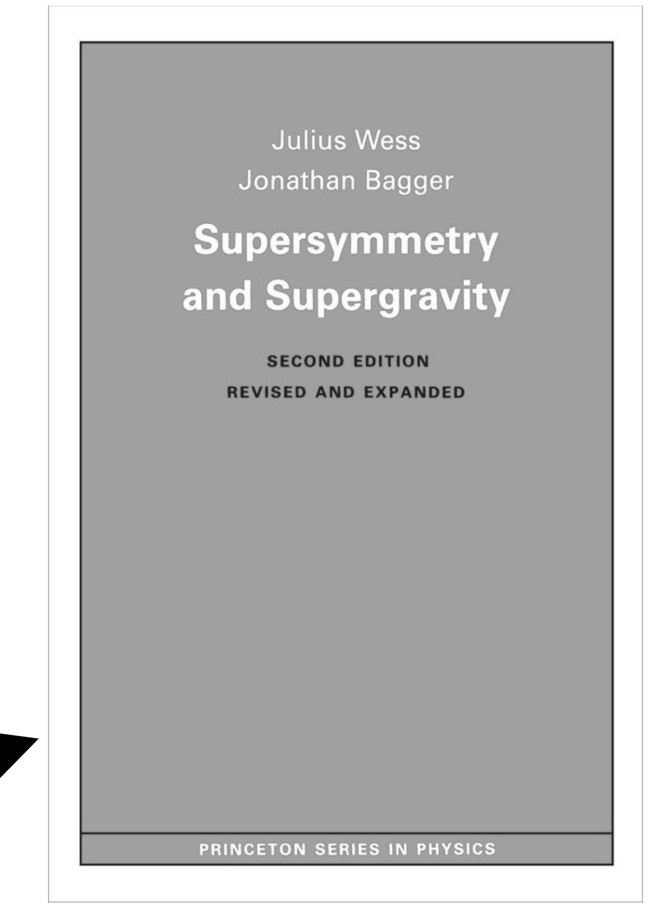


The main focus is on multi-loop Feynman Integrals computations
Significant progress after 2010

Applications of new analytic results on Feynman integrals



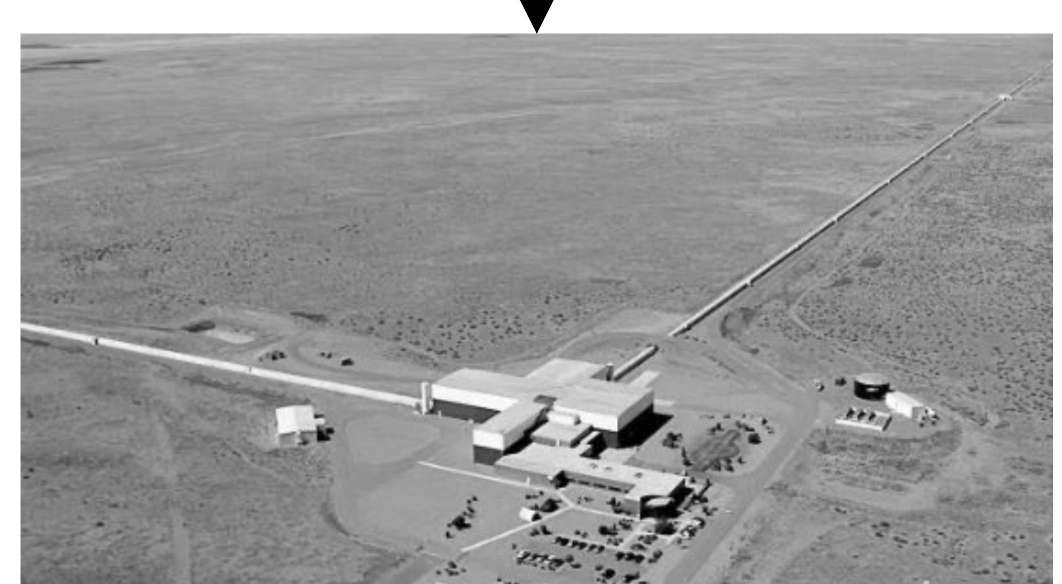
precision
physics



formal
theory

Feynman
integrals

N=8 supergravity
is UV-finite until
five loop

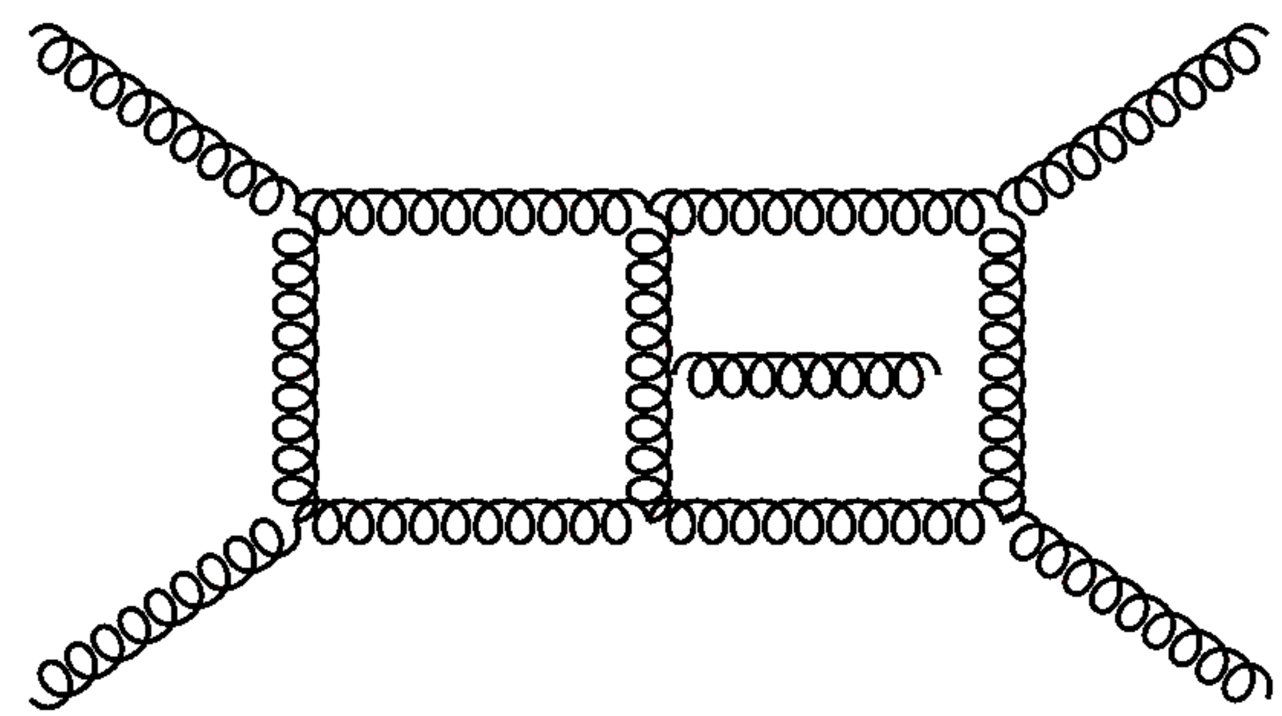


Gravitational wave
template computations

Why analytic Feynman integrals?

- Once the analytic expression is obtained, the phase point generation is extremely fast
- Avoid unstable numeric phase points
- Understand the deep structure and hidden symmetry in quantum field theory

- and yes, we can.



The State of Art

recent analytic results

- 2-loop qqbar → ttbar nonplanar integrals
  (Di Vita, Gehrmann, Laporta, Mastrolia, Pierpaolo; Primo, Schubert, Ulrich, 2019)
- 4-loop form factor planar integrals
  (von Manteuffel, Schabinger, 2019)
- 2-loop Higgs + one jet production (with t quark mass dep.) nonplanar integrals (almost all)
  (Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov, 2019)
- 2-loop five-point massless planar and nonplanar integrals
  (Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019)

….

one example in this talk

and there would be many more in the near future

uncharted territory

Feynman integral with elliptic polylogarithms,
usually in multi-loop nonplanar diagrams with "large" massive internal loop

new transcendental functions (new, even for mathematicians)

see, eg, Broedel, Duhr, Dulat, Tancredi, 2018

Mainstream Analytic Methods

Most used

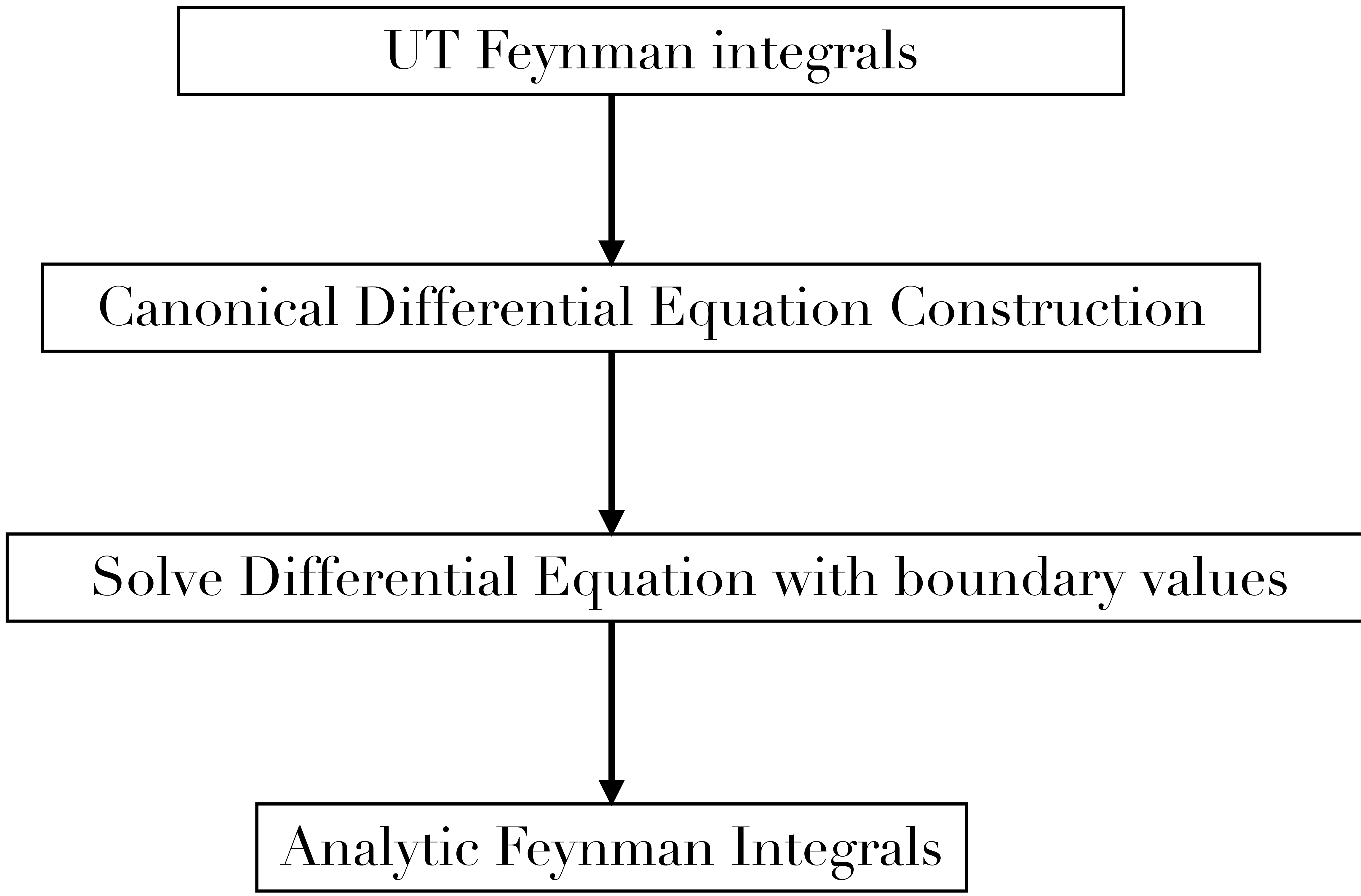
- Canonical differential equation (Henn 2013), in terms of polylagarithm
- Partial fraction + recursive integration (Panzer 2015), package: **HyperInt**, in terms of polylagarithm
  very useful for dual conformally invariant Feynman integrals
- Elliptic Canonical Differetial Equation (Broedel, Duhr, Dulat, Tancredi, 2018), in terms of elliptic polylagarithm

Sometimes magic

- Mellin-Barnes
- Dimension Recursion Relations (Lee, Smirnov 2012),
  usually numeric, but sometimes provide analytic result for complicated integrals
- Integral Bootstrap (Chicherin, Henn, Mitev 2017)
  can easily get the "nice" integrals, eg. conformal, in an integral family

Recent development in UT integrals (and solving canonical differential equation)

Canonical Differential Equation for Analytic Feynman integrals



D-dimensional cuts
Module lift
INITIAL algorithm

Alphabet searching
Differential equation reconstruction

Determine boundary values
from the singularity of Feynman integrals

New Ideas

Baikov representation
Computational Algebraic geometry
Finite fields
Large-scale parallelization

Differential Equation for Feynman Integrals

$d = 4 - 2\epsilon$ $I(\bar{x}, \epsilon)$ master integrals as a column vector

$$\frac{\partial}{\partial x_i} I(\bar{x}, \epsilon) = A_i(\bar{x}, \epsilon) I(\bar{x}, \epsilon)$$ (Kotikov 1991)

Can be used both numerically and analytically
eg. Gehrmann, Remiddi 1999, Papadopoulos 2014, Liu, Ma, Wang, 2018 …

Different choices of the master integrals change the DE dramatically. The simplest choice is the integrals with uniform transcendental (UT) weights, which gives

Canonical Differential Equation

Canonical Differential Equation Henn 2013

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

$$= \epsilon\left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l\right)\tilde{I}(\bar{x},\epsilon)$$

Proportional to $\boldsymbol{\epsilon}$

Symbol letters

Constant rational number matrix

The first line ensures that the equation can be solved perturbatively in $\boldsymbol{\epsilon}$

The second line ensures that the solution is the polylogarithm function in symbol letters

Analogy of the interaction pictures in quantum mechanics

$$\frac{\partial}{\partial x_i}I(\bar{x},\epsilon) = A_i(\bar{x},\epsilon)I(\bar{x},\epsilon)$$

$$\tilde{I}(\bar{x},\epsilon) = T(\bar{x},\epsilon)I(\bar{x},\epsilon)$$

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \boldsymbol{\epsilon}\tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (H_0 + \epsilon H_1)|\psi\rangle$$

$$|\psi\rangle_I = e^{iH_0 t}|\psi\rangle$$

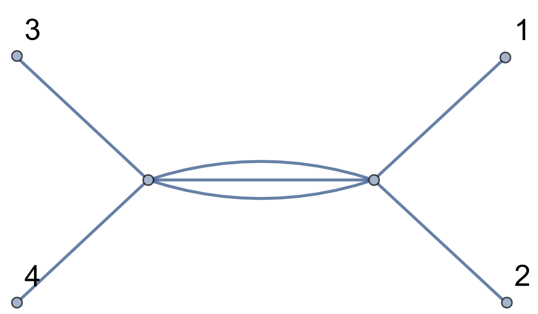
$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = \boldsymbol{\epsilon} H_I(t)|\psi\rangle_I$$

Uniformly transcendental (UT) integrals

$$\mathcal{T}(\log) = 1, \mathcal{T}(\pi) = 1, \mathcal{T}(\zeta_n) = n, \mathcal{T}(\mathrm{Li}_n) = n, \ldots, \mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$$

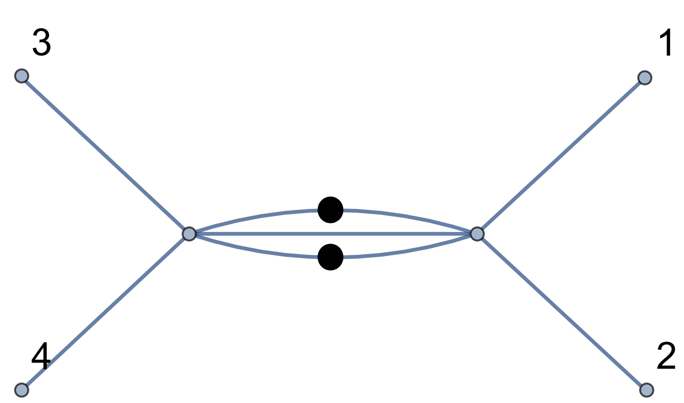
also require the 1st derivative has the transcendental weight n-1

$$I = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$



$$(s_{12})^{1-2\epsilon}\left(\frac{1}{4\epsilon} + \frac{13}{8} + \frac{1}{48}\left(345 - 2\pi^2\right)\epsilon + \frac{1}{96}\left(-256\zeta(3) + 2595 - 26\pi^2\right)\epsilon^2 + O\left(\epsilon^3\right)\right)$$

not UT



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{\epsilon^2} + \frac{\pi^2}{6} + \frac{32\zeta(3)\epsilon}{3} + \frac{19\pi^4\epsilon^2}{120} + O\left(\epsilon^3\right)\right)$$

UT

UT basis is also good for numeric computations

UT Integral ⇒ Canonical DE

$$\tilde{I} = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$

$$\boxed{\frac{\partial}{\partial x_i} \tilde{I}(\bar{x}, \epsilon) = \epsilon \tilde{A}_i(\bar{x}) \tilde{I}(\bar{x}, \epsilon)}$$

Feynman integrals are the iterated integration of rational functions ⇒ polylogarithm functions

$$\tilde{I}(x) = P \exp\left(\epsilon \int_{\mathcal{C}} dA\right) \tilde{I}(x_0)$$

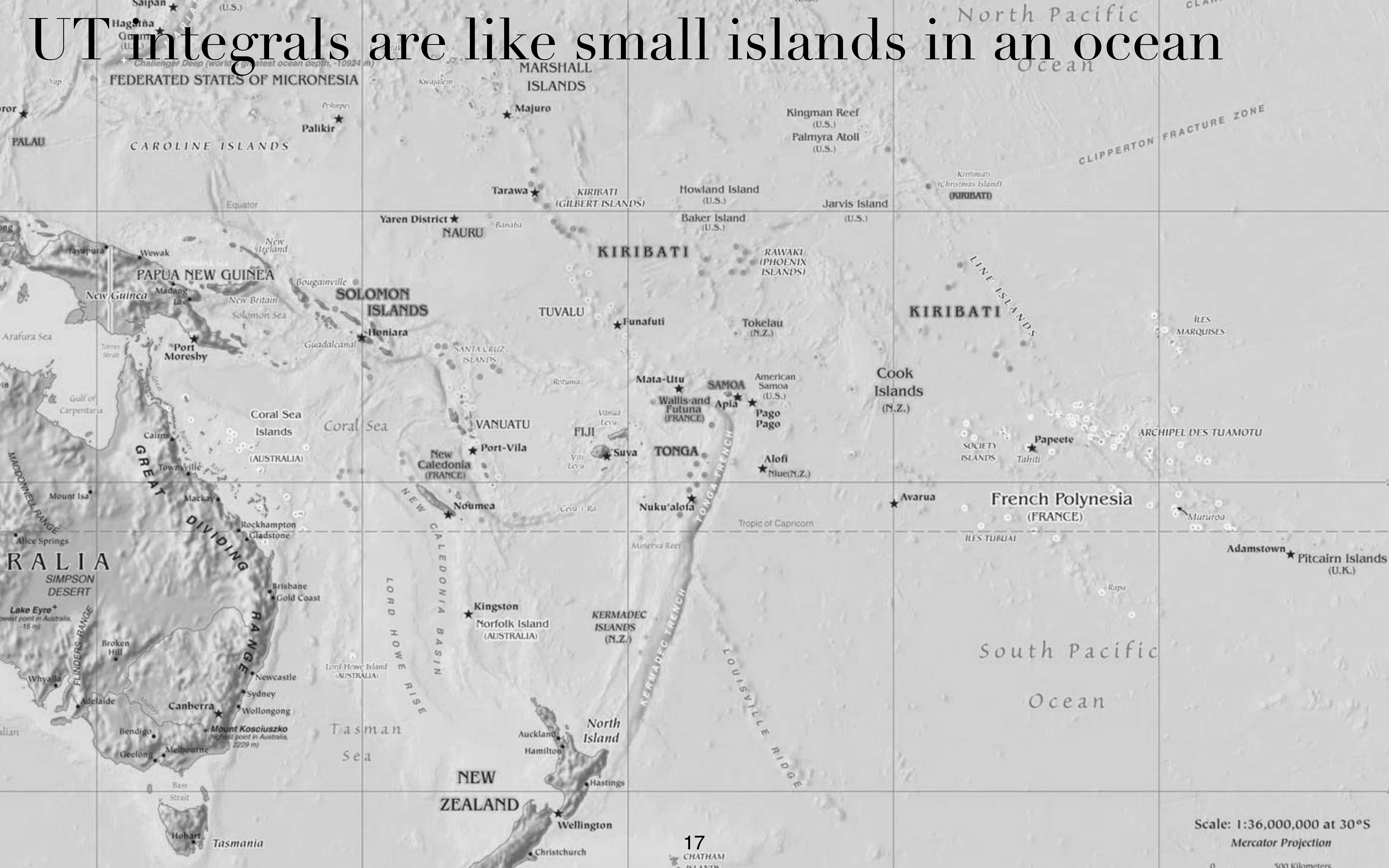
path-ordered

Analogy of perturbation theory of QFT (Dyson series)

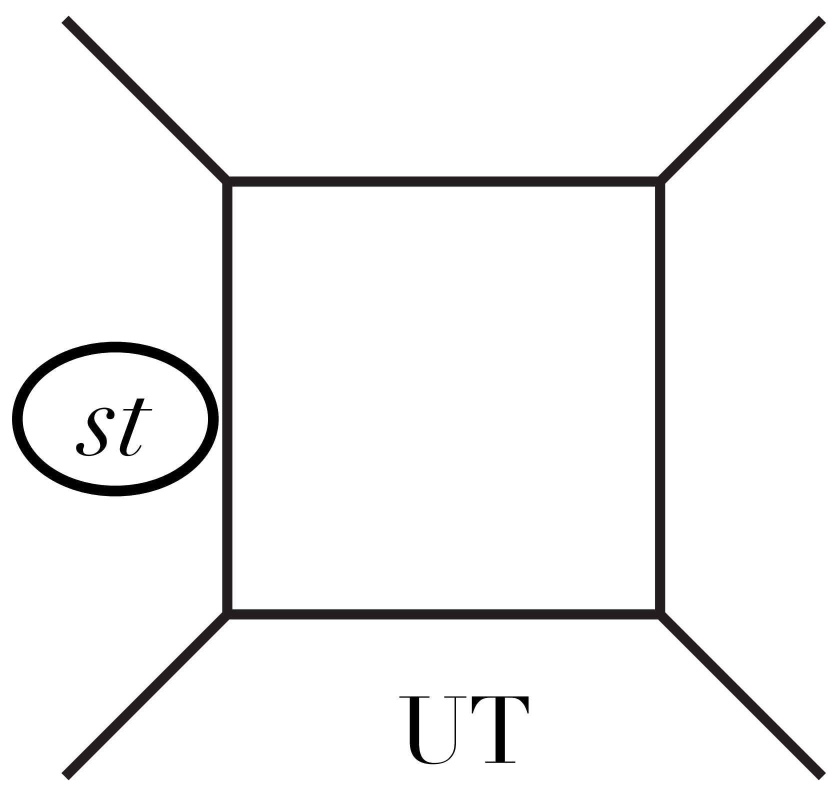
Chen's (陈国才) iterated integrals, homotopically invariant

For finite UT integrals, iterated integration is further truncated and simplified. (Caron-Huot, Henn 2014)

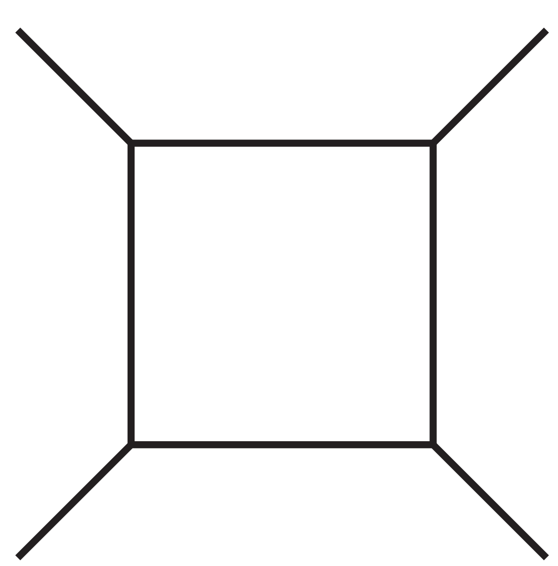
UT integrals are like small islands in an ocean



To find UT integrals



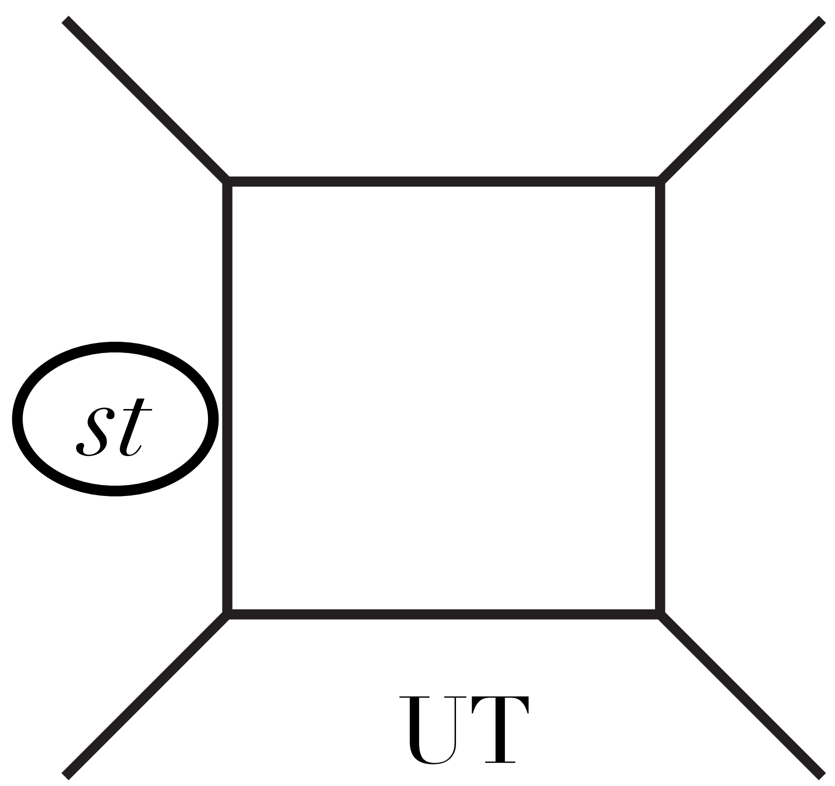
UT



$$= \frac{4}{\underline{s\,t}\,\epsilon^2} - \frac{2\,\mathrm{Log}\left[\frac{t}{s}\right]}{(\underline{s\,t})\,\epsilon} - \frac{4\,\pi^2}{3\,(\underline{s\,t})} + \left(\frac{\mathrm{Log}\left[\frac{t}{s}\right]\left(13\,\pi^2 + 6\,\mathrm{Log}\left[\frac{t}{s}\right]\left(i\,\pi + \mathrm{Log}\left[\frac{t}{s}\right]\right)\right)}{6\,\underline{s\,t}} - \frac{2\,\pi\left(\pi + i\,\mathrm{Log}\left[\frac{t}{s}\right]\right)\mathrm{Log}\left[1 + \frac{t}{s}\right]}{\underline{s\,t}} + \frac{2\,i\,\pi\,\mathrm{Log}\left[1 + \frac{t}{s}\right]^2}{\underline{s\,t}} + \frac{2\,\mathrm{Log}\left[\frac{t}{s}\right]\mathrm{PolyLog}\left[2, -\frac{s}{t}\right]}{\underline{s\,t}} + \frac{2\,\mathrm{PolyLog}\left[3, 1 + \frac{s}{t}\right]}{\underline{s\,t}} + \frac{4\,\mathrm{PolyLog}\left[3, -\frac{t}{s}\right]}{\underline{s\,t}} + \frac{2\,\mathrm{PolyLog}\left[3, 1 + \frac{t}{s}\right]}{\underline{s\,t}} - \frac{40\,\mathrm{Zeta}[3]}{3\,\underline{s\,t}}\right)\epsilon + O[\epsilon]^2$$

But we want to “guess" UT integrals before we get the analytic result

To find UT integrals: 4D integrand analysis



Integrals may be UT if

- 4D residues (leading singularity) are rational constants
- or 4D integrand can be written as a "dlog" product

Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

$$\text{Res}\left(\frac{st}{D_1 \ldots D_4}\right) = \pm 1$$

$$st \int d^4 l \frac{1}{D_1 D_2 D_3 D_4} = \int d\log\left(\frac{F}{D_1}\right) \wedge d\log\left(\frac{F}{D_2}\right) \wedge d\log\left(\frac{F}{D_3}\right) \wedge d\log\left(\frac{F}{D_4}\right)$$

Wasser algorithm for dlog (master thesis 2017)

Consider the partial fraction in x_1,

$$\sum_i \frac{dx}{x_1 - a_i} \wedge \Omega_i = \sum_i d\log(x_1 - a_i) \wedge \Omega_i$$

Integrand with simple residues construction

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

- Require that the integral numerators are "almost" polynomials in Mandelstam variables and irreducible scalar product, except for a pseudo scalar as a divisor

to be solved for $$N = \sum f_\alpha(s_{ij}) \times (\text{scalar product})^\alpha$$

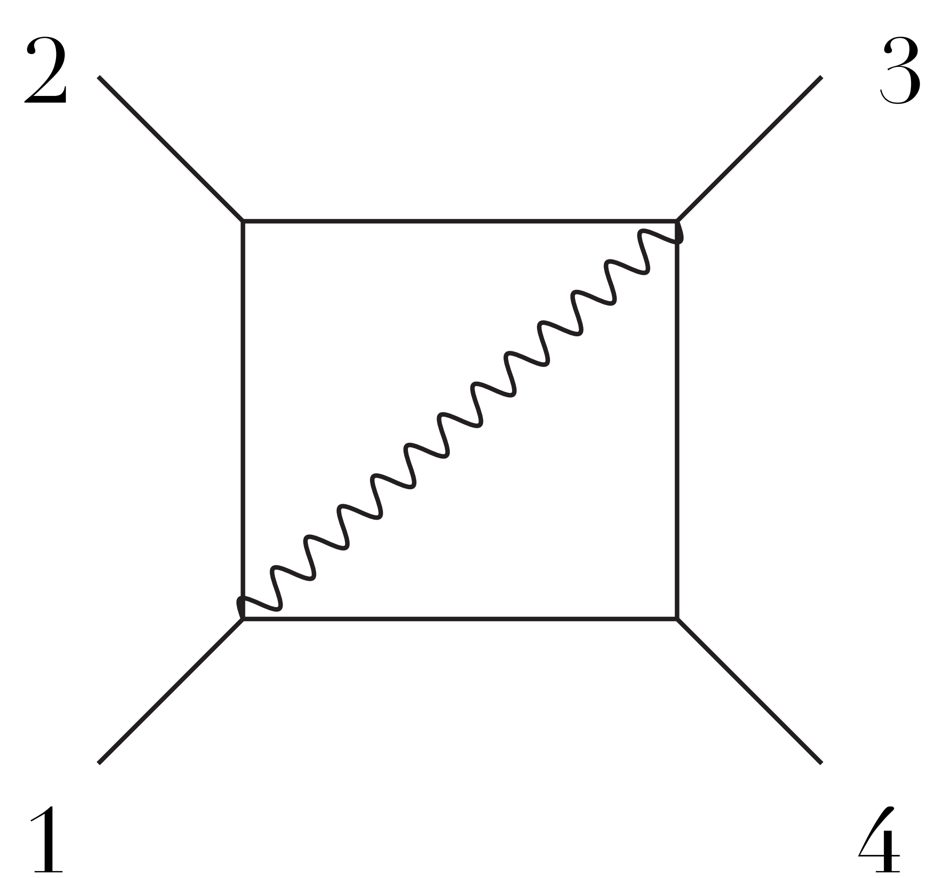
$$\sum_\alpha f_a \times \text{Res}\left(\frac{(\text{Scalar Product})^\alpha}{D_1 \ldots D_n}\right) = (1, 0, \ldots, 0, \ldots 0)$$

Module Lift Method

easily solved by **Singular codes** (computational algebraic geometry software)
systematic analysis and public code on the way!

Other methods to find UT integrals

Chiral Numerator



$$\left(l - \frac{[34]}{[13]} 4\tilde{1}\right)^2$$

Arkani-Hamed, Bourjaily, Cachazo, Trnka (2012)
Bourjaily, Trnka (2015)

chiral numerator vanishes
in the region the integral becomes IR div.

Integrals with chiral numerators are usually UT

Lee's algorithm (2015)

- First reduce the high-multiplicity poles in DE to get a Fuchsian form
- Use matrix transformation to make the DE proportional to ϵ

Mostly for one-variable cases, heavy computation

Fuchsia (Gituliar, Magerya, 2016), epsilon (Prausa 2016)

Advanced D-dimensional leading singularity

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603
Chen, Jiang, Xu, Yang, Phys.Lett.B 814 (2021) 136085
Dlapa, Li, YZ, 2103.04638

- Baikov representation

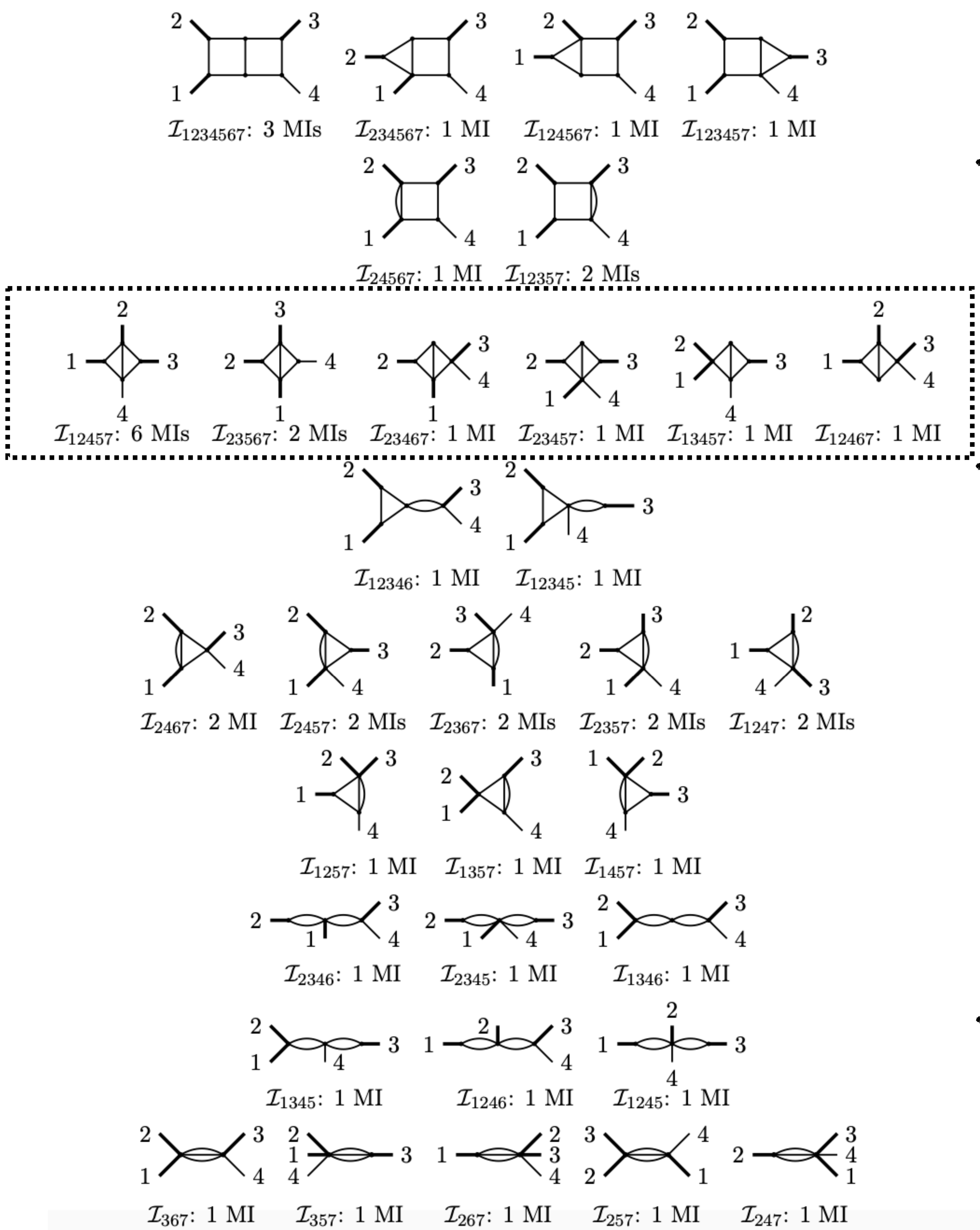
$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = C(s_{ij}, L, E, D) \int dz_1 \dots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \dots z_k^{\alpha_k}}$$

- Consider residue/dlog in Baikov representation

Keeps the d-dimensional information
It usually contains more poles for the residues analysis

Examples: 2-loop 4-point with 3 external mass

Dlapa, Li, YZ, 2103.04638



easy part for the UT searching
with Baikov Leading Singularity

Most difficult part for the UT searching
Baikov Leading Singularity +
Super-sector backtrace

easy part for the UT searching
with Baikov Leading Singularity

Examples: 2-loop 4-point with 3 external mass

Dlapa, Li, YZ, 2103.04638

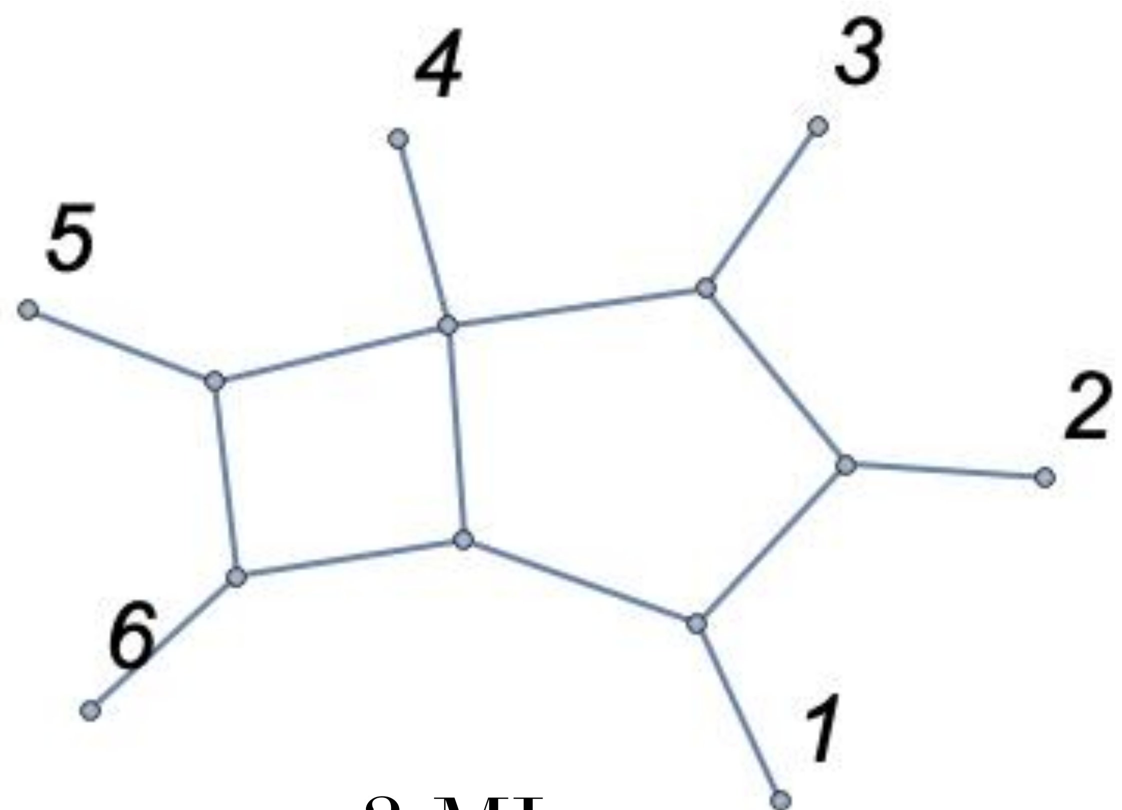
$$
\begin{aligned}
\mathcal{I}_{1234567}: & \ g_1 = -\epsilon^2 s(st - m_1^2 m_3^2) I_{1,1,1,1,1,1,1,0,0}, \\
& \ g_2 = \epsilon^2 r_1 (-m_3^2 I_{0,1,1,1,1,1,1,0,0} + s I_{1,1,1,1,1,1,1,-1,0}), \\
& \ g_3 = \epsilon^2 (m_3^2 - s)(-m_1^2 I_{1,1,1,0,1,1,1,0,0} - m_2^2 I_{1,1,1,1,1,0,1,0,0} + s I_{1,1,1,1,1,1,1,0,-1}), \\
\mathcal{I}_{234567}: & \ g_4 = \epsilon^2 (m_1^2 m_3^2 + m_2^2 s - m_2^2 m_3^2 - st) I_{0,1,1,1,1,1,1,0,0}, \\
\mathcal{I}_{124567}: & \ g_5 = \epsilon^2 s(m_1^2 - t) I_{1,1,0,1,1,1,1,0,0}, \\
\mathcal{I}_{123457}: & \ g_6 = \epsilon^2 r_2 I_{1,1,1,1,1,0,1,0,0}, \\
\mathcal{I}_{24567}: & \ g_7 = \epsilon(2\epsilon - 1)(m_3^2 - s) I_{0,1,0,1,1,1,1,0,0}, \\
\mathcal{I}_{12357}: & \ g_8 = \epsilon(2\epsilon - 1) r_1 I_{1,1,1,0,1,0,1,0,0}, \quad g_9 = \epsilon(m_1^2 m_3^2 - st) I_{1,1,1,0,2,0,1,0,0}, \\
\mathcal{I}_{12457}: & \ g_{10} = -\epsilon m_1^2 (m_3^2 - s) I_{2,1,0,1,1,0,1,0,0}, \quad g_{11} = -\epsilon m_1^2 r_3 I_{1,2,0,1,1,0,1,0,0}, \\
& \ g_{12} = -\epsilon m_3^2 r_1 I_{1,1,0,2,1,0,1,0,0}, \quad g_{13} = -\epsilon m_3^2 (m_1^2 - t) I_{1,1,0,1,2,0,1,0,0}, \\
& \ g_{14} = \epsilon(m_3^2 m_1^2 - st) I_{1,1,0,1,1,0,2,0,0}, \quad g_{15} = \epsilon^2 r_4 I_{1,1,0,1,1,0,1,0,0}, \\
\mathcal{I}_{23567}: & \ g_{16} = \epsilon^2 (m_1^2 + m_3^2 - s - t) I_{0,1,1,0,1,1,1,0,0}, \quad g_{17} = 2\epsilon m_2^2 (t - m_1^2) I_{0,2,1,0,1,1,1,0,0}, \\
\mathcal{I}_{23467}: & \ g_{18} = \epsilon^2 r_1 I_{0,1,1,1,0,1,1,0,0}, \\
\mathcal{I}_{23457}: & \ g_{19} = \epsilon^2 r_3 I_{0,1,1,1,1,0,1,0,0}, \\
\mathcal{I}_{13457}: & \ g_{20} = \epsilon^2 (m_3^2 - s) I_{1,0,1,1,1,0,1,0,0}, \\
\mathcal{I}_{12467}: & \ g_{21} = \epsilon^2 r_1 I_{1,1,0,1,0,1,1,0,0}, \\
\mathcal{I}_{12345}: & \ g_{22} = \epsilon^2 (m_3^2 - s) r_1 I_{1,1,1,1,1,1,0,0,0}, \\
\mathcal{I}_{12346}: & \ g_{23} = \epsilon r_1 m_3^2 (-2\epsilon I_{1,1,1,1,1,1,0,0,0} + s I_{1,1,1,2,1,1,0,0,0}), \\
\mathcal{I}_{2467}: & \ g_{24} = \epsilon r_1 I_{0,2,0,1,0,1,1,0,0}, \quad g_{25} = 3\epsilon(s - m_1^2 - m_2^2) I_{0,2,0,1,0,1,1,0,0} + m_2^2 m_1^2 I_{0,2,0,1,0,1,2,0,0}, \\
\mathcal{I}_{2457}: & \ g_{26} = \epsilon r_3 I_{0,2,0,1,1,0,1,0,0}, \quad g_{27} = 3\epsilon(m_3^2 - m_2^2 - t) I_{0,2,0,1,1,0,1,0,0} + m_2^2 t I_{0,2,0,1,1,0,2,0,0}, \\
\mathcal{I}_{2367}: & \ g_{28} = \epsilon r_1 I_{0,1,1,0,0,2,1,0,0}, \quad g_{29} = 3\epsilon(m_2^2 - m_1^2 - s) I_{0,1,1,0,0,2,1,0,0} + m_1^2 t I_{0,1,1,0,0,2,2,0,0}, \\
\mathcal{I}_{2357}: & \ g_{30} = \epsilon r_3 I_{0,1,1,0,2,0,1,0,0}, \quad g_{31} = 3\epsilon(m_2^2 - m_3^2 - t) I_{0,1,1,0,2,0,1,0,0} + m_3^2 t I_{0,1,1,0,2,0,2,0,0}, \\
\mathcal{I}_{1247}: & \ g_{32} = \epsilon r_1 I_{1,1,0,2,0,0,1,0,0}, \quad g_{33} = 3\epsilon(m_1^2 - m_2^2 - s) I_{1,1,0,2,0,0,1,0,0} + s m_2^2 I_{1,1,0,2,0,0,2,0,0}, \\
\mathcal{I}_{1257}: & \ g_{34} = (2\epsilon - 1)(3\epsilon - 1) I_{1,1,0,0,1,0,1,0,0}, \\
\mathcal{I}_{1357}: & \ g_{35} = (2\epsilon - 1)(3\epsilon - 1) I_{1,0,1,0,1,0,1,0,0}, \\
\mathcal{I}_{1457}: & \ g_{36} = (2\epsilon - 1)(3\epsilon - 1) I_{1,0,0,1,1,0,1,0,0}, \\
\mathcal{I}_{2346}: & \ g_{37} = (1 - 2\epsilon)^2 I_{0,1,1,1,0,1,0,0,0}, \quad \mathcal{I}_{2345}: \ g_{38} = (1 - 2\epsilon)^2 I_{0,1,1,1,1,0,0,0,0}, \\
\mathcal{I}_{1346}: & \ g_{39} = (1 - 2\epsilon)^2 I_{1,0,1,1,0,1,0,0,0}, \quad \mathcal{I}_{1345}: \ g_{40} = (1 - 2\epsilon)^2 I_{1,0,1,1,1,0,0,0,0}, \\
\mathcal{I}_{1246}: & \ g_{41} = (1 - 2\epsilon)^2 I_{1,1,0,1,0,1,0,0,0}, \quad \mathcal{I}_{1245}: g_{42} = (1 - 2\epsilon)^2 I_{1,1,0,1,1,0,0,0,0}, \\
\mathcal{I}_{367}: & \ g_{43} = s I_{0,0,2,0,0,2,1,0,0}, \quad \mathcal{I}_{357}: \ g_{44} = m_3^2 I_{0,0,2,0,2,0,1,0,0}, \quad \mathcal{I}_{267}: \ g_{45} = m_1^2 I_{0,2,0,0,0,2,1,0,0}, \\
\mathcal{I}_{247}: & \ g_{46} = t I_{0,2,0,0,2,0,1,0,0}, \quad \mathcal{I}_{247}: \ g_{47} = m_2^2 I_{0,2,0,2,0,0,1,0,0}.
\end{aligned}
$$

All UT integrals found with four square roots

$$
r_1 = \sqrt{\lambda(m_1^2, m_2^2, s)}, \quad r_2 = \sqrt{(m_2^2 m_3^2 - m_3^2 s + st)^2 - 4 m_1^2 m_2^2 m_3^2 s},
$$
$$
r_3 = \sqrt{\lambda(m_2^2, m_3^2, t)}, \quad r_4 = \sqrt{\lambda(m_1^2, m_3^2, u)},
$$

Examples: 2-loop 6-point UT

Henn, Peraro, Xu, **YZ**, to appear



3 MIs

$$N_1^{\text{pentabox}} = s_{12}\left(s_{16} + \frac{\langle 26\rangle[23][61]}{[13]}\right)s_{56}\boxed{(l_1 - w_1)^2}$$

$$N_2^{\text{pentabox}} = s_{12}\left(s_{16} + \frac{[26]\langle 23\rangle\langle 61\rangle}{\langle 13\rangle}\right)s_{56}\boxed{(l_1 - w_2)^2}$$

$$N_3^{\text{pentabox}} = 2s_{12}s_{56}\boxed{l_1 \cdot (w_1 - w_2)}(l_1 + p_6)^2$$

$$w_1 = -\frac{Q_{456} \cdot \tilde{\lambda}_3 \tilde{\lambda}_1}{[13]}, \quad w_2 = w_1^* .$$

chiral numerators
other factors fixed by
D-dimensional integrand
analysis

Symbol letters to full DE

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon\tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$
$$= \epsilon\left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l\right)\tilde{I}(\bar{x},\epsilon)$$

Symbol letters

Constant rational number matrix

If all the symbol letters W_l are known, then the DE (constant matrix m_l) can be fitted easily by numeric IBPs instead of using analytic IBPs

to find symbol letters:

- Compute DE analytic on the top sector with maximal cut and then permute
- Include the known sub-topology integral symbol letters, lower-loop order symbol letters
- Algorithm to find odd letters in term of even letters (under development)

Chen's iterated integrals

$$\tilde{I}(s_{ij},\epsilon) = \epsilon^{-4}\sum_{m}^{\infty}\epsilon^m \tilde{I}^{(m)}(s_{ij})$$

$$\epsilon^4 \tilde{I}(s_{ij},\epsilon) = B^{(0)} + \epsilon\left(B^{(1)} + \int_\gamma dA(s_{ij})B^{(0)}\right) + \epsilon^2\left(B^{(2)} + \int_\gamma dA(s_{ij})\left(B^{(1)} + \int_{\gamma'} dA(s_{ij})B^{(0)}\right)\right) + \dots$$

boundary value

$$\epsilon^4 \tilde{I}(e_{ij},\epsilon) = \sum_{m=0}^{\infty}\epsilon^m B^{(m)}$$

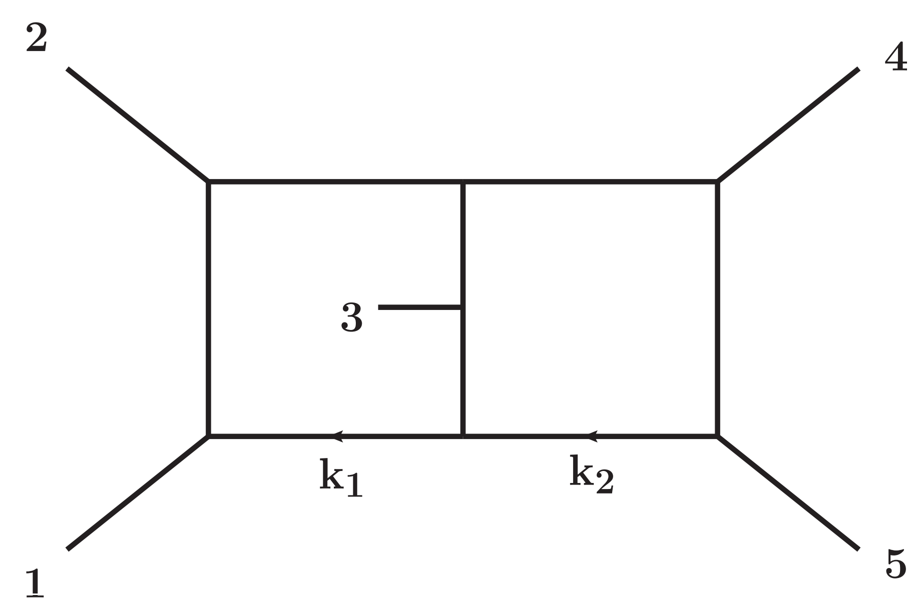
boundary point

To find the analytic boundary value is not easy …

Boundary value

Choice a boundary point

- should be in the kinematic region under consideration
- Use simple and symmetric numbers



$$\{e_{12}, e_{23}, e_{34}, e_{45}, e_{15}\} = \{3, -1, 1, 1, -1\}$$

108 master integrals → 49 independent integrals

These two conditions usually determine the boundary value analytically

- Many subtopology integrals are known analytically
  or can be computed by direct integration/ dimension recursion relation
- Study the behaviour of the general solution on spurious poles

Analytic Feynman integral: Solution

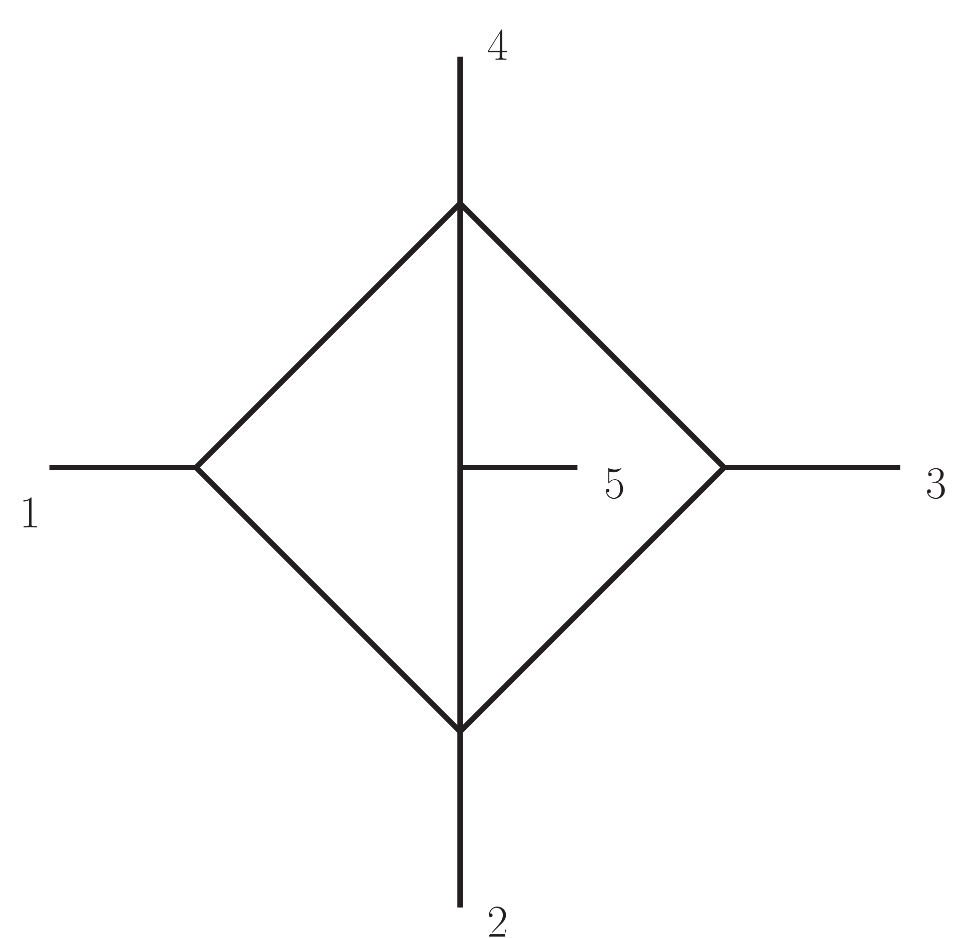
implemented in **Ginac**

$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\left(\log z\right)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t)$$

If all square roots in one iterative are rationalised, then the result is a GPL function

A tiny example:

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603



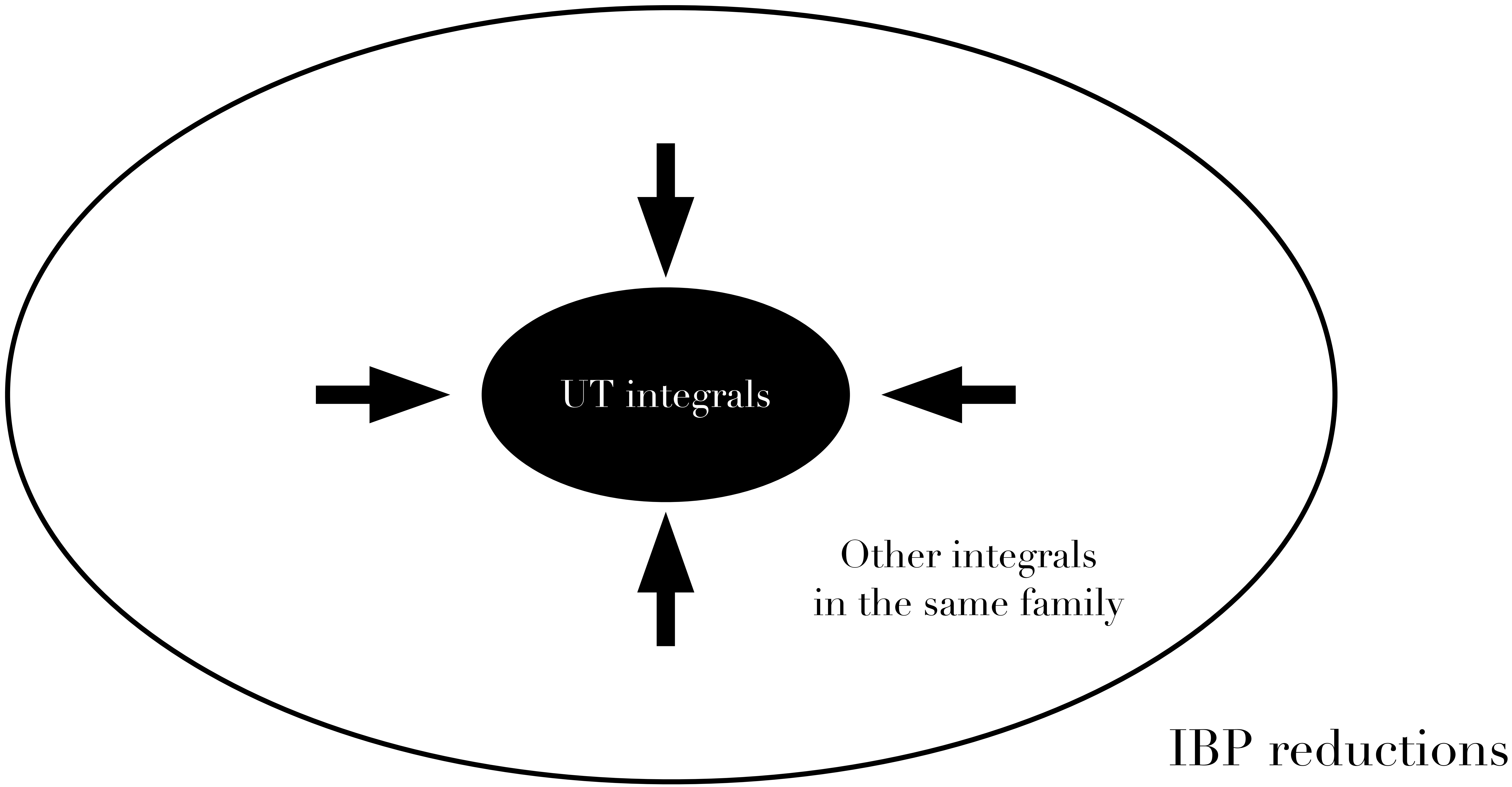
$$I = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)\,.$$

Leading part:

$$f^{(2)} = -3\Big[\mathrm{Li}_2\left(\frac{1}{W_{27}}\right) - \mathrm{Li}_2\left(W_{27}\right) + \mathrm{Li}_2\left(\frac{1}{W_{28}}\right) - \mathrm{Li}_2\left(\frac{1}{W_{27}W_{28}}\right) - \mathrm{Li}_2\left(W_{28}\right) + \mathrm{Li}_2\left(W_{27}W_{28}\right)\Big]\,.$$

To Simplify the IBP coefficients

What about the other Feynman integrals?



IBP reduction coefficients **made simple**

Boehm, Wittmann, Xu, Wu and YZ, JHEP 12 (2020) 054
Bendle, Boehm, Heymann, Ma, Rahn, Wittman, Ristau, Wu, **YZ**, 2104.06866

Use UT basis for the reduction

Use multivariate partial fraction to simplify the expression

We can shorten the coefficients by a factor as large as 10~100.

Why do we prefer a UT integral basis for the reduction ?

A good master integral choice can make the reduction result simpler

IBP reduction coefficients may contain unphysical poles …

If one can avoid these poles, the coefficients are getting physical and simpler

---

Smirnov, Smirnov, Nucl.Phys.B 960 (2020) 115213
Klappert, Lange, Maierhöfer, Usovitsch Comput.Phys.Commun. 266 (2021) 108024

<u>no dimension/kinematics mixed pole</u>

$$(s + dt - 3t)$$

Boehm, Wittmann, Xu, Wu and **YZ,** *JHEP 12 (2020) 054*

<u>no dimension/kinematics mixed pole , no unphysical pole</u>

$$(s + dt - 3t) \qquad (s^2 + t^2 + st)$$

Choose a UT basis

(four-point massless example)

A close look at UT/non-UT integrals

$$\frac{G(1,0,1,1,2,0,1,0,0)\left(4x^2\epsilon+2x^2\right)}{6\left(x^3+x\right)\epsilon^2}+\frac{G(1,0,1,2,1,0,1,0,0)\left(\epsilon-x^2\epsilon\right)}{6\left(x^3+x\right)\epsilon^2}+\frac{\left(x^2-1\right)G(1,0,1,1,1,0,1,0,0)}{x^3+x}$$

UT integral

Symbol letters $x, x-1, x+1$

$$-\frac{1}{6\epsilon^4}-\frac{\mathrm{Log}[x]}{3\epsilon^3}+\frac{-3\pi^2+8\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]-12\,\mathrm{Log}[x]^2+8\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]+8\,\mathrm{PolyLog}[2,-x]+8\,\mathrm{PolyLog}[2,x]}{12\epsilon^2}+$$
$$\frac{1}{18\epsilon}\Big(-4\pi^2\,\mathrm{Log}[2]+8\,\mathrm{Log}[2]^3+4\pi^2\,\mathrm{Log}[1-x]-12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1-x]+12\,\mathrm{Log}[2]\,\mathrm{Log}[1-x]^2-4\,\mathrm{Log}[1-x]^3-13\pi^2\,\mathrm{Log}[x]-$$
$$36\,\mathrm{Log}[1-x]^2\,\mathrm{Log}[x]+12\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]^2-20\,\mathrm{Log}[x]^3+4\pi^2\,\mathrm{Log}[1+x]-12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1+x]-24\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]+$$
$$12\,\mathrm{Log}[x]^2\,\mathrm{Log}[1+x]+12\,\mathrm{Log}[2]\,\mathrm{Log}[1+x]^2-12\,\mathrm{Log}[-x]\,\mathrm{Log}[1+x]^2-12\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]^2-48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,1-x]-$$
$$48\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,-x]-24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,-x]-48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,x]-48\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,x]-$$
$$24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,1+x]-24\,\mathrm{PolyLog}\Big[3,\frac{1-x}{2}\Big]+48\,\mathrm{PolyLog}[3,1-x]+96\,\mathrm{PolyLog}[3,-x]+120\,\mathrm{PolyLog}[3,x]+$$
$$24\,\mathrm{PolyLog}\Big[3,-\frac{2x}{1-x}\Big]-24\,\mathrm{PolyLog}\Big[3,\frac{x}{1+x}\Big]+24\,\mathrm{PolyLog}\Big[3,\frac{2x}{1+x}\Big]-24\,\mathrm{PolyLog}\Big[3,\frac{1+x}{2}\Big]+24\,\mathrm{PolyLog}[3,1+x]-16\,\mathrm{Zeta}[3]\Big)$$

UT integrals only contains constants and transcendental functions
no rational functions

A close look at UT/non-UT integrals

a common master integral chosen by Laporta algorithm $\quad G(1,0,1,2,1,0,1,0,0) \quad$ non-UT integral

$$
\begin{aligned}
&-\frac{1+x^2}{2\,x\,\epsilon^2}+\frac{1+x^2+\mathrm{Log}[x]-x^2\,\mathrm{Log}[x]}{x\,\epsilon}+\frac{\pi^2-12\,\mathrm{Log}[x]-12\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]+6\,\mathrm{Log}[x]^2-12\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]-12\,\mathrm{PolyLog}[2,-x]-12\,\mathrm{PolyLog}[2,x]}{3\,x}-\\
&\frac{\left(1+x^2\right)\left(8+3\,\pi^2-8\,\mathrm{Log}[x]-8\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]+12\,\mathrm{Log}[x]^2-8\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]-8\,\mathrm{PolyLog}[2,-x]-8\,\mathrm{PolyLog}[2,x]\right)}{4\,x}+\\
&\epsilon\left(\frac{4\left(\pi^2\,\mathrm{Log}[x]+\mathrm{Log}[x]^3\right)}{x}-\frac{1}{6\,x}\left(1+x^2\right)\left(-24-9\,\pi^2+4\,\pi^2\,\mathrm{Log}[2]-8\,\mathrm{Log}[2]^3-4\,\pi^2\,\mathrm{Log}[1-x]+12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1-x]-12\,\mathrm{Log}[2]\,\mathrm{Log}[1-x]^2+\right.\right.\\
&\quad 4\,\mathrm{Log}[1-x]^3+24\,\mathrm{Log}[x]+13\,\pi^2\,\mathrm{Log}[x]+24\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]+36\,\mathrm{Log}[1-x]^2\,\mathrm{Log}[x]-36\,\mathrm{Log}[x]^2-12\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]^2+\\
&\quad 20\,\mathrm{Log}[x]^3-4\,\pi^2\,\mathrm{Log}[1+x]+12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1+x]+24\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]+24\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]-12\,\mathrm{Log}[x]^2\,\mathrm{Log}[1+x]-\\
&\quad 12\,\mathrm{Log}[2]\,\mathrm{Log}[1+x]^2+12\,\mathrm{Log}[-x]\,\mathrm{Log}[1+x]^2+12\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]^2+48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,1-x]+24\,\mathrm{PolyLog}[2,-x]+\\
&\quad 48\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,-x]+24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,-x]+24\,\mathrm{PolyLog}[2,x]+48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,x]+48\,\mathrm{Log}[x]\,\mathrm{PolyLog}[2,x]+\\
&\quad 24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,1+x]+24\,\mathrm{PolyLog}\left[3,\frac{1-x}{2}\right]-48\,\mathrm{PolyLog}[3,1-x]-96\,\mathrm{PolyLog}[3,-x]-120\,\mathrm{PolyLog}[3,x]-\\
&\quad \left.24\,\mathrm{PolyLog}\left[3,-\frac{2\,x}{1-x}\right]+24\,\mathrm{PolyLog}\left[3,\frac{x}{1+x}\right]-24\,\mathrm{PolyLog}\left[3,\frac{2\,x}{1+x}\right]+24\,\mathrm{PolyLog}\left[3,\frac{1+x}{2}\right]-24\,\mathrm{PolyLog}[3,1+x]+16\,\mathrm{Zeta}[3]\right)+\\
&\quad \frac{1}{3\,x}\left(-2\,\pi^2+4\,\pi^2\,\mathrm{Log}[2]-8\,\mathrm{Log}[2]^3-4\,\pi^2\,\mathrm{Log}[1-x]+12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1-x]-12\,\mathrm{Log}[2]\,\mathrm{Log}[1-x]^2+4\,\mathrm{Log}[1-x]^3+24\,\mathrm{Log}[x]-\right.\\
&\quad 3\,\pi^2\,\mathrm{Log}[x]+24\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]+36\,\mathrm{Log}[1-x]^2\,\mathrm{Log}[x]-12\,\mathrm{Log}[x]^2-12\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]^2-4\,\pi^2\,\mathrm{Log}[1+x]+12\,\mathrm{Log}[2]^2\,\mathrm{Log}[1+x]+\\
&\quad 24\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]+24\,\mathrm{Log}[1-x]\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]-12\,\mathrm{Log}[x]^2\,\mathrm{Log}[1+x]-12\,\mathrm{Log}[2]\,\mathrm{Log}[1+x]^2+12\,\mathrm{Log}[-x]\,\mathrm{Log}[1+x]^2+\\
&\quad 12\,\mathrm{Log}[x]\,\mathrm{Log}[1+x]^2+48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,1-x]+24\,\mathrm{PolyLog}[2,-x]+24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,-x]+24\,\mathrm{PolyLog}[2,x]+\\
&\quad 48\,\mathrm{Log}[1-x]\,\mathrm{PolyLog}[2,x]+24\,\mathrm{Log}[1+x]\,\mathrm{PolyLog}[2,1+x]+24\,\mathrm{PolyLog}\left[3,\frac{1-x}{2}\right]-48\,\mathrm{PolyLog}[3,1-x]-24\,\mathrm{PolyLog}[3,x]-\\
&\quad \left.\left.24\,\mathrm{PolyLog}\left[3,-\frac{2\,x}{1-x}\right]+24\,\mathrm{PolyLog}\left[3,\frac{x}{1+x}\right]-24\,\mathrm{PolyLog}\left[3,\frac{2\,x}{1+x}\right]+24\,\mathrm{PolyLog}\left[3,\frac{1+x}{2}\right]-24\,\mathrm{PolyLog}[3,1+x]+24\,\mathrm{Zeta}[3]\right)\right)
\end{aligned}
$$

Rational functions lead to unphysical poles in the IBP reduction coefficients

A close look at reduction to UT/non-UT basis

$G(3,1,1,1,1,1,1,0,-1)$ reduced to a UT basis

$$\left\{0,\ 0,\ \frac{2\,ep\,(1+2\,ep)\,(1+4\,ep)\,(1+x^2)}{(-1+2\,ep)\,(-1+x)\,(1+x)},\ -\frac{3\,ep\,(1+ep)\,(1+6\,ep)}{2\,(3+2\,ep)},\ 0,\right.$$

$$\left(ep\left(-ep-ep^2+9\,ep\,x^2+28\,ep^2\,x^2+20\,ep^3\,x^2-8\,ep^4\,x^2+15\,x^4+122\,ep\,x^4+338\,ep^2\,x^4+376\,ep^3\,x^4+136\,ep^4\,x^4+9\,x^6+78\,ep\,x^6+242\,ep^2\,x^6+304\,ep^3\,x^6+128\,ep^4\,x^6+ep^2\,x^8+4\,ep^3\,x^8\right)\right)\Big/\left(8\,(-1+2\,ep)\,(1+2\,ep)\,(3+2\,ep)\,(-1+x)\,x^4\,(1+x)\right),$$

$$\frac{3\,ep\,\left(ep+ep^2-8\,ep\,x^2-26\,ep^2\,x^2-16\,ep^3\,x^2+8\,ep^4\,x^2-6\,x^4-52\,ep\,x^4-122\,ep^2\,x^4-88\,ep^3\,x^4-8\,ep\,x^6-26\,ep^2\,x^6-16\,ep^3\,x^6+8\,ep^4\,x^6+ep\,x^8+ep^2\,x^8\right)}{4\,(-1+2\,ep)\,(1+2\,ep)\,(3+2\,ep)\,x^4},$$

$$-\frac{3\,ep\,\left(ep+ep^2-8\,ep\,x^2-23\,ep^2\,x^2-12\,ep^3\,x^2+8\,ep^4\,x^2-3\,x^4-26\,ep\,x^4-61\,ep^2\,x^4-44\,ep^3\,x^4-3\,ep^2\,x^6-4\,ep^3\,x^6\right)}{4\,(-1+2\,ep)\,(1+2\,ep)\,(3+2\,ep)\,x^4},$$

$$\frac{3\,ep\,(1+2\,ep)\,(1+3\,ep)\,(2+3\,ep)\,(1+x^2)}{2\,(2+ep)\,(-1+2\,ep)\,(-1+x)\,(1+x)},\ \frac{2\,ep^2\,(1+4\,ep)}{-1+2\,ep},\ 0,\ 0,$$

$$\left.\frac{3\,ep\,\left(-2\,ep^3+x^2+18\,ep\,x^2+65\,ep^2\,x^2+72\,ep^3\,x^2+20\,ep^4\,x^2-2\,ep^3\,x^4\right)}{4\,(1+ep)\,(-1+2\,ep)\,(3+2\,ep)\,x^2}\right\}$$

$G(3,1,1,1,1,1,1,0,-1)$ reduced to a Laporta basis

Unphysical pole x = i, -i

$$\left\{0,\ 0,\ \frac{2\,(1+2\,ep)\,(-1+4\,ep)\,(1+4\,ep)\,x\,(1+x^2)}{ep\,(-1+x)^2\,(1+x)^2},\ 0,\right.$$

$$\frac{-ep+2\,ep^2+9\,x^2+69\,ep\,x^2+132\,ep^2\,x^2+60\,ep^3\,x^2+30\,x^4+184\,ep\,x^4+308\,ep^2\,x^4+136\,ep^3\,x^4+9\,x^6+69\,ep\,x^6+132\,ep^2\,x^6+60\,ep^3\,x^6-ep\,x^8+2\,ep^2\,x^8}{8\,(-1+2\,ep)\,(3+2\,ep)\,x^3\,(1+x^2)},$$

$$\frac{ep+ep^2-8\,ep\,x^2-26\,ep^2\,x^2-16\,ep^3\,x^2+8\,ep^4\,x^2-6\,x^4-52\,ep\,x^4-122\,ep^2\,x^4-88\,ep^3\,x^4-8\,ep\,x^6-26\,ep^2\,x^6-16\,ep^3\,x^6+8\,ep^4\,x^6+ep\,x^8+ep^2\,x^8}{4\,ep\,(-1+2\,ep)\,(3+2\,ep)\,x^3\,(1+x^2)},$$

$$\frac{3\,(1+ep)\,(-1+6\,ep)\,(1+6\,ep)}{2\,(3+2\,ep)},\ \frac{3\,ep\,(1+2\,ep)\,(-1+x)^2\,(1+x)^2\,\left(ep+3\,x^2+7\,ep\,x^2+2\,ep^2\,x^2+ep\,x^4\right)}{4\,(-1+2\,ep)\,(3+2\,ep)\,x^3\,(1+x^2)},\ \frac{3\,(1+2\,ep)\,(1+3\,ep)\,(2+3\,ep)\,(1+x^2)}{2\,(2+ep)\,x},$$

$$\left.-\frac{(-1+4\,ep)\,(1+4\,ep)\,\left(1+2\,x^2+8\,ep\,x^2+x^4\right)}{ep\,(-1+2\,ep)\,(-1+x)^2\,(1+x)^2},\ 0,\ 0,\ \frac{3\,(-2+3\,ep)\,(-1+3\,ep)\,\left(-2\,ep^3+x^2+18\,ep\,x^2+65\,ep^2\,x^2+72\,ep^3\,x^2+20\,ep^4\,x^2-2\,ep^3\,x^4\right)}{2\,ep^2\,(1+ep)\,(3+2\,ep)\,x^2}\right\}$$

To reduce the number (and types) of denominators

Hilbert Nullstellensatz

less frequent

Input: $\frac{f}{g} = \frac{f}{q_1^{e_1} \cdots q_m^{e_m}}$

If $q_1 = \cdots = q_m = 0$ (thus $q_1^{e_1} = \cdots = q_m^{e_m} = 0$) has no solution, then

$$\exists h_i, \sum_{i=1}^{m} h_i q_i^{e_i} = 1 \quad \Rightarrow \quad \frac{f}{q_1^{e_1} \cdots q_m^{e_m}} = \frac{f}{q_1^{e_1} \cdots q_m^{e_m}} \sum_{i=1}^{m} h_i q_i^{e_i} = \sum_{i=1}^{m} \frac{f h_i}{\prod_{j=1, j\neq i}^{m} q_j^{e_j}}$$

Algebraic independence

more frequent

$$0 = p(q_1^{e_1}, \cdots, q_m^{e_m}) = c_{\boldsymbol{\alpha}} \prod_{i=1}^{m} (q_i^{e_i})^{\alpha_i} + \sum_j c_{\boldsymbol{\beta}_j} \prod_{i=1}^{m} (q_i^{e_i})^{\beta_{ji}}$$

$\sum_i \alpha_i \leq \sum_i \beta_{ji}$

$$\Rightarrow 1 = -\sum_j \frac{c_{\boldsymbol{\beta}_j} \prod_{i=1}^{m} (q_i^{e_i})^{\beta_{ji}}}{c_{\boldsymbol{\alpha}} \prod_{i=1}^{m} (q_i^{e_i})^{\alpha_i}}$$

$$\Rightarrow \frac{f}{q_1^{e_1} \cdots q_m^{e_m}} = -\frac{f}{q_1^{e_1} \cdots q_m^{e_m}} \frac{c_{\boldsymbol{\beta}_j} \prod_{i=1}^{m} (q_i^{e_i})^{\beta_{ji}}}{c_{\boldsymbol{\alpha}} \prod_{i=1}^{m} (q_i^{e_i})^{\alpha_i}} = \sum_j \frac{f_j}{q_{j1}^{e_{j1}} \cdots q_{jm_j}^{e_{jm_j}}}$$

Further simplification of the partial fraction coefficients

$$\frac{f}{g} = \frac{f}{q_1^{e_1} \cdots q_m^{e_m}}$$

$q_1, \cdots, q_m$ have common zero point & algebraic independent

Polynomial division

$$f = \sum_{i=1}^{m} b_i q_i + r \quad \Rightarrow \quad \frac{f}{q_1^{e_1} \cdots q_m^{e_m}} = \frac{r}{q_1^{e_1} \cdots q_m^{e_m}} + \sum_{i=q}^{m} \frac{b_i}{q_i^{e_i-1} \prod_{j=1, j\neq i}^{m} q_j^{e_j}}$$

unique, low degree

non-unique, degree out of control

Syzygy division

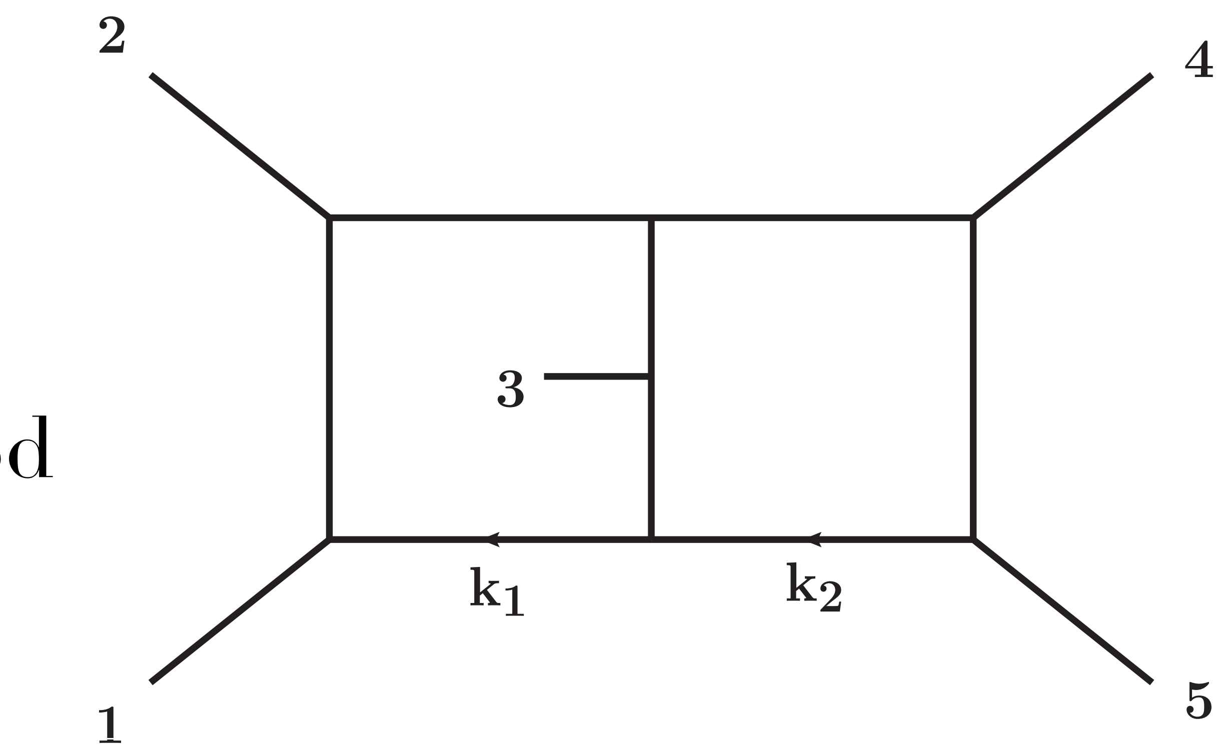
$$\sum_{i=1}^{m} a_i^{\alpha} q_i = 0$$

syzygy generators

$$(b_1, \ldots, b_m) = \sum_{\alpha} P_{\alpha} \times (a_1^{(\alpha)}, \ldots, a_m^{(\alpha)}) + \underline{(B_1, \ldots B_m)}$$

only keep this list

deg-5 numerator reduction



Integral relations provides by auxiliary mass flow method

Guan, Liu, Ma, Chin. Phys. C 44(2020) 9, 093106

If we want the analytic integral reduction, base on Peking U. Group's result

Klappert, and Lange and Maierhofer, and Usovitsch size 25GB !

It is not necessary to find the analytic reduction,
Nowadays, if one wants analytic amplitudes,
one uses numeric IBP coefficient and then interpolate the IR safe "hard part"

Here we uses the complicated coefficients from this example
to show the power of our algorithm

Degree-5

module intersection

Cut	# relations	# integrals	size
{1,5,7}	2723	2749	1.4 MB
{1,5,8}	2753	2777	1.6 MB
{1,6,8}	2817	2822	2.1 MB
{2,4,8}	2918	2921	2.1 MB
{2,5,7}	2796	2805	1.5 MB
{2,6,7}	2769	2814	1.2 MB
{2,6,8}	2801	2821	1.6 MB
{3,4,7}	2742	2771	1.4 MB
{3,4,8}	2824	2849	1.9 MB
{3,6,7}	2662	2674	1.5 MB
{1,3,4,5}	1600	1650	0.72MB

17.2 MB in total

Reduced to a UT basis with an analytic coefficients with integer-valued interpolation
307200 semi-numeric points

RREF about 3 month on Kaiserslautern's cluster
would be improved with pure numeric (finite-field) RREF

Improve Leintars

Use our partial fraction package “pfd” + UT integrals to simplify the analytic reduced IBP

	reduced IBP size	after pfd	Compression Rate
deg-4	700 MB	20 MB	35
deg-5	20 GB	190 MB	105

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, **YZ**
JHEP 02 (2020) 079

Bendle, Boehm, Heymann Ma, Rahn, Wittmann, Wu, **YZ**
2104.06866

partial fraction:
All entries with external parallelisation, 350 cores, about 72 hours; the “hardest” entry took about 72 hou
with internal parallelisation: 16 cores, 19 hours to finish the “hardest” entry (under development)

Summary

- Significant progress of analytic Feynman evaluation
- Canonical differential equation: lastest development
- Simplify IBP results by UT and computational algebraic geometry