Hadronic light-by-light contribution to muon $g - 2$

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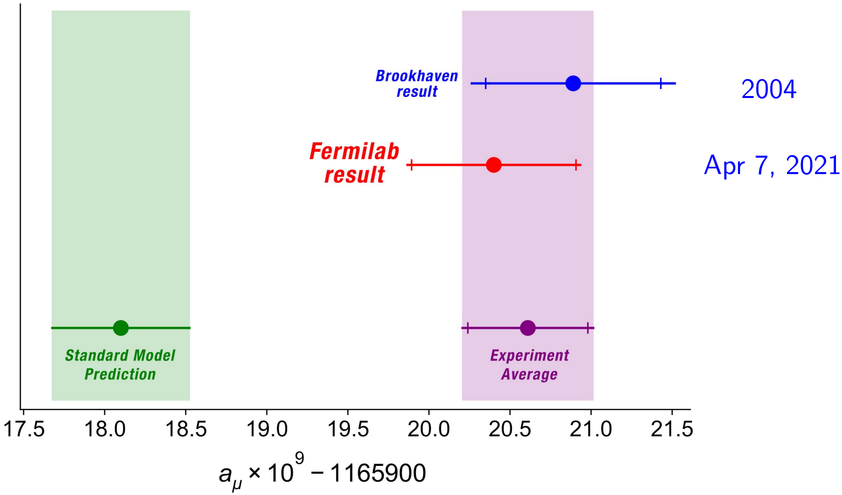
Chiral Dynamics 2021

Institute of High Energy Physics (IHEP), CAS

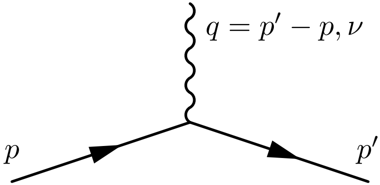
Outline

Muon anomalous magnetic moment ($g - 2$)



- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g - 2$ experiment.

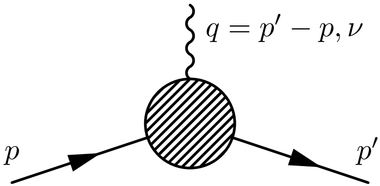
Anomalous magnetic moments



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

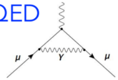
- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

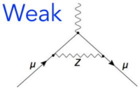
$$\boxed{a_\mu = a_\mu(\mathrm{QED}) + a_\mu(\mathrm{Weak}) + a_\mu(\mathrm{Hadronic})}$$

QED



$+\ldots$ $116\,584\,718.9\,(1) \times 10^{-11}$ 0.001 ppm

Weak

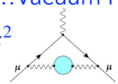


$+\ldots$ $153.6\,(1.0) \times 10^{-11}$ 0.01 ppm

Hadronic...

...Vacuum Polarization (HVP)

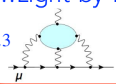
α^2



$+\ldots$ $6845\,(40) \times 10^{-11}$ [0.6%] 0.34 ppm

...Light-by-Light (HLbL)

α^3



$+\ldots$ $92\,(18) \times 10^{-11}$ [20%] 0.15 ppm

- Two methods: dispersive + data $\leftrightarrow$ lattice QCD

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.

- Heroic efforts by

  Johan Bijnens, Gilberto Colangelo, Antoine Gérardin, Nils Hermansson-Truedsson, Martin Hoferichter, Bai-Long Hoid, Bastian Kubis, Stefan Leupold, Pere Masjuan, Kirill Melnikov, Harvey B. Meyer, Andreas Nyffeler, Massimiliano Procura, Antonio Rodríguez-Sánchez, Pablo Sanchez-Puertas, Sebastian P. Schneider, Peter Stoffer, Arkady Vainshtein, etc

  Published in a series of works, which are summarized in the community muon $g-2$ white paper.

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[20] G. Colangelo, M. Hoferichter, M. Procura, and P. Stoffer, JHEP **04**, 161 (2017), arXiv:1702.07347 [hep-ph].
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[23] J. Bijnens, N. Hermansson-Truedsson, and A. Rodríguez-Sánchez, Phys. Lett. **B798**, 134994 (2019), arXiv:1908.03331 [hep-ph].
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[28] M. Knecht, S. Narison, A. Rabemananjara, and D. Rabetiarivony, Phys. Lett. **B787**, 111 (2018), arXiv:1808.03848 [hep-ph].
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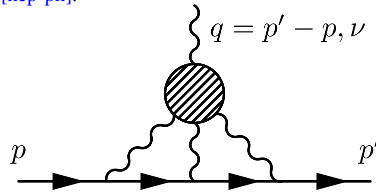
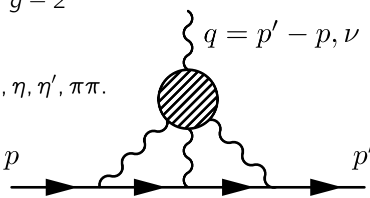


Contribution	PdRV(09) [471]	N/JN(09) [472, 573]	J(17) [27]	Our estimate
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	$\}$ − 1(3)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

Table 15: Comparison of two frequently used compilations for HLbL in units of 10^{-11} from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.

- Values in the table is in unit of 10^{-11}.
- The total HLbL contribution is on the order of 10×10^{-10}.
- "Short-distance constraints in hadronic-light-by-light for the muon $g-2$"
  Talk by: Johan Bijnens
- Make use of experimental inputs on many processes: $\gamma^{(*)}\gamma^* \to \pi^0, \eta, \eta', \pi\pi$.
  New experiments results (Jlab, KLOE, BESIII, etc).
  Talk by: Ilya Larin, Igal Jaegle, Elena P. D. Rio



Contribution	7) [27]	Our estimate
π^0, η, η'-poles	;(12.40)	93.8(4.0)
π, K-loops/boxes	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	)8(1.20)	−8(1)
subtotal	.5(13.4)	69.4(4.1)
scalars	–	} − 1(3)
tensors	1.1(1)	
axial vectors	;5(2.71)	6(6)
u, d, s-loops / short-distan	20(4)	15(10)
c-loop	2.3(2)	3(1)
total	.4(28.2)	92(19)

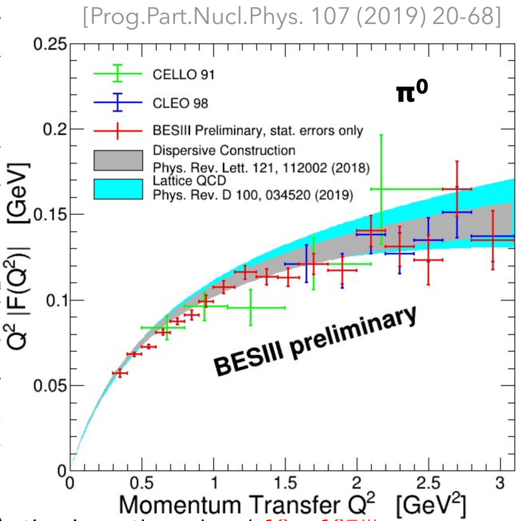


Table 15: Comparison of two frequen ... cent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ... Jegerlehner.

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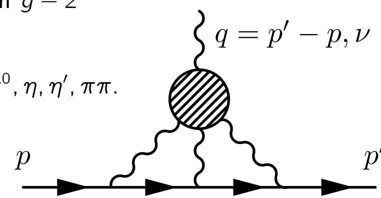

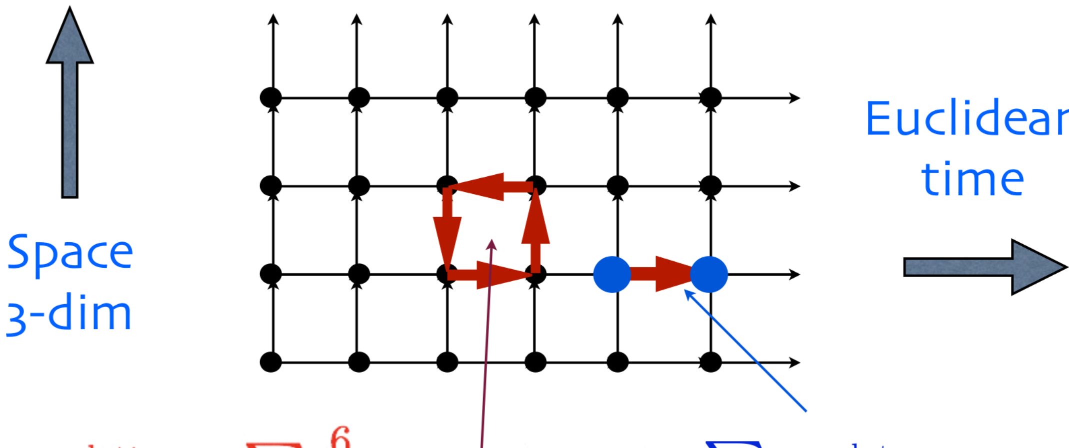

$$S_E^{\mathrm{latt}} = -\sum_{\Box} \frac{6}{g^2}\,\mathrm{Re}\,\mathrm{tr}_N(U_{\Box,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\mathrm{lat}}\gamma_\mu + a m_q) q$$

Wilson gauge action   Lattice fermion action

Figure credit: Stephen R. Sharpe.

PHYSICAL REVIEW LETTERS **124,** 132002 (2020)

Editors' Suggestion   Featured in Physics

Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

Thomas Blum,[1,2] Norman Christ,[3] Masashi Hayakawa,[4,5] Taku Izubuchi,[6,2]
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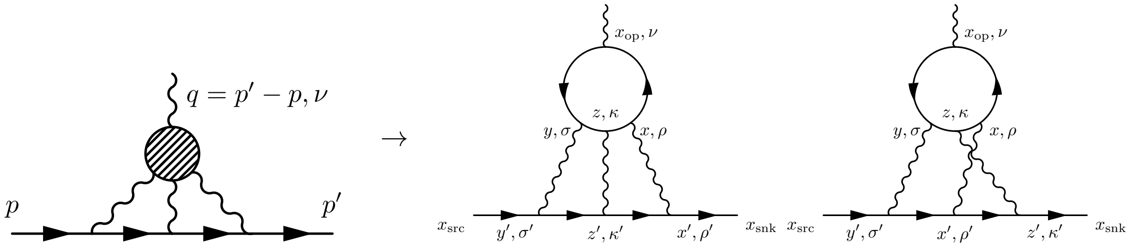
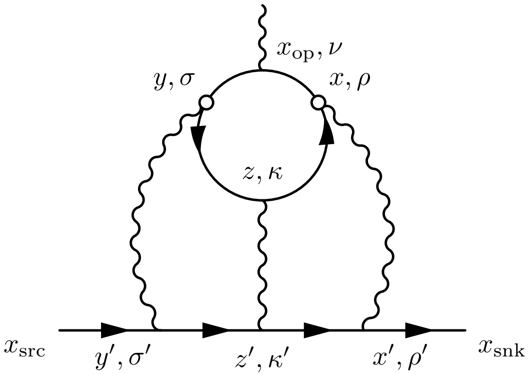
(Received 18 December 2019; accepted 27 February 2020; published 1 April 2020)

- First lattice result for the hadronic light-by-light scattering contribution to the muon $g-2$ with all errors systematically controlled.
- Lattice calculation directly at the physical pion mass and no Chiral extrapolation is needed.

HLbL: diagrams



- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional four different permutations of photons not shown.
- The photons can be connected to different quark loops. These are referred to as the disconnected diagrams. They will be discussed later.
- First results are obtained by T. Blum et al. 2015 (PRL 114, 012001).



- Two point sources at x, y: randomly sample x and y.
- Importance sampling: focus on small $|x-y|$.
- Complete sampling for $|x-y| \leq 5a$ upto discrete symmetry.

$$\frac{a_\mu}{m_\mu} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_{r=x-y} \sum_{z} \sum_{x_{\mathrm{op}}} \frac{1}{2}(\vec{x}_{\mathrm{op}} - \vec{x}_{\mathrm{ref}}) \times \bar{u}_{s'}(\vec{0}) i \vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\mathrm{op}}) u_s(\vec{0})$$

$$\vec{\mu} = \sum_{\vec{x}_{\mathrm{op}}} \frac{1}{2}(\vec{x}_{\mathrm{op}} - \vec{x}_{\mathrm{ref}}) \times \vec{J}(\vec{x}_{\mathrm{op}})$$

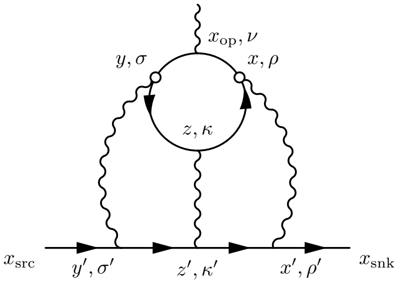
Reorder summation

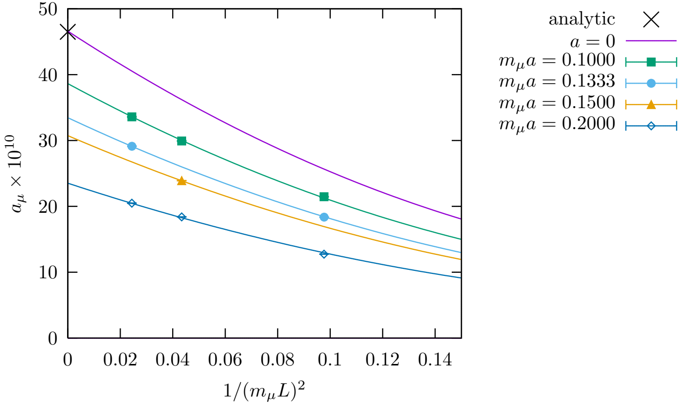
$|x-y| \leq \min(|y-z|, |x-z|)$

- Muon is plane wave, $x_{\mathrm{ref}} = (x+y)/2$.
- Sum over time component for x_{op}.
- Only sum over $r = x - y$.

T. Blum et al 2016. (PRD 93, 1, 014503)

- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.



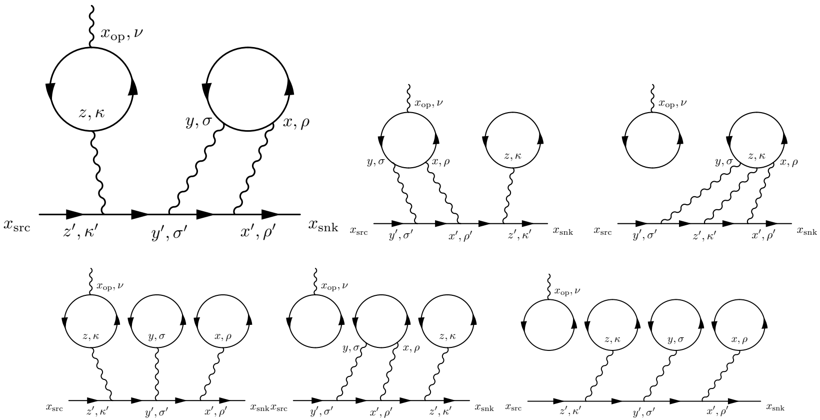


$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2 a^2 + c_2' a^4) \quad \rightarrow \quad F_2 = 46.6(2) \times 10^{-10} \qquad (19)$$

- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

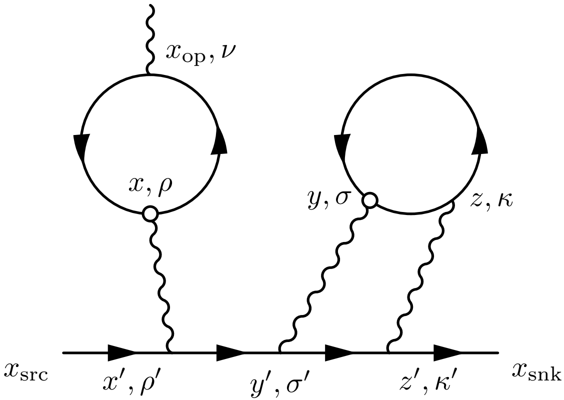
HLbL: disconnected diagrams

- One diagram (the biggest diagram below) do not vanish even in the $\mathrm{SU}(3)$ limit.
- We extend the method and computed this leading disconnected diagram as well.



- Permutations of the three internal photons are not shown.
- **Gluons exchange between and within the quark loops are not drawn.**
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

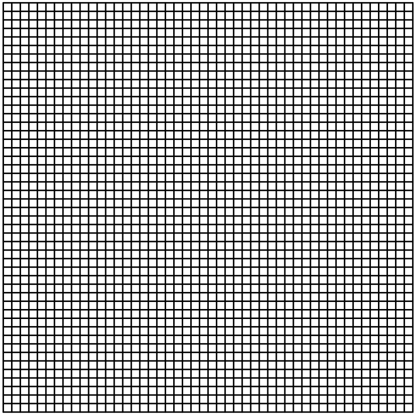
HLbL: disconnected formula



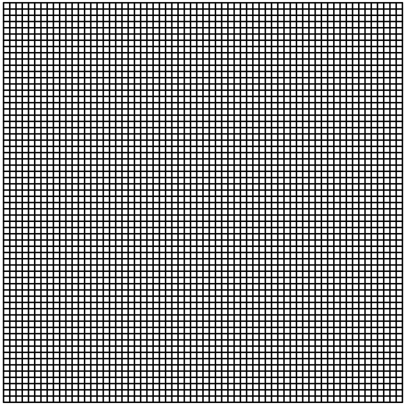
- Point x is used as the reference point for the moment method.
- We can use two point source photons at x and y, which are chosen randomly. The points $x_{\rm op}$ and z are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = x - y$.

T. Blum et al 2017. (PRL 118, 2, 022005)

48I



64I

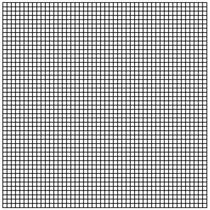


- Domain wall fermion action (preserves Chiral symmetry, no $\mathcal{O}(a)$ lattice artifacts).
- Iwasaki gauge action.
- $M_\pi = 135$ MeV *, $L = 5.5$ fm box, $1/a_{48\mathrm{I}} = 1.73$ GeV, $1/a_{64\mathrm{I}} = 2.359$ GeV.
  PRD 93, 074505 (2016) RBC-UKQCD

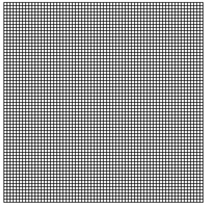
*: Valence pion mass. Slightly different from the 139 MeV unitary pion mass used in the ensemble generation.

T. Blum et al 2020. (PRL 124, 13, 132002)

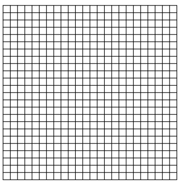
48I



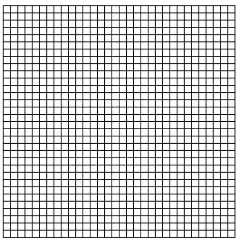
64I



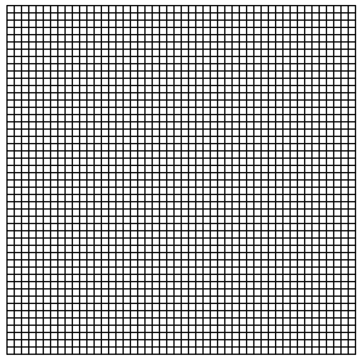
24D



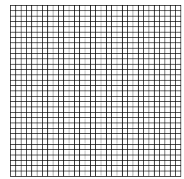
32D



48D



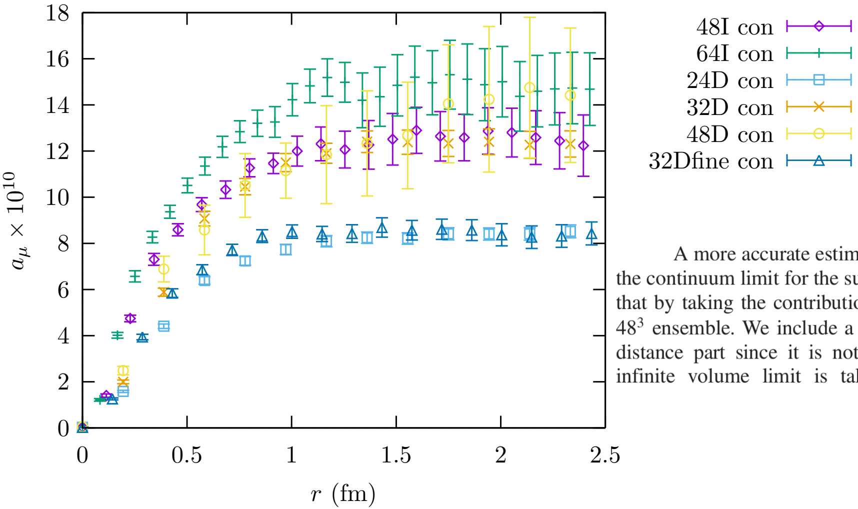
32Dfine



- For 24D, 32D, 48D, 32Dfine, $M_\pi \approx 140$ MeV

T. Blum et al 2020. (PRL 124, 13, 132002)

$$\frac{a_\mu}{m_\mu} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_{r=x-y} \sum_{z} \sum_{x_{\text{op}}} \frac{1}{2} (\vec{x}_{\text{op}} - \vec{x}_{\text{ref}}) \times \bar{u}_{s'}(\vec{0}) i \vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\text{op}}) u_s(\vec{0})$$

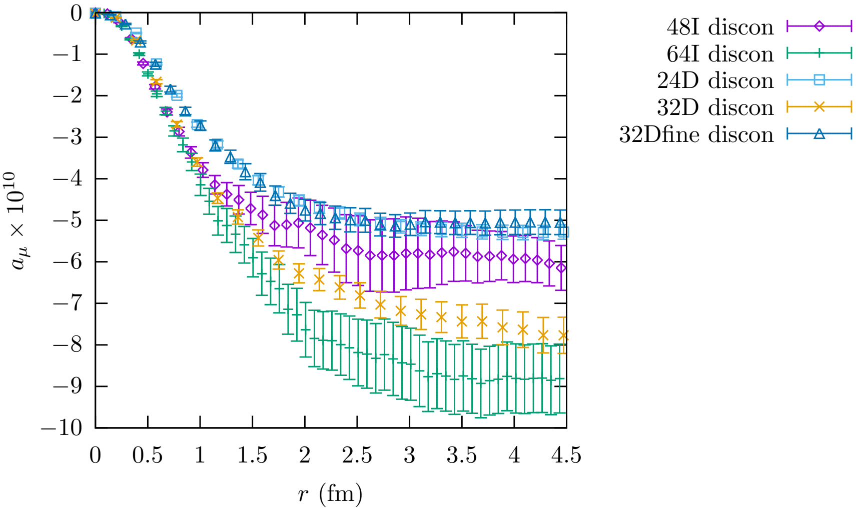


A more accurate estimate can be obtained by taking the continuum limit for the sum up to $r = 1$ fm, and above that by taking the contribution from the relatively precise 48^3 ensemble. We include a systematic error on this long distance part since it is not extrapolated to $a = 0$. The infinite volume limit is taken as before.

Partial sum is plotted above. Full sum is the right most data point.

T. Blum et al 2020. (PRL 124, 13, 132002)

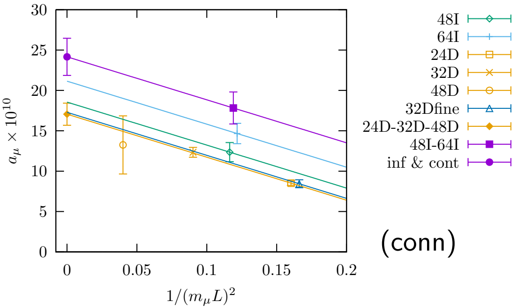
$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\text{op}}}\frac{1}{2}(\vec{x}_{\text{op}} - \vec{x}_{\text{ref}}) \times \bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\text{op}})u_s(\vec{0})$$



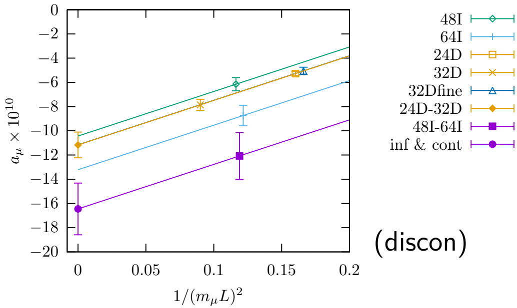
Partial sum is plotted above. Full sum is the right most data point.

T. Blum et al 2019. (PRL 124, 13, 132002)

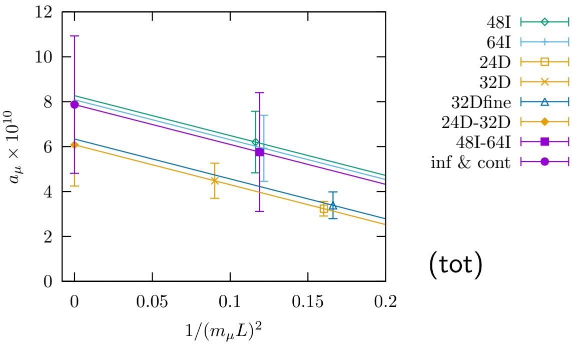
$$a_\mu(L, a^\mathrm{I}, a^\mathrm{D}) \quad = \quad a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^\mathrm{I}(a^\mathrm{I}\ \mathrm{GeV})^2 - c_1^\mathrm{D}(a^\mathrm{D}\ \mathrm{GeV})^2 + c_2^\mathrm{D}(a^\mathrm{D}\ \mathrm{GeV})^4\right)$$



(conn)



(discon)



(tot)

T. Blum et al 2020. (PRL 124, 13, 132002)

	con	discon	tot
a_μ	24.16(2.30)	-16.45(2.13)	7.87(3.06)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.88(0.31)	0.71(0.28)	0.95(0.92)
sys $\mathcal{O}(a^2\log(a^2))$	0.23(0.08)	0.25(0.09)	0.02(0.11)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.49(1.37)	1.08(1.57)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.11(1.32)	3.99(1.29)	1.77(1.13)

- Same method is used for esimating the systematic error of individual and total contribution.
- Systematic error has some cancellation between the connected and disconnected diagrams.

T. Blum et al 2020. (PRL 124, 13, 132002)

Hadronic light-by-light contribution to $(g-2)_\mu$ from lattice QCD: a complete calculation

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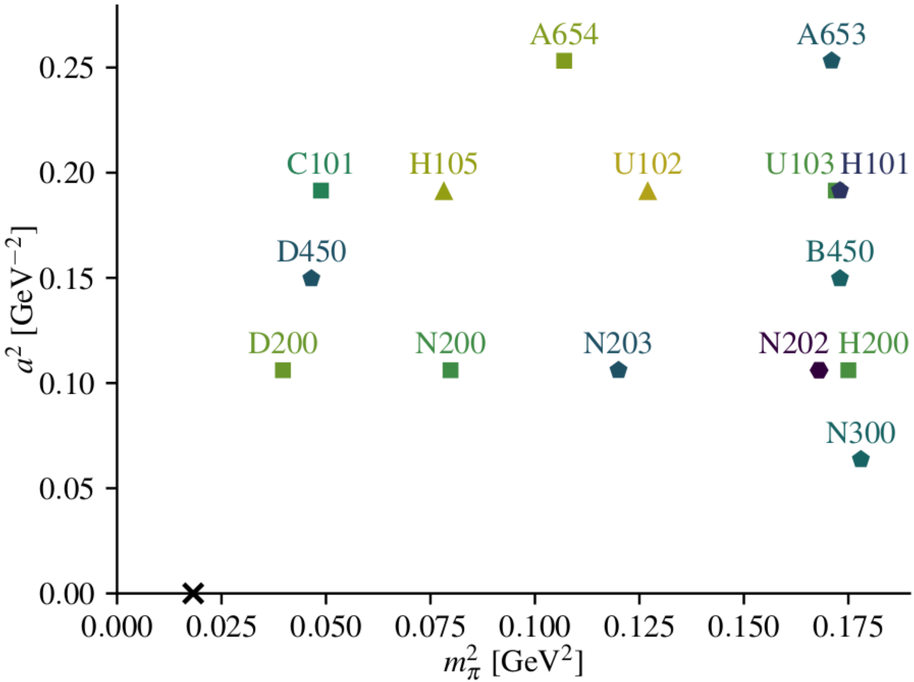
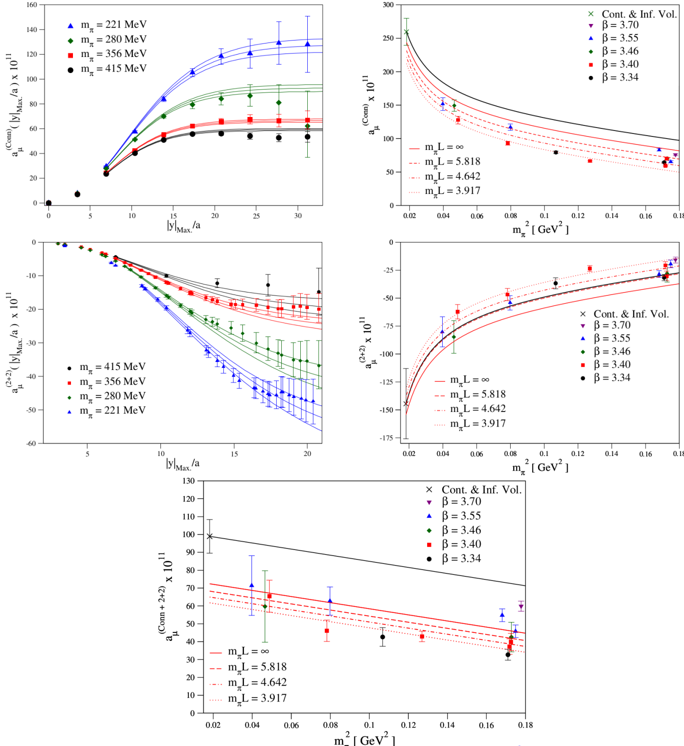
- Mainz pioneered in using the infinite volume QED method in HLbL. The QED weighting function can be saved to disk and reuse.
- Use 4D rotational symmetry of the Euclidean space-time when combining the hadronic 4-point function with the QED weighting function.
- The other aspect of the method is similar to the one used in the RBC/UKQCD calculation. It is developed to a very large degree independently.
- Use the subtraction method for the QED weighting funcion invented by RBC-UKQCD based on the QED_∞: T. Blum et al 2017. PRD 96, 3, 034515



E.H. Chao et al 2021. (EPJC 81 7, 651)

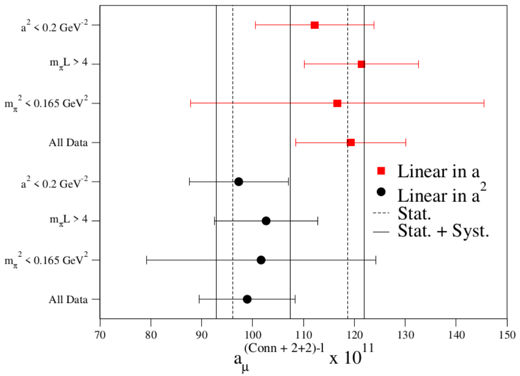
- Pion masses are heavier than physical value and Chiral extrapolation is used.
- For the connected and disconnected diagrams' contributions individually:
$$a_\mu(m_\pi^2, m_\pi L, a^2) = A\, e^{-m_\pi L/2} + B\, a^2 + C\ S(m_\pi^2) + D + E\, m_\pi^2\ , \tag{21}$$
$$\begin{aligned} \text{Pole} &:: \ \frac{1}{m_\pi^2} \\ \text{Log} &:: \ \log m_\pi^2 \\ \text{Log2} &:: \ \log^2\left(m_\pi^2\right) \\ \text{m2Log} &:: \ m_\pi^2 \log\left(m_\pi^2\right)\ . \end{aligned} \tag{22}$$
- For the total contribution:
$$a_\mu(m_\pi^2, m_\pi L, a^2) = a_\mu(0, \infty, 0)(1 + A m_\pi^2 + B e^{-m_\pi L/2} + C a^2), \tag{23}$$
  - Note: more stringent Chiral fitting form for the total contribution due to weaker pion mass dependence for the total contribution.
- Long distance (separation of the two vertices locations) contribution obtained by fitting an ansatz: $f(|y|) = |y|^3 A e^{-B|y|}$.
Results only weakly depend on the form of the ansatz.

E.H. Chao et al 2021. (EPJC 81 7, 651)



E.H. Chao et al 2021. (EPJC 81 7, 651)

- Systematic uncertainty of the continuum extrapolation



Root-mean-squared deviation (9.2×10^{-11}) is estimated to be the uncertainty.

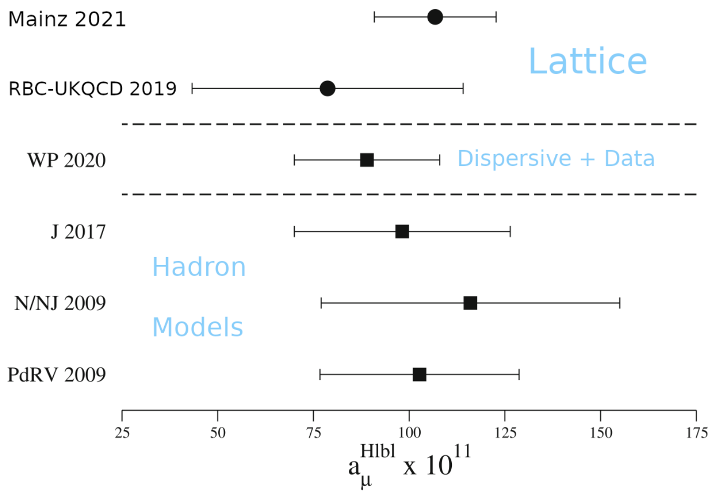
- Systematic uncertainty from Chiral extrapolation:

$$a_\mu(m_\pi^2, m_\pi L, a^2) = a_\mu(0, \infty, 0)(1 + Am_\pi^2 + Be^{-m_\pi L/2} + Ca^2), \tag{23}$$

With $Am_\pi^2 \to A_l \log(m_\pi^2/\mathrm{GeV}^2)$, half difference ($6.0 \times 10^{-11}$) is used as the estimate of the uncertainty. E.H. Chao et al 2021. (EPJC 81 7, 651)

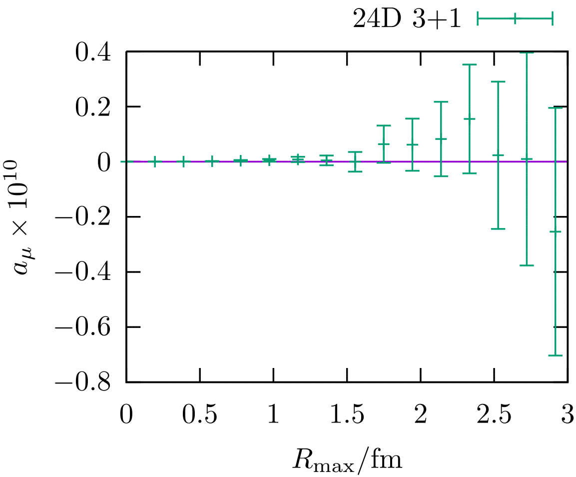
Muon $g-2$ HLbL summary

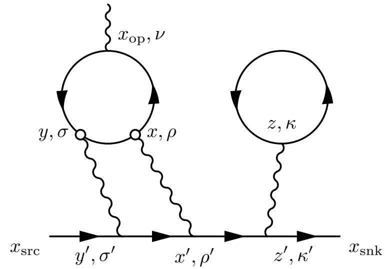
- Mainz 2021 is the most recent lattice result. It uses heavier pion mass with infinite volume QED kernel and extrapolate to the physical pion mass.
- RBC-UKQCD 2019 is the first lattice result. It uses physical pion mass in the finite volume QED_L scheme and extrapolate to the infinite volume.



- WP 2020 result uses dispersive relations and data. It is the sum of the contributions from different cuts and poles. High energy contributions are the major source of uncertainties.
- These three results have different systematics and agree well with each other. Uncorrelated average gives: $a_\mu^{\mathrm{HLbL}} = 9.77(1.16) \times 10^{-10}$.
- Hadronic light-by-light contribution cannot be the source of the muon $g-2$ puzzle.

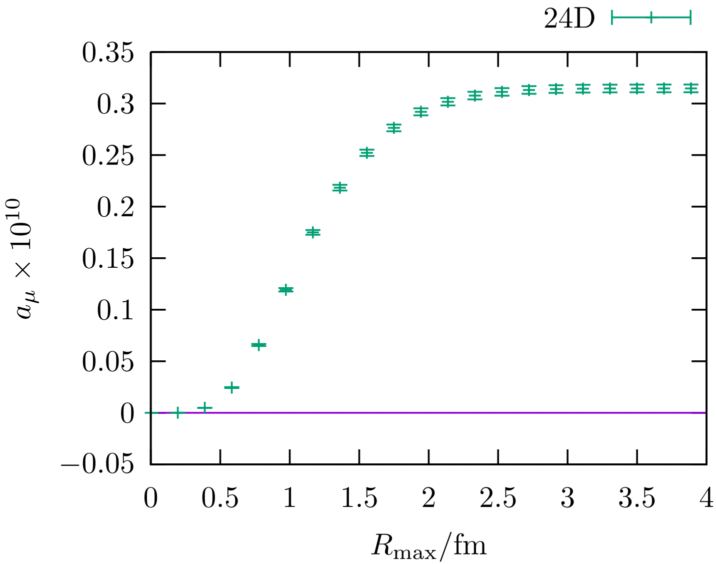
Thank You!

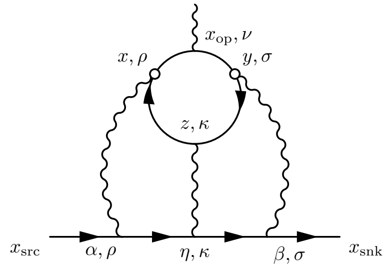




- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum upto $R_{\max}$
  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$
- The tadpole part comes from C. Lehner et al. 2016 (PRL 116, 232002)
- Systematic error (subdiscon): 0.5×10^{-10}



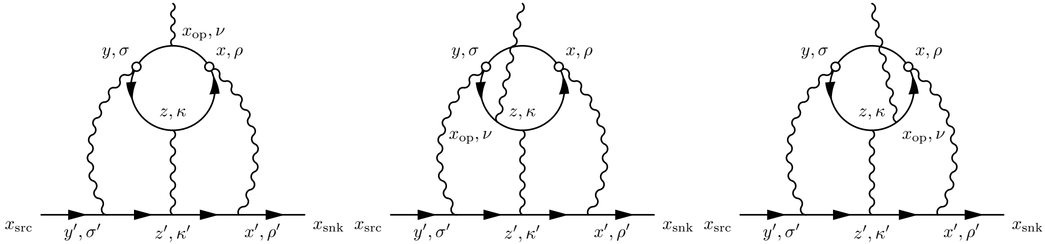


- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum upto $R_{\max}$
  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$
- Systematic error (strange con): 0.3×10^{-10}

Contribution	Value $\times 10^{11}$
Light-quark fully-connected and $(2+2)$	107.4(11.3)(9.2)
Strange-quark fully-connected and $(2+2)$	$-0.6(2.0)$
$(3+1)$	0.0(0.6)
$(2+1+1)$	0.0(0.3)
$(1+1+1+1)$	0.0(0.1)
Total	106.8(14.7)

TABLE VII. A breakdown of our result for a_μ^{Hlbl}.



- The three internal vertex attached to the quark loop are equivalent (all permutations are included).
- We can pick the closer two points as the point sources x, y.

$$\sum_{x,y,z} \to \sum_{x,y,z} \begin{cases} 3 & \text{if } |x-y| < |x-z| \text{ and } |x-y| < |y-z| \\ 3/2 & \text{if } |x-y| = |x-z| < |y-z| \\ 3/2 & \text{if } |x-y| = |y-z| < |x-z| \\ 1 & \text{if } |x-y| = |y-z| = |x-z| \\ 0 & \text{others} \end{cases}$$

Split the a_μ^{con} into two parts:

$$a_\mu^{\text{con}} = a_\mu^{\text{con,short}} + a_\mu^{\text{con,long}}$$

- $a_\mu^{\text{con,short}} = a_\mu^{\text{con}}(r \leq 1\text{fm})$:
  most of the contribution, small statistical error.
- $a_\mu^{\text{con,long}} = a_\mu^{\text{con}}(r > 1\text{fm})$:
  small contribution, large statistical error.

Perform continuum extrapolation for short and long parts separately.

- $a_\mu^{\text{con,short}}$: conventional a^2 fitting.
- $a_\mu^{\text{con,long}}$: simply use 48I value.
  Conservatively estimate the relative $\mathcal{O}(a^2)$ error: it may be as large as for $a_\mu^{\text{con,short}}$ from 48I.

Sys error from difference of fits

$$
\begin{aligned}
a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad & a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\
& -c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)
\end{aligned}
$$

$\mathcal{O}(1/L^3)$

$$
\begin{aligned}
a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad & a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^3} \\
& -c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)
\end{aligned}
$$

$\mathcal{O}(a^2 \log(a^2))$

$$
\begin{aligned}
a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad & a_\mu\Bigg(1 - \frac{b_2}{(m_\mu L)^2} \\
& -\Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big) \\
& \times\Big(1 - \frac{\alpha_S}{\pi} \log\big((a\ \mathrm{GeV})^2\big)\Big)\Bigg)
\end{aligned}
$$

Sys error from difference of fits

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\ -c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^4)$ (maximum of the following two)

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\ -c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2(a\ \mathrm{GeV})^4\Big)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) \quad = \quad a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} \\ -c_1(a\ \mathrm{GeV})^2 + c_2^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^4 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

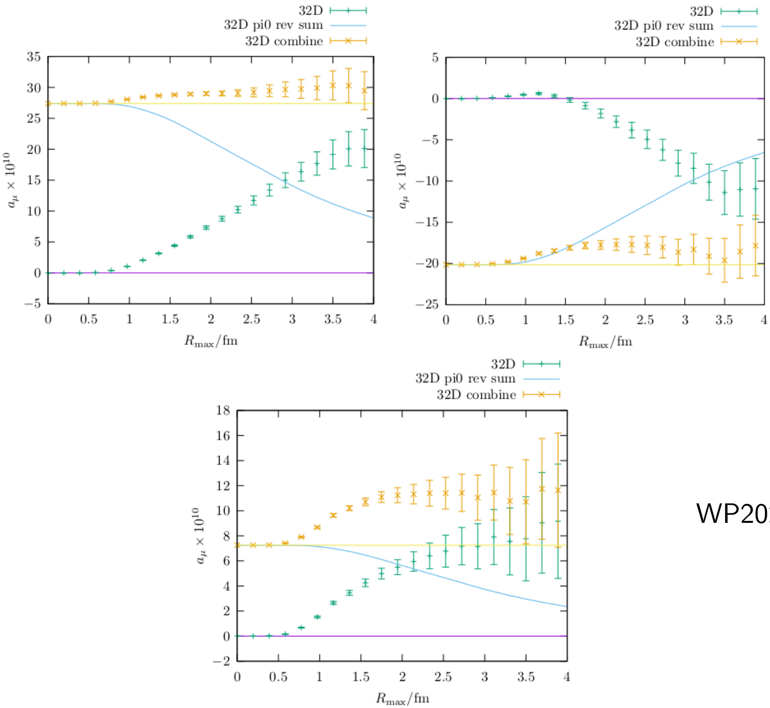
Sys error from difference of fits

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$

$\mathcal{O}(a^2/L)$ (maximum of the following two)

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - \Big(c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 + c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 - c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)\Big(1 - \frac{1}{m_\mu L}\Big)\right)$$

$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\Big(1 - \frac{b_2}{(m_\mu L)^2}\Big) \times\Big(1 - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\Big)$$



WP2020

Figure 91: Combined lattice and pion-pole contributions to the HLbL scattering part of the muon anomaly. Partial sums for the hadronic contributions, connected (top-left), leading disconnected (top-right), and total (bottom), computed with QED$_\infty$. $a^{-1} = 1$ GeV, $L = 6.4$ fm, and $M_\pi = 142$ MeV. Lines denote the π^0-pole contribution computed from the LMD model and are summed right-to-left.