

Dispersive determination of low energy πK interactions

10th IW on Chiral Dynamics



WILLIAM & MARY

CHARTERED 1693



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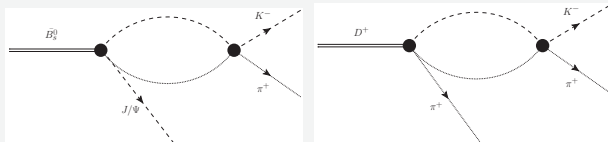
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Motivation: πK

- πK scattering \rightarrow final state in hadronic strange processes
- Heavy decays, CP violation, τ decays JHEP 09 031, JHEP 09 042, PLB 804 135371
- $\pi\pi \rightarrow K\bar{K} \Rightarrow$ new physics, $g-2$...

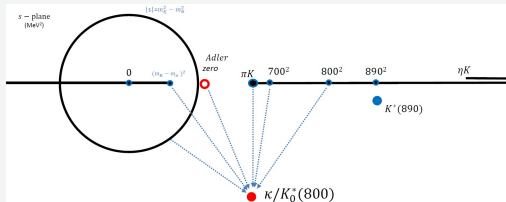
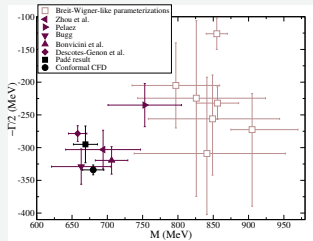
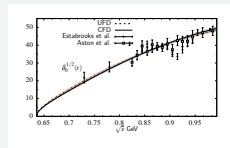
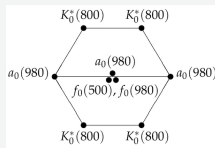


- π, K pseudo-Goldstone Bosons \rightarrow ChPT \rightarrow Scattering Lengths
- UChPT \rightarrow Good description, not suited for high precision
Nucl.Phys.B 587 331-362, Phys.Rev.D 65 054009
- Experimental groups need robust params \rightarrow LHCb for CP
- New experiment \rightarrow KLF



Motivation: κ

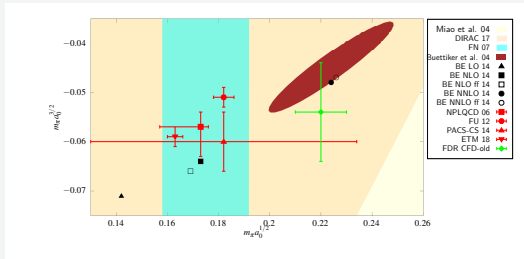
- Debated for decades
- Not $q\bar{q} \leftrightarrow \pi K$ $S^{1/2}$ threshold
- One of the broadest resonances



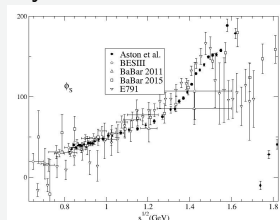
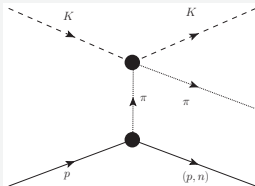
- Most of its determinations \rightarrow simple models
- Threshold behavior (ChPT), Adler Zero and LHC play a role
- Same problems in Lattice QCD at low m_π mass

πK scattering lengths

- Tension between Lattice and ChPT calculations



- Experiment cannot access πK directly



- No precise data at **threshold**
- For all these reasons \Leftrightarrow **Dispersion Relations**

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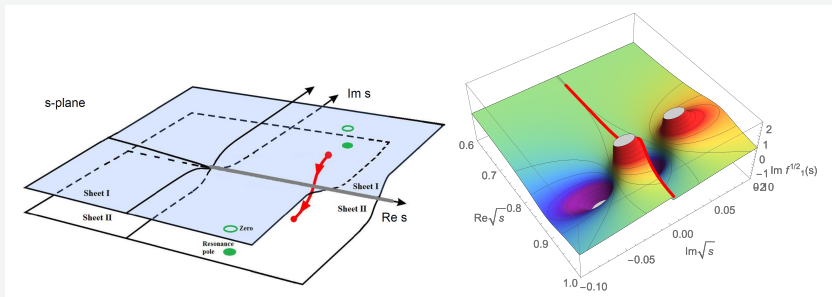
2.2 Applications

3 Summary



S-matrix principles: Unitarity

- **UNITARITY** \Leftrightarrow probability $\sum |\langle f | S | i \rangle|^2 = 1$
- Both right and left branch cuts $SS^\dagger = I \Rightarrow F - F^\dagger = iFF^\dagger$.
- Elastic unitarity $\rightarrow S^{II}(z) = \frac{1}{S^I(z)}$
- Zero of $S^I(z) \rightarrow$ pole of $S^{II}(z)$



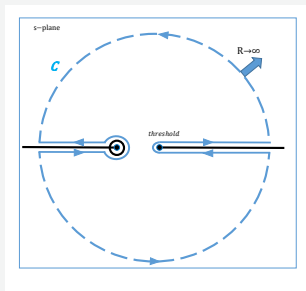
S-matrix principles: Analyticity and Crossing

- CAUSALITY \Leftrightarrow ANALYTICITY

- No poles in the first sheet

$$F(s, t) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\text{Im} F(s', t)}{s' - s} + LHC$$

- Structures \rightarrow unitarity, bound states, cusp



- Together with CROSSING \rightarrow Mandelstam analyticity

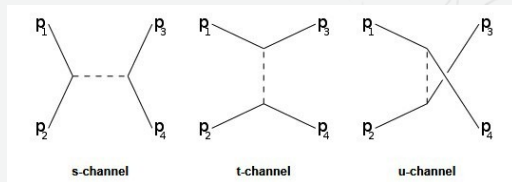


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Amplitudes

- Two independent amplitudes $l=1/2, 3/2$.
- s -channel πK and t -channel $\pi\pi \rightarrow K\bar{K}$

$$F^+(s,t) = \frac{1}{3}F^{1/2}(s,t) + \frac{2}{3}F^{3/2}(s,t) = \frac{G^{l_t=0}(t,s)}{\sqrt{6}},$$

$$F^-(s,t) = \frac{1}{3}F^{1/2}(s,t) - \frac{1}{3}F^{3/2}(s,t) = \frac{G^{l_t=1}(t,s)}{2}.$$

- Symmetric and antisymmetric amplitudes under $s \leftrightarrow u$ exchange
- Customary decomposition in partial waves

$$F^I(s,t) = 16\pi \sum_{\ell} (2\ell + 1) f_{\ell}^I(s) P(z_s(t)),$$

$$G^I(t,s) = 16\pi\sqrt{2} \sum_{\ell} (2\ell + 1) (q_{\pi}q_K)^{\ell} g_{\ell}^I(t) P(z_t(s)).$$

Forward dispersion relations

Peláez, AR (Phys.Rev. D93 074025)

- Combining the **First Principles**
- Example, amplitude DR, $t = 0$

$$\text{Re} F^I(s) = F^I(s_{th}) + \frac{(s - s_{th})}{\pi} PV \int_{s_{th}}^{\infty} ds' \left[\frac{\text{Im} F^I(s')}{(s' - s)(s' - s_{th})} + (-1)^I \frac{\text{Im} F^I(s')}{(s' + s - 2\Sigma\pi K)(s' + s_{th} - 2\Sigma\pi K)} \right],$$

- If we project $F^I(s) \rightarrow f_\ell^I(s)$

- We need **Input**
 $\rightarrow F^I(s), f_\ell^I(s)$
- We get **DR**

- We recover **Re** $F^I(s)$

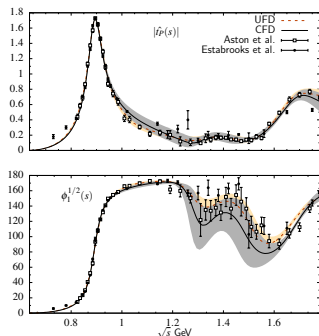
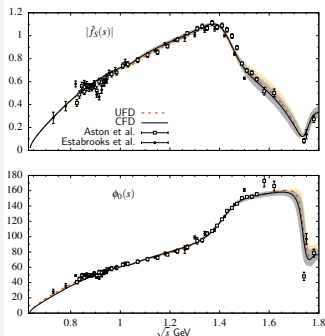
- Stringent constrains
- Perform stable analytic continuation

UFD Input: Elastic region

- Unitarity for partial waves
- with $\cot \delta_l^I(s) = \frac{\sqrt{s}}{2q^{2l+1}} \sum B_n \omega(s)^n \rightarrow$ conformal map
- Inelastic region \rightarrow pheno fits
- 8 πK PW ~ 1.8 GeV
- 5 $\pi\pi \rightarrow K\bar{K}$ PW ~ 2 GeV

Peláez, AR (Phys.Rev. D93 074025)

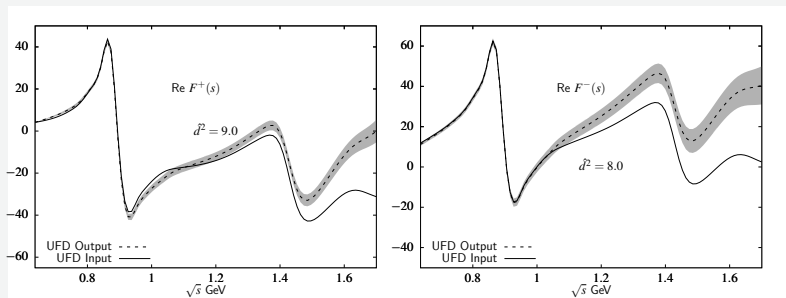
$$f_l^I(s) = \frac{1}{\sigma(s)} \frac{1}{\cot \delta_l^I(s) - i},$$



Forward Dispersion relations

Peláez,AR (Phys.Rev. D93 074025)

- Amplitudes built using the whole tower of partial waves
- Two independent amplitudes F^+ and F^-
- We define a penalty function $\hat{d}^2 = \frac{1}{N} \sum_i^N \left(\frac{\text{Re}(F_{out}^I - F_{fit}^I)(s_i)}{\Delta \text{Re}(F_{out}^I - F_{fit}^I)(s_i)} \right)^2$
- Above 1.8 GeV discrepancies too big

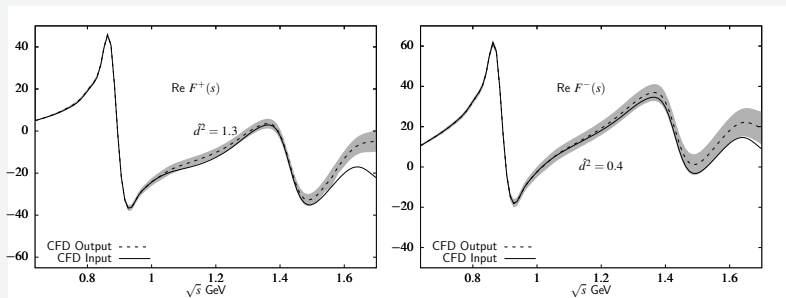


- Room for improvement \rightarrow Constrained fits

Forward Dispersion relations

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- Above 1.8 GeV discrepancies too big

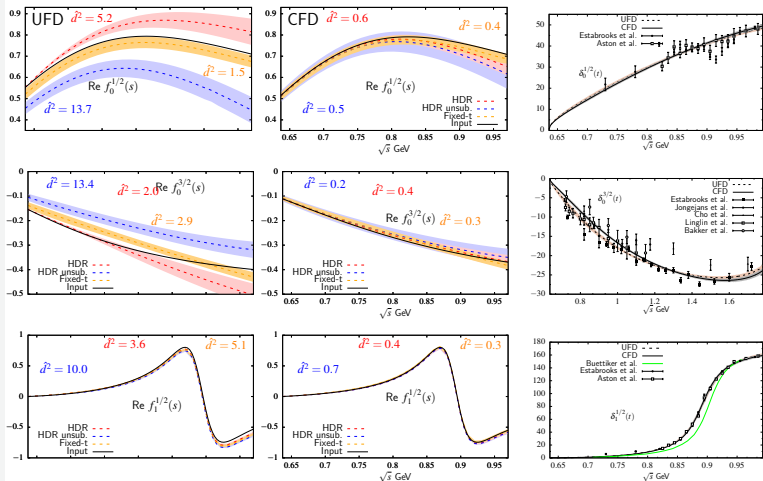


- Very good agreement

DR for πK and $\pi\pi \rightarrow K\bar{K}$

Peláez,AR (2010.11222)

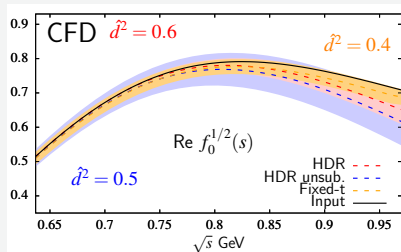
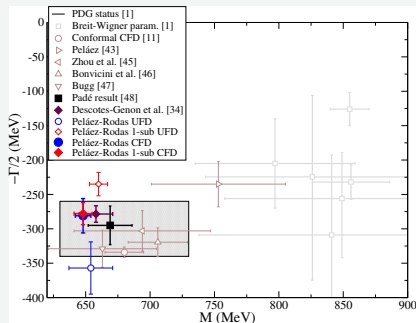
- Average $\hat{d}^2/DR \simeq 5.5$ (UFD) \rightarrow 0.6 (CFD)
- 13 partial waves $\rightarrow \chi^2/dof \simeq 1$ (UFD) \rightarrow 1.6 (CFD)



CFD $K_0^*(700)/\kappa$ pole

Peláez,AR (Phys.Rev.Lett. 172001)

- Stable result **AFTER** constraining
- All uncertainties have been taken into account



$$\sqrt{s_p} = (648 \pm 7) - i(560 \pm 32)/2 \text{ MeV} \quad \text{HDR}$$

$$\sqrt{s_p} = (658 \pm 13) - i(557 \pm 24)/2 \text{ MeV} \quad \text{Descotes-Genon, Moussallam}$$

$$\sqrt{s_p} = (680 \pm 50) - i(600 \pm 80)/2 \text{ MeV} \quad \text{PDG}$$

πK scattering lengths

$$F^I(s_{th}, 0) = 8\pi m_+ a_0^I$$

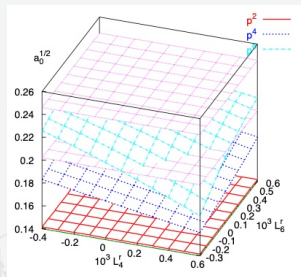
where $m_+ = m_\pi + m_K$

- At LO

$$a_0^- \propto \frac{1}{f_\pi^2} \quad a_0^+ = \mathcal{O}(m_+^4)$$

- NLO \rightarrow LECs L_{1-8}

$$a_0^- \propto \frac{L_5}{f_\pi^4} \quad a_0^+ \rightarrow 7L_i$$



- NNLO $\rightarrow 32C_i$, $a_0^- \rightarrow 10C_i$, $a_0^+ \rightarrow 23C_i$ Bijmans et al. (JHEP 05 036)

Sum rules

Peláez,AR (2010.11222)

- Sum rules can be obtained for the LEP
- For $F^I(s) = \int_{m_+^2}^{\infty} \dots \leftrightarrow a_0^I \propto F(s_{th}), b_0^I \propto \frac{dF(s_{th})}{ds} \dots$

$$a_0^- = \frac{m_\pi m_K}{2\pi^2 m_+} \int_{m_+^2}^{\infty} \frac{\text{Im} F^-(s')}{(s' - m_-^2)(s' - m_+^2)} ds',$$

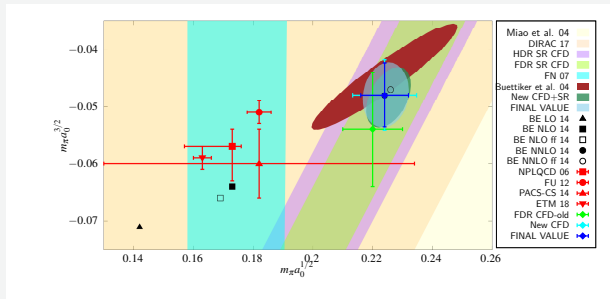
$$a_0^- = \frac{m_\pi m_K}{2\pi^2 m_+} \left(\frac{1}{2} \int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} \text{Im} \frac{G^1(t', s'_b)}{\sqrt{(t' - 4m_\pi^2)(t' - 4m_K^2)}} + \int_{m_+^2}^{\infty} ds' \frac{\text{Im} F^-(s', t'_b)}{\lambda_{s'}} \right)$$

- Combining all DRs \rightarrow dozens of SR for a_ℓ^I, b_ℓ^I

Adler Zeroes and Scattering lengths

Peláez,AR (2010.11222)

	FINAL	CFD	Paris group
$a_0^{1/2}$	0.224 ± 0.008	0.224 ± 0.011	0.224 ± 0.022
$a_0^{3/2}$	-0.0480 ± 0.0056	-0.048 ± 0.006	-0.0448 ± 0.0077



	UFD $I = 1/2$	CFD $I = 1/2$	UFD $I = 3/2$	CFD $I = 3/2$
$\sqrt{s_A, \text{fixed-}t}$	$0.477^{+0.0010}_{-0.007}$	$0.466^{+0.006}_{-0.005}$	$0.530^{+0.013}_{-0.016}$	$0.549^{+0.008}_{-0.0010}$
$\sqrt{s_A, \text{HDR}}$	$0.473^{+0.011}_{-0.009}$	$0.466^{+0.007}_{-0.005}$	$0.537^{+0.016}_{-0.019}$	$0.551^{+0.009}_{-0.0010}$
$\sqrt{s_A, \text{HDR-sub}}$	$0.481^{+0.008}_{-0.008}$	$0.470^{+0.010}_{-0.005}$	$0.532^{+0.013}_{-0.016}$	$0.552^{+0.008}_{-0.010}$
LO ChPT	486		516	

LEP

Peláez,AR (2010.11222)

	This work sum rules with CFD input				This work direct CFD	Sum rules Büttiker et al. (2004)	NNLO ChPT Bijens et al. (2004,14)
	PWFTR	PWHDR	PWHDR _{sub}	Final Value			
$m_{\pi}^3 b_0^{1/2} \times 10$	1.04±0.04	1.05±0.07	1.15±0.04	1.08±0.08	0.95±0.04	0.85±0.04	1.278
$m_{\pi}^3 b_0^{3/2} \times 10$	-0.42±0.02	-0.41±0.03	-0.44±0.02	-0.43±0.03	-0.36±0.04	-0.37±0.03	-0.326
$m_{\pi}^3 a_1^{1/2} \times 10$	0.228±0.010	0.218±0.008	0.222±0.006	0.222±0.009	0.20±0.04	0.19±0.01	0.152
$m_{\pi}^5 b_1^{1/2} \times 10^2$	0.58±0.03	0.59±0.03	0.60±0.03	0.59±0.02	0.5±0.2	0.18±0.02	0.032
$m_{\pi}^3 a_1^{3/2} \times 10^2$	0.15±0.05	0.19±0.05	0.17±0.04	0.17±0.05	0.15±0.11	0.065±0.044	0.293
$m_{\pi}^5 b_1^{3/2} \times 10^3$	-0.94±0.09	-0.97±0.08	-1.03±0.07	-0.99±0.09	-1.04±0.8	-0.92±0.17	0.544
$m_{\pi}^5 a_2^{1/2} \times 10^3$	0.60±0.13	0.54±0.03	0.55±0.02	0.55±0.05	0.53±0.05	0.47±0.03	0.142
$m_{\pi}^7 b_2^{1/2} \times 10^4$	-0.89±0.10	-0.96±0.09	-0.95±0.09	-0.94±0.09	0.20±0.02	-1.4±0.3	-1.98
$m_{\pi}^5 a_2^{3/2} \times 10^4$	-0.05±0.60	-0.11±0.16	-0.18±0.15	-0.14±0.17	-0.09±0.03	-0.11±0.27	-0.45
$m_{\pi}^7 b_2^{3/2} \times 10^4$	-1.12±0.10	-1.13±0.09	-1.14±0.09	-1.13±0.06	-0.03±0.01	-0.96±0.26	0.61

	This work sum rules with CFD input			Sum rules Büttiker et al. (2004)	NNLO ChPT Bijens et al. (2004)	Sum rules Lang et al. (1980)
	Fixed- t	HDR	HDR _{sub}			
C_{00}^+	1.52±0.56	like fixed- t		2.01±1.10	0.278	-0.52±2.03
C_{10}^+	0.96±0.11	1.04±0.11		0.87±0.08	0.898	0.55±0.07
C_{01}^+	2.34±0.05	like fixed- t		2.07±0.10	3.8	2.06±0.22
C_{11}^+	-0.047±0.006	-0.050±0.006		-0.066±0.010	-0.10	-0.04±0.02
C_{00}^-	9.11±0.35	9.54±0.38	9.04±0.39	8.92±0.38	8.99	7.31±0.90
C_{10}^-	0.45±0.05	0.38±0.02	0.39±0.02	0.31±0.01	0.088	0.21±0.04
C_{01}^-	0.68±0.02	0.66±0.02	0.68±0.02	0.62±0.06	0.71	0.51±0.10
F_{CD}^+	3.55±0.64	3.71±0.64		3.90±1.50	2.11	

πK dispersive analysis: Summary

- DR analysis on almost all πK available data
 1. Pruning on the data
 2. Result \rightarrow simple params compatible with both **Data and DR**

- DR applications
 1. Extraction of the $\kappa/K_0^*(700)$ with 2 DR \rightarrow exists
 2. Model independent determination of > 20 LEP

Spare slides!

Mandelstam Analyticity in Relativistic scattering

If one combines analyticity and crossing \rightarrow **Mandelstam Hypothesis**

- Only one analytic function which

$$T(s, t, u) = \begin{cases} T_{12 \rightarrow 34}(s, t, u), & s \geq (m_1 + m_2)^2, \quad t \leq 0, \quad u \leq 0, \\ T_{1\bar{3} \rightarrow \bar{2}4}(t, s, u), & t \geq (m_1 + m_3)^2, \quad s \leq 0, \quad u \leq 0, \\ T_{1\bar{4} \rightarrow 3\bar{2}}(u, t, s), & u \geq (m_1 + m_4)^2, \quad s \leq 0, \quad t \leq 0. \end{cases}$$

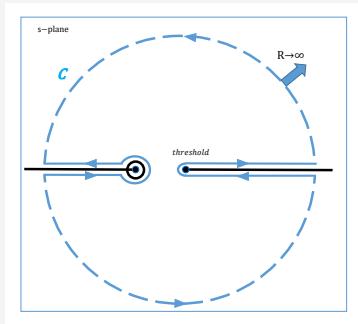
- No more non-analytic structures
- Cauchy theorem: Let D be a domain of the complex plane where the function $f(z)$ is analytic and let C be the closed curve defined by its boundary. Then, for any $z \in D$

$$f(z) = \oint_C \frac{f(z')}{z' - z} dz'$$

Analyticity in Relativistic scattering: $\pi\pi$

- Fixed- t right and left hand cuts starting at $s = 4m_\pi^2$ and $s = -t$
- If $T(s,t) \rightarrow 1/s$ when $s \rightarrow \infty$ then

$$T(s,t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}T(s',t)}{(s' - s)} + \frac{1}{\pi} \int_{-\infty}^{-t} ds' \frac{\text{Im}T(s',t)}{(s' - s)}$$



- If not \rightarrow subtractions

$$T(s,t) = T(s_0,t) + \frac{(s - s_0)}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}T(s',t)}{(s' - s)(s' - s_0)} + \frac{(s - s_0)}{\pi} \int_{-\infty}^{-t} ds' \frac{\text{Im}T(s',t)}{(s' - s)(s' - s_0)}$$

Analyticity in Relativistic scattering: $\pi\pi$

- If we make the change of variables $s' \rightarrow u' = 4m_\pi^2 - t - s'$

$$T(s, t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds' \left(\frac{\text{Im}T(s', t)}{(s' - s)} - \frac{\text{Im}T(4m_\pi^2 - s' - t, t)}{(u' - u)} \right)$$

- u' is a dummy variable
- The LHC can be always rewritten as RHC terms
- Due to crossing $T^{I_s}(s, t, u) = \sum_{I_t} C_{su} T^{I_u}(u, t, s)$ and

$$T(s, t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds' \left(\frac{\text{Im}T(s', t)}{(s' - s)} - \frac{\sum C_{su}^{II'} \text{Im}T^{I'}(s', t)}{(s' - u)} \right)$$

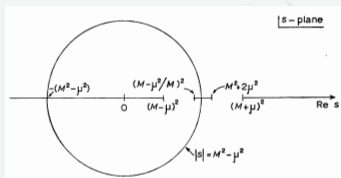
- Here we have our closed dispersion relation

Analyticity in Relativistic scattering: $\pi\pi$

- However this is a “toy DR”, we actually need more elaborated stuff.
- Sometimes we will not fix t , but move it as a function of the other two (s, u) variables.
- In particular, by using $T(s, t) = 32\pi \sum_{\ell} (2\ell + 1) P_{\ell}(z_s) t_{\ell}(s)$ we can project

$$t_{\ell}(s) = \frac{1}{32\pi} \int_0^1 dz_s T(s, t) P_{\ell'}(z_s),$$

- $P_{\ell'}(z_s)$ are the so called Legendre Polynomials (project the amplitude into defined angular momentums).



Analyticity in Relativistic scattering: $\pi\pi$

- The most sound dispersion relations for meson-meson scattering \rightarrow Roy-Steiner eqs.

$$\vec{T}(s, t, u) = \text{S.T.} + \int_{4m_\pi^2}^{\infty} ds' g_2(s, t; s') \text{Im} \vec{T}(s', 0, u') \\ + \int_{4m_\pi^2}^{\infty} ds' g_3(s, t; s') \text{Im} \vec{T}(s', t, u')$$

$$\text{Re} \vec{t}_J(s) = \frac{1}{32\pi} \int_0^1 dx P_J(x) \vec{T}(s, t(x)) = \frac{1}{32\pi} \int_0^1 dx P_J(x) \text{S.T.} + \\ \sum_{J'} (2J' + 1) \int_{4m_\pi^2}^{\infty} ds' \int_0^1 dx P_J(x) [g_2(s, t(x); s') \\ + P_{J'}(x) g_3(s, t(x); s')] \text{Im} \vec{t}_{J'}(s')$$

- g_2, g_3 are matrices of polynomials in the Mandelstam variables

- Simple set of DR, $t = 0$

$$\text{Re}F^I(s) = F^I(s_{th}) + \frac{(s - s_{th})}{\pi}$$

$$PV \int_{s_{th}}^{\infty} ds' \left[\frac{\text{Im}F^I(s')}{(s' - s)(s' - s_{th})} + (-1)^I \frac{\text{Im}F^I(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],$$

- For the symmetric $s \leftrightarrow u$ amplitude one subtraction is needed

$$\text{Re}F^+(s) = F^+(s_{th}) + \frac{(s - s_{th})}{\pi}$$

$$P \int_{s_{th}}^{\infty} ds' \left[\frac{\text{Im}F^+(s')}{(s' - s)(s' - s_{th})} - \frac{\text{Im}F^+(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],$$

where $\Sigma_{\pi K} = m_{\pi}^2 + m_K^2$.

- For the antisymmetric amplitude no subtraction is needed

$$\text{Re}F^-(s) = \frac{(2s - 2\Sigma_{\pi K})}{\pi} P \int_{s_{th}}^{\infty} ds' \frac{\text{Im}F^-(s')}{(s' - s)(s' + s - 2\Sigma_{\pi K})}.$$

UFD: Inelastic region

Peláez, AR (Phys.Rev. D93 074025)

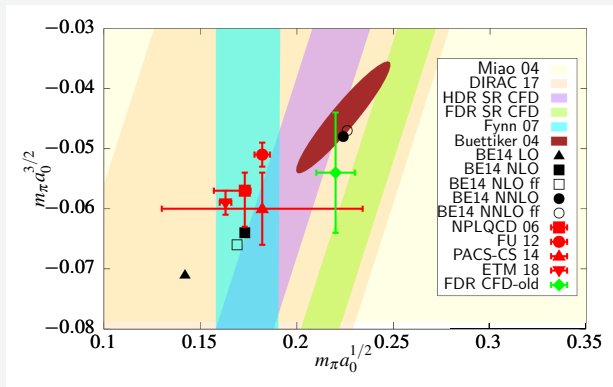
- In the inelastic region $f_l^I = \frac{\eta_l^I(s)e^{2i\delta_l^I(s)} - 1}{2i} = |f_l^I|e^{i\phi_l^I}$.
- We use complex rational functions that near each resonance look like BW.
- Focusing on simple parameterizations, no *EFT* included here.
- We impose matching conditions on the inelastic ηK threshold.
- We use up to $G^{1/2} \rightarrow 8$ partial waves.
- Although we use for our analysis the $P^{3/2}, D^{3/2}, F^{1/2}$ and $G^{1/2}$ their contribution is small. Not shown here.

πK scattering lengths

- Sum rule from FDR $\rightarrow a_0^- = 0.292 \pm 0.01$
- However, sum rule coming from G^1 channel :

$$a_0^- = 0.253 \pm 0.015$$

- New sum rule closer to Lattice



- HDR used for both πK and $\pi\pi \rightarrow K\bar{K}$ channels
- $(s - a_c)(u - a_c) = b$ with a_s, a_t used to maximize to applicability region

$$\begin{aligned}
 f_0^\pm(s) &= a_0^\pm + \frac{1}{\pi} \sum_l \int_{s_{th}}^\infty ds' K_{0l}^\pm(s, s') \text{Im} f_l^\pm(s') \\
 &+ \frac{1}{\pi} \sum_l \int_{4m_\pi^2}^\infty dt' G_{0(2l-2), (2l-1)}^\pm(s, t') \text{Im} g_{(2l-2), (2l-1)}^{0,1}(t') \\
 g_0^0(t) &= \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{t}{\pi} \int_{4m_\pi^2}^\infty \frac{\text{Im} g_0^0(t')}{t'(t'-t)} dt' \\
 &+ \frac{t}{\pi} \sum_l \int_{4m_\pi^2}^\infty \frac{dt'}{t'} G_{0, 2l-2}^0(t, t') \text{Im} g_{2l-2}^0(t') + \sum_l \int_{m_+^2}^\infty ds' G_{0,l}^+(t, s') \text{Im} f_l^+(s').
 \end{aligned}$$

- Fixed- t only used for $\pi K \rightarrow \pi K$ inputs dominate

- Fixed- t only used for $\pi K \rightarrow \pi K$ inputs dominate
- HDR used for both πK and $\pi\pi \rightarrow K\bar{K}$ channels
- $(s - a_i)(u - a_i) = b$ with a_s, a_t used to maximize to applicability region

$$f_0^+(s) = a_0^+ + \frac{1}{\pi} \sum_l \left(\int_{s_{th}}^{\infty} ds' K_{0l}^+(s, s') \text{Im} f_l^+(s') + \int_{4m_\pi^2}^{\infty} dt' G_{02l}^+(s, t') \text{Im} g_{2l}^0(t') \right)$$

$$g_0^0(t) = \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{1}{\pi} \sum_l \left(\int_{m_+^2}^{\infty} ds' G_{0,l}^+(t, s') \text{Im} f_l^+(s') + t \int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} G_{0,2l}^0(t, t') \text{Im} g_{2l}^0(t') \right)$$

- Both channels are coupled

DR for πK and $\pi\pi \rightarrow K\bar{K}$

Peláez, AR (2010.11222)

1. We build DR

2. Define Penalty function $\hat{d}_{DR}^2 = \frac{1}{N} \sum_i \left(\frac{\text{Re}(f_{out} - f_{fit})(s_i)}{\Delta \text{Re}(f_{out} - f_{fit})(s_i)} \right)^2$

3. We minimize a global $\chi^2 = W_1 \chi_{data}^2 + W_2 \hat{d}_{DR}^2$

4. Weights (W_i) \sim d.o.f

✓ Forward Dispersion Relations

Phys.Rev.D 93 074025

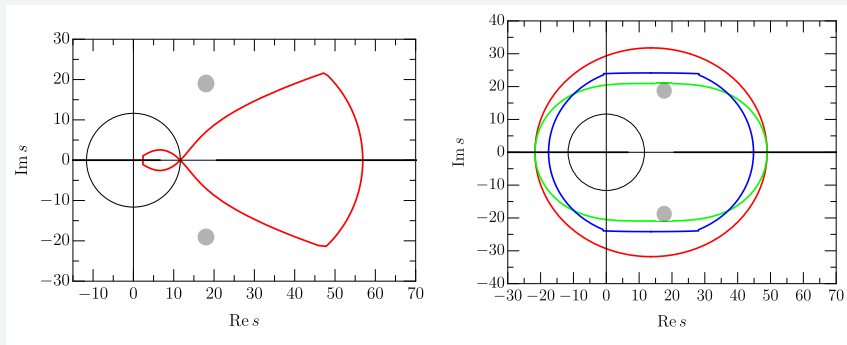
1. Very simple
2. Applicable to arbitrary high energies

✓ PWDR for πK and $\pi\pi \rightarrow K\bar{K}$

2010.11222, Eur.Phys.J.C 78 897

1. Fixed- t DR for πK only
2. Hyperbolic dispersion relations for both
3. Omnès-Muskhelishvili problem
4. Applicable $\sim \mathcal{O}(1)$ GeV

- Tension between FDR, HDR and Lattice.
- Scarcity of πK data \rightarrow SL poorly determined.
- $K_0^*(700)$ pole out of FDR/fixed-t range of validity \rightarrow only HDR here



$\pi\pi \rightarrow K\bar{K}$

Peláez, AR Eur.Phys.J.C 78

JHEP 06 (2012) 043

- Crossed channel HDR partial wave with one subtraction

$$\begin{aligned}
 g_0^0(t) &= \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im}g_0^0(t')}{t'(t'-t)} dt' \\
 &+ \frac{t}{\pi} \sum_l \int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} G_{0,2l-2}^0(t, t') \text{Im}g_{2l-2}^0(t') + \sum_l \int_{m_+^2}^{\infty} ds' G_{0,l}^+(t, s') \text{Im}f_l^+(s') \\
 &= \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im}g_0^0(t')}{t'(t'-t)} dt' + \Delta_0^0(t)
 \end{aligned}$$

- $\Delta_0^0(t)$ contains the left cut.
- Unknown value of $|g_l^I(t)|$ below $K\bar{K}$ threshold
- Phase shift below $K\bar{K} \rightarrow$ [Watson Theorem](#)

- Define $\hat{g}_l^I(t) = \frac{g_l^I(t) - \Delta_l^I(t)}{\Omega_l^I(t)}$ with $\Omega_l^I(t) = e^{\frac{t}{\pi} \int_{4m_\pi^2}^{t_m} \frac{\phi_l^I(t')}{t'(t'-t)} dt'}$.

- We develop a DR for the new function $\hat{g}_l^I(t)$

Omnès-Muskhelishvili problem

Peláez, AR Eur.Phys.J.C 78

- The set of final Omnès-Muskhelishvili DR:

$$\begin{aligned}
 g_0^0(t) &= \Delta_0^0(t) + \frac{t\Omega_0^0(t)}{t_m - t} \left[\alpha + \frac{t}{\pi} \int_{4m_\pi^2}^{t_m} dt' \frac{(t_m - t')\Delta_0^0(t') \sin \phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t' - t)} \right. \\
 &\quad \left. + \frac{t}{\pi} \int_{t_m}^{\infty} dt' \frac{(t_m - t')|g_0^0(t')| \sin \phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t' - t)} \right] \\
 g_1^1(t) &= \Delta_1^1(t) + \Omega_1^1(t) \left[\frac{1}{\pi} \int_{4m_\pi^2}^{t_m} dt' \frac{\Delta_1^1(t') \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right. \\
 &\quad \left. + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{|g_1^1(t')| \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right]
 \end{aligned}$$

- When s real we obtain $|g_l^I(t)|_{out}$.

Omnès-Muskhelishvili equations

- Omnès-Muskhelishvili DR with as less subtractions as possible
- S-channel and T-channel coupled in a complicated non-linear way

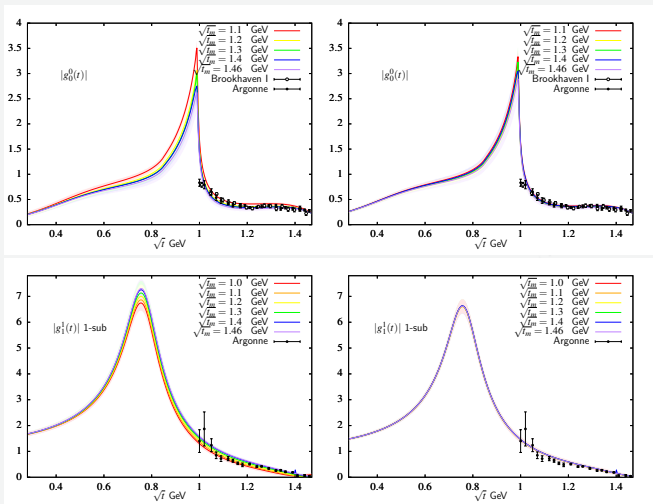
$$g_0^0(t) = \Delta_0^0(t) + \frac{t\Omega_0^0(t)}{t_m - t} \left[\alpha + \frac{t}{\pi} \int_{4m_\pi^2}^{t_m} dt' \frac{(t_m - t')\Delta_0^0(t') \sin \phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t' - t)} \right. \\ \left. + \frac{t}{\pi} \int_{t_m}^{\infty} dt' \frac{(t_m - t')|g_0^0(t')| \sin \phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t' - t)} \right],$$

$$g_1^1(t) = \Delta_1^1(t) + \Omega_1^1(t) \left[\frac{1}{\pi} \int_{4m_\pi^2}^{t_m} dt' \frac{\Delta_1^1(t') \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right. \\ \left. + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{|g_1^1(t')| \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right].$$

- If more subtractions \Rightarrow scalar and vector partial waves coupled in a non-linear way.

Omnès-Muskhelishvili matching condition

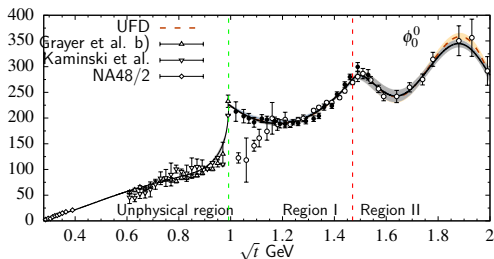
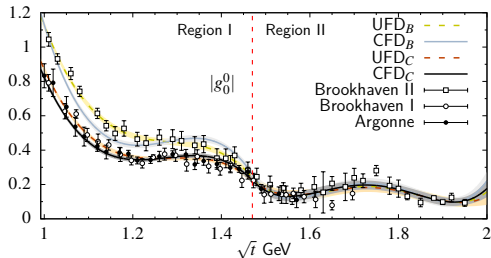
- $\Omega_\ell^I(t) = \exp\left(\frac{t}{\pi} \int_{4m_\pi^2}^{t_m} \frac{\phi_\ell^I(t') dt'}{t'(t'-t)}\right)$
- Unique/Perfect solution \rightarrow not t_m dependence



$\pi\pi \rightarrow K\bar{K}$

Peláez, AR 2010.11222

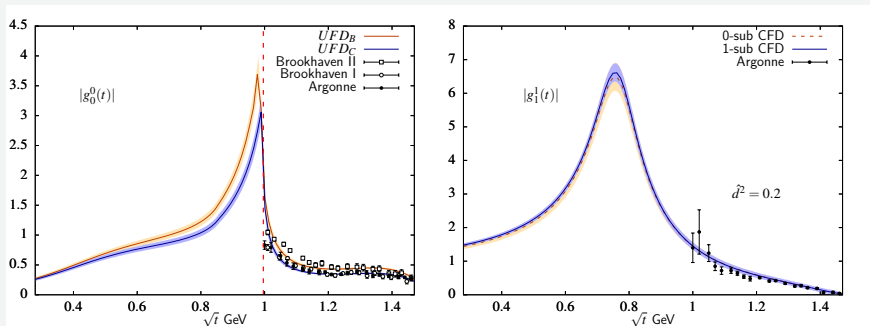
- There are 2 possible $g_0^0(t)$ even after imposing the DR



$\pi\pi \rightarrow K\bar{K}$

Peláez, AR 2010.11222

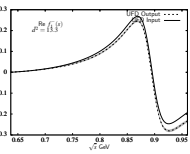
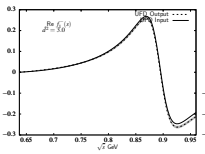
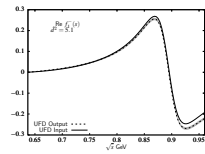
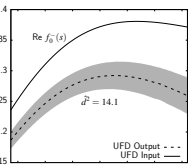
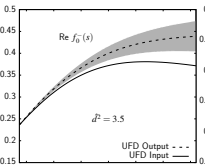
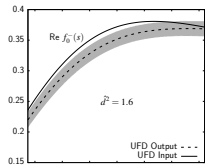
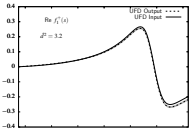
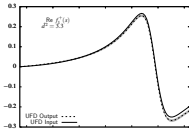
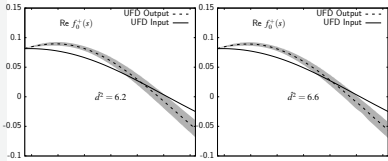
- Different $f_0(980)$ behaviors yet almost same πK and $\kappa/K_0^*(700)$ results
- Both $g_1^1(t)$ fully compatible in the pseudo-threshold region



πK UFD

Peláez, AR 2010.11222

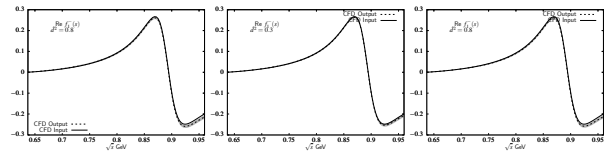
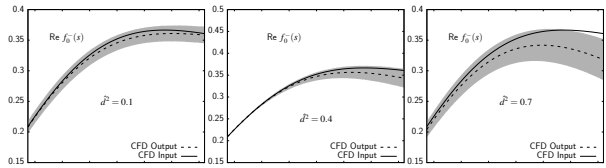
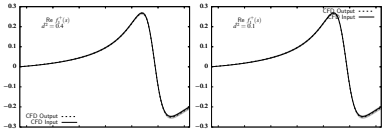
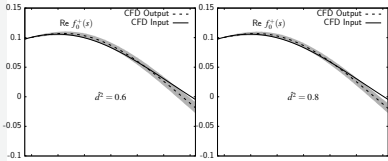
- Some of the dispersion relations are severely deviated
- The scattering lengths are not compatible with the DR



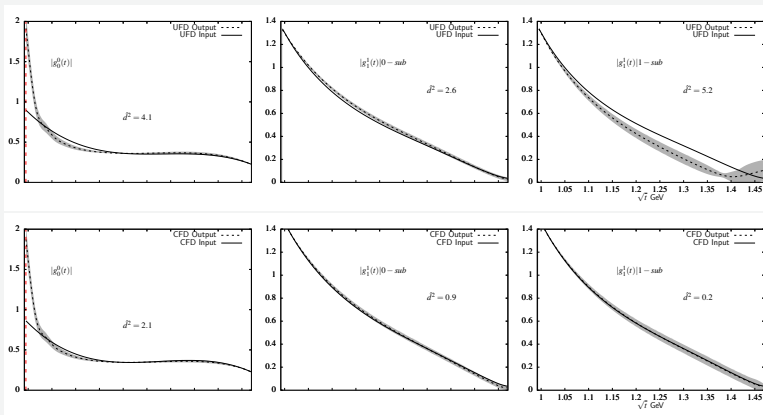
πK CFD

Peláez, AR 2010.11222

- Remarkable agreement
- All DR now compatible from threshold on



- Again, much better agreement after the constrains



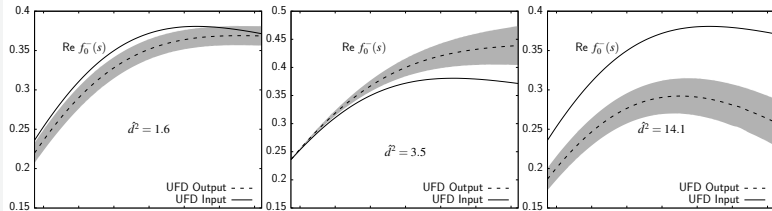
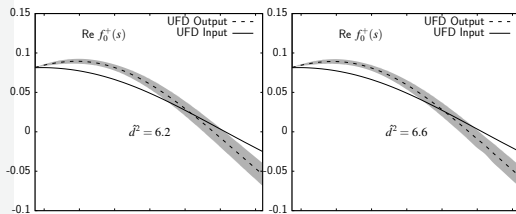
2010.11222, Invited to Phys.Rep.

- 16 dispersion relations \rightarrow 2 FDR, 4 OM, 4 fixed-t, 6 HDR.
- HDR with less subtractions \rightarrow worst discrepancies.
- UFD deviations of more than 3 sigmas.
- Up to 8 low energy parameters can be obtained with high precision.
- Up to 13 partial waves included in this analysis \rightarrow 7 constrained

πK UFD

Peláez, AR 2010.11222

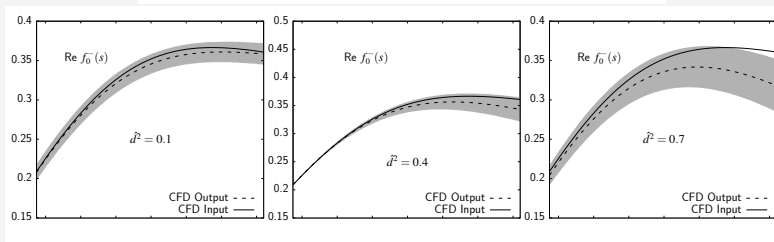
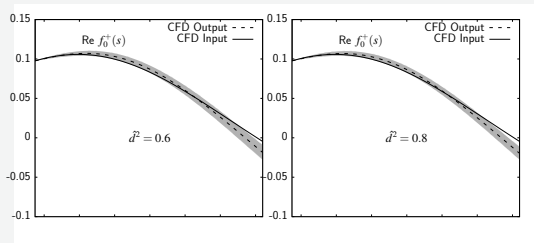
- Some of the dispersion relations are severely deviated
- The scattering lengths are not compatible with the DR



πK CFD

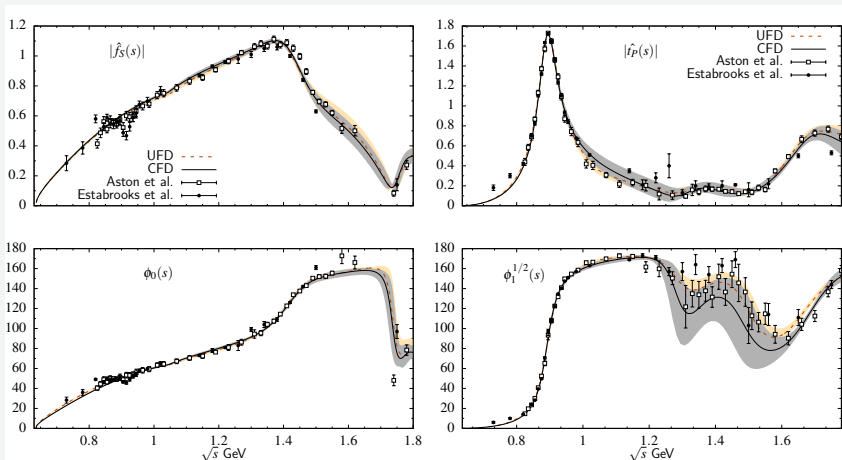
Peláez, AR 2010.11222

- Remarkable agreement
- All DR now compatible from threshold on



Preliminary: πK CFD

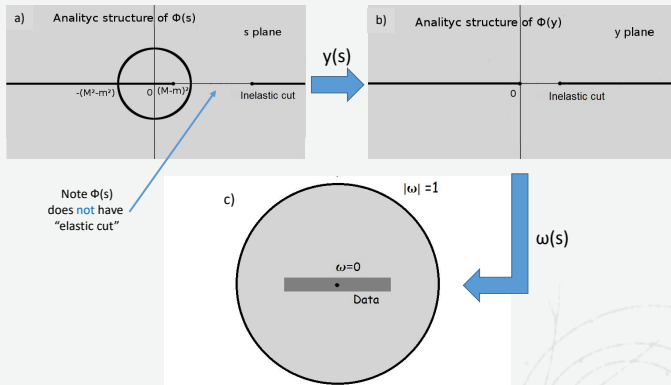
- The χ^2/dof worsen by a 30% on average.



- Most regions for most partial waves \rightarrow nice data description

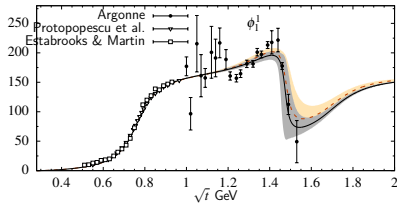
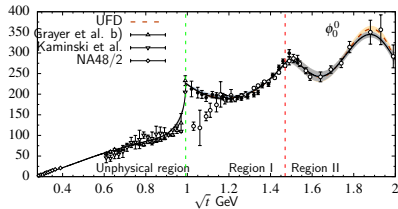
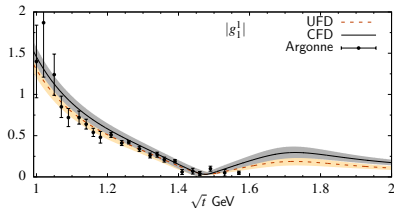
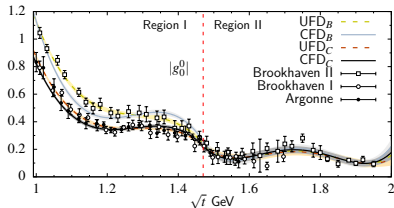
Conformal map

- Simple, yet powerful in the elastic region



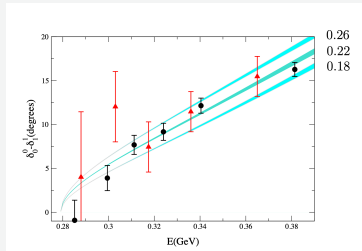
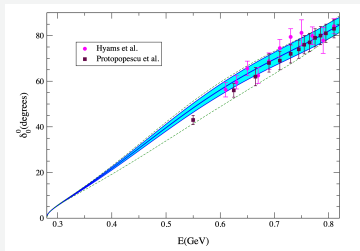
- $\cot \delta_l(s) = \frac{\sqrt{s}}{2q^{2l+1}} F(s) \sum_n B_n \omega(s)^n$, where $F(s)$ can have zeroes or poles.
- Can mimic the LHC \rightarrow fit/poles should be more stable

Preliminary: $\pi\pi \rightarrow K\bar{K}$ CFD



$\pi\pi$ and σ : Fixed-t

- Commonly known as Roy Eqs. (2-sub Bern group) Phys.Lett.B 36 , Nucl.Phys.B 603
- Approach:
 1. Matching conditions \rightarrow unique solution below 850 MeV_{Eur.Phys.J.C}
10
 2. Numerical matching \rightarrow Analyticity, Crossing and Unitarity



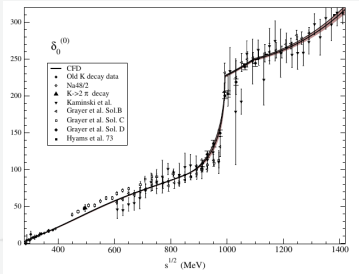
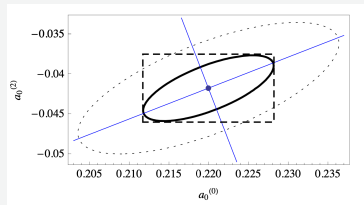
$\pi\pi$ and σ : Fixed-t

- Or GPKY Eqs. (1-sub Madrid group)

Phys.Rev.D 83

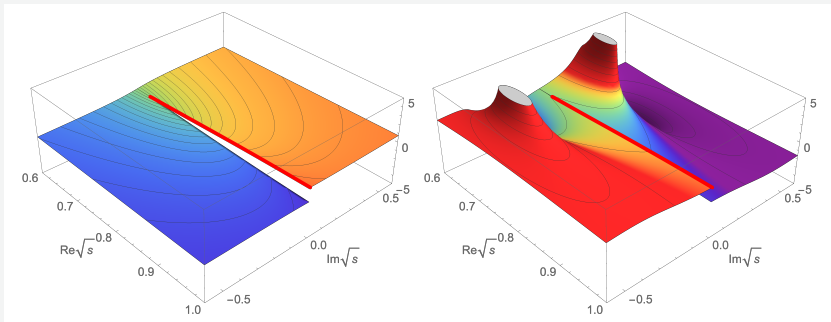
$$\begin{aligned} \text{Re } F^{(l)}(s, t) &= \sum_{l'} C_{st}^{II'} F^{(l')}(4M_\pi^2, 0) + \frac{s}{\pi} \text{P.P.} \int_{4M_\pi^2}^\infty ds' \left[\frac{\text{Im } F^{(l)}(s', t)}{s'(s' - s)} - \frac{\sum_{l''} C_{su}^{II''} \text{Im } F^{(l'')}(s', t)}{(s' + t - 4M_\pi^2)(s' + s + t - 4M_\pi^2)} \right] \\ &+ \frac{t - 4M_\pi^2}{\pi} \text{P.P.} \int_{4M_\pi^2}^\infty ds' \sum_{l''} C_{st}^{II''} \left[\frac{\text{Im } F^{(l'')}(s', 0)}{(s' - t)(s' - 4M_\pi^2)} - \frac{\sum_{l'''} C_{su}^{I'''} \text{Im } F^{(l''')}(s', 0)}{s'(s' + t - 4M_\pi^2)} \right] \\ t_\ell^{(l)}(s) &= \overline{ST}_\ell^l(s) + \sum_{l'=0}^2 \sum_{\ell'=0}^{\ell_{\max}} \int_{4M_\pi^2}^{s_{\max}} ds' \overline{K}_{\ell\ell'}^{II''}(s, s') \text{Im } t_{\ell'}^{II''}(s') + \overline{DT}_\ell^l(s), \end{aligned}$$

- Approach:
 - Use data as constrain
 - Numerical minimization of the distances
 - Very precise determination of LEP



Reminder: Unitarity

- **UNITARITY** \Leftrightarrow probability $\sum |\langle f|S|i\rangle|^2 = 1$
- Elastic unitarity $\rightarrow S^{II}(z) = \frac{1}{S^I(z)}$
- Zero of $S^I(z) \rightarrow$ pole of $S^{II}(z)$

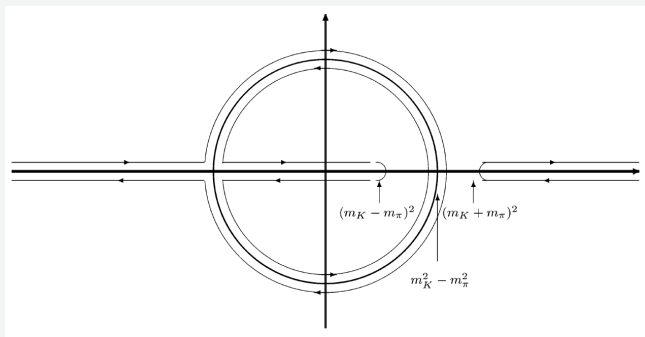


The κ resonance

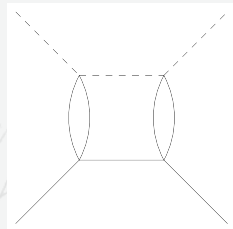
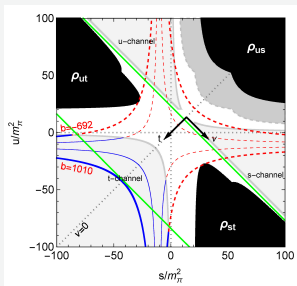
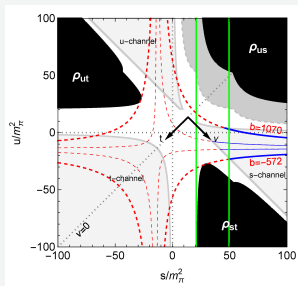
Peláez,AR (Phys.Rev.Lett. 172001)

- Several different models and methods used to determine its parameters.
- Clear convergence with the use of analytic techniques.
- Model dependent determinations not suitable for this scenario.
- Model independent: \rightarrow Padé (before), HDR (next)

$$S^H(s) = \frac{1}{S^L(s)}.$$



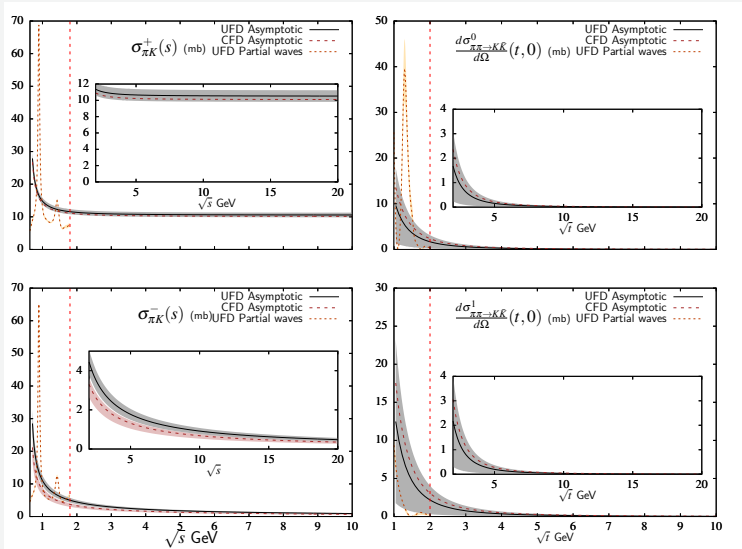
- Dispersion relations obeying $(s-a)(u-a) = b$. Most previous works $\rightarrow a = 0$.
- This work: a used to maximize applicability region.



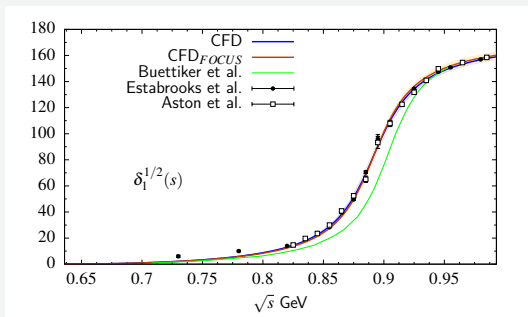
$\pi\pi \rightarrow K\bar{K}$

Peláez, AR 2010.11222

■ Regge physics constrains



- Compatible with $D^+ \rightarrow K^- \pi^+ \mu^+ \nu$ by the FOCUS collab.
- Compatible with previous dispersive approaches to τ decays and form factors
- Compatible with $K_{\ell 3}$ decays.



Scattering lengths: OLD CFD

SL	UFD	CFD	Roy-Steiner result
$m_\pi a_0^{1/2}$	0.222 ± 0.014	0.218 ± 0.014	0.224 ± 0.022
$m_\pi a_0^{3/2}$	-0.101 ± 0.03	-0.054 ± 0.014	-0.0448 ± 0.0077
$m_\pi^3 a_1^{1/2}$	0.031 ± 0.008	0.024 ± 0.005	0.019 ± 0.001

- Dirac collaboration measured the difference between the scalar scattering lengths.

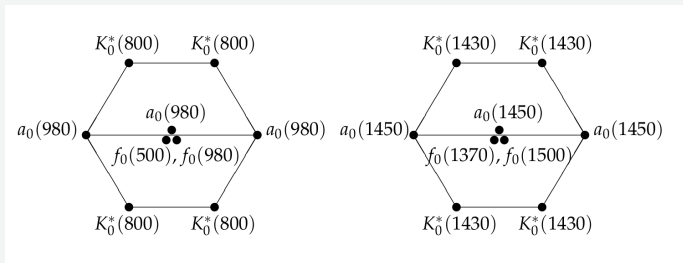
$$\frac{1}{3} \left(a_0^{1/2} - a_0^{3/2} \right) = 0.11_{-0.04}^{+0.09} m_\pi^{-1}, \quad (\text{DIRAC})$$

- Our results are compatible with Roy-Steiner equations, although there is tension with $\pi\pi \rightarrow K\bar{K}$ Sum Rule

$$\frac{1}{3} \left(a_0^{1/2} - a_0^{3/2} \right) = 0.091_{-0.005}^{+0.006} m_\pi^{-1}. \quad (\text{CFD})$$

$$\frac{1}{3} \left(a_0^{1/2} - a_0^{3/2} \right) = 0.075 \pm 0.006 m_\pi^{-1}. \quad (\text{Sum rule})$$

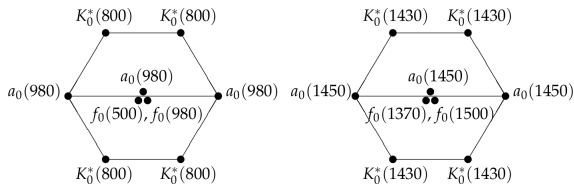
Spectroscopy for strange states



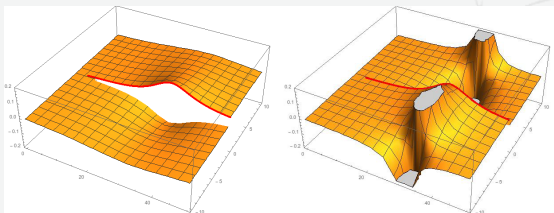
- Precise determination using model independent techniques.
- We can study more than **6 resonances** appearing in πK .
- Another 4 appearing in $\pi\pi \rightarrow K\bar{K}$ scattering.
- Used to determine the $f_0(500)/\sigma$, the $K_0^*(700)/\kappa$, etc...

Meson Spectroscopy

Eur.Phys.J. C77 91

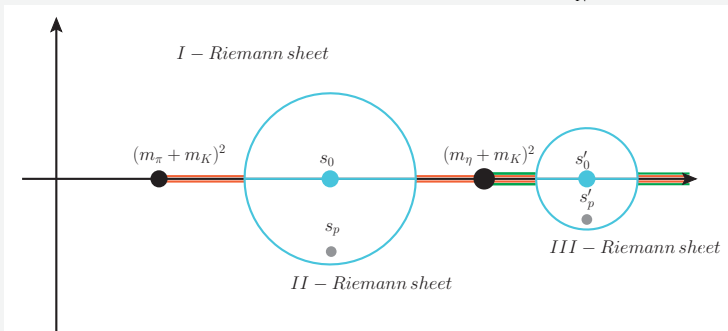


- Resonances \rightarrow poles in **unphysical sheets**
- Analytic continuation is usually model dependent \rightarrow precise and model independent determination using **S-matrix principles**.

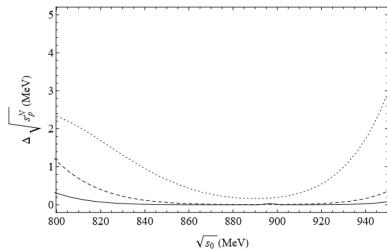
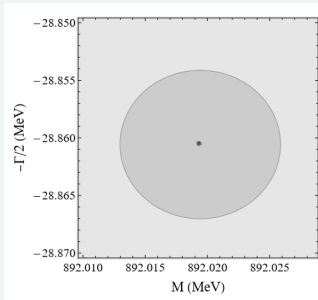


- High L or broad resonance parameters not stable when using simple models. Customary $(q(s)/q(s_r))^L$ and $B_L(q, q_r) \Rightarrow$ deviations.
- Rigorous dispersive techniques cannot get to the poles at higher energies.
- Partial wave is described by a Padé approximant Eur.Phys.J.C 73 2594

$$t_l(s) \simeq P_1^N(s, s_0) = \sum_{k=0}^{N-1} a_K (s - s_0)^k + \frac{a_N (s - s_0)^N}{1 - \frac{a_{N+1}}{a_N} (s - s_0)}.$$



- We stop at a N ($N + 1$ derivatives) where the systematic uncertainty is smaller than the statistical one (usually $N = 4$ is enough).
- s_0 fixed \rightarrow gives the minimum difference between N and $N + 1$.
- Run a Montecarlo for every fit to calculate the parameters and errors of each resonance.
- Different fitting functions included as systematics.



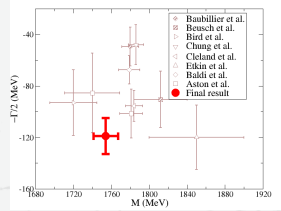
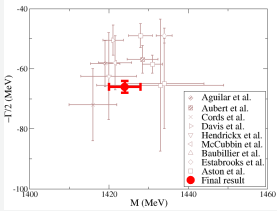
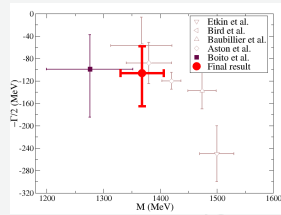
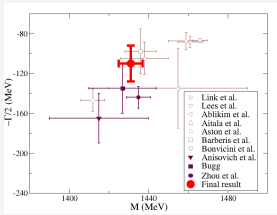
Meson Spectroscopy

Eur.Phys.J. C77 91

- $K_0^*(700)$ Padé → triggewmtwilight the change of name from $K_0^*(800)$.

$$\sqrt{s_p} = (670 \pm 18) - i(295 \pm 28) \text{ MeV}$$

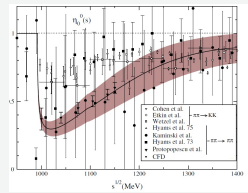
$$\sqrt{s_p} = (682 \pm 29) - i(274 \pm 12) \text{ MeV (PDG)}$$



- $K^*(1430)$, $K^*(1410)$, $K^*(1420)$ and $K^*(1780)$ vs PDG list

Very preliminary: $f_0(1370)$

- Original $\pi\pi$ CFD \rightarrow a pole exists \rightarrow too unstable

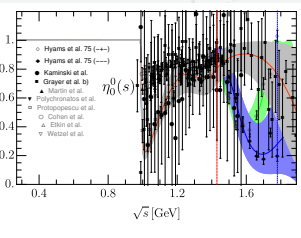
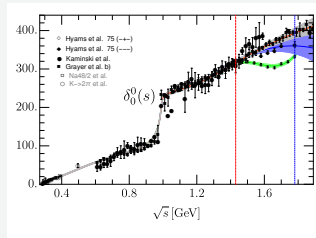


- Analytic parameterization (Eur.Phys.J.C 79 12)

- Padè extraction $\sqrt{s_p} \simeq (1.23 \pm 0.02) - i(0.21 \pm 0.02)$ GeV
- Continuous fractions $\sqrt{s_p} \simeq (1.24 \pm 0.02) - i(0.22 \pm 0.02)$ GeV

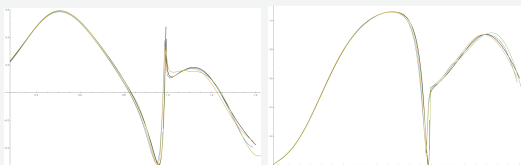
Phys.Lett.B 774 411-416

- However the systematics are large \rightarrow deviations from this particular param.
- Could the pole even disappear?

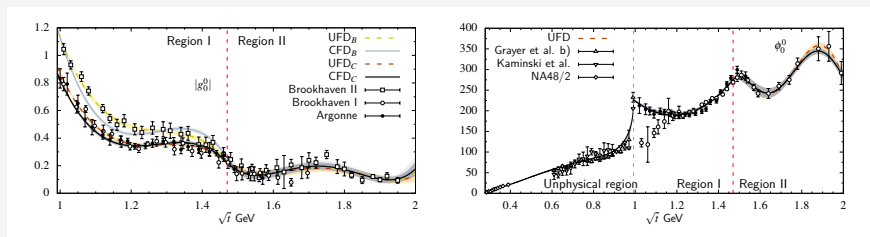


Very preliminary: $f_0(1370)$

- We extend $\pi\pi$ DR beyond original region $\sqrt{s_{max}} = 1.15 \rightarrow 1.3$ GeV



- Original and new CFD $\rightarrow \sqrt{s_p} \simeq (1.31 \pm 0.04) - i(0.22 \pm 0.03)$ GeV
- Crossed channel $\pi\pi \rightarrow K\bar{K} \rightarrow$ another stable pole



- CFD $\rightarrow \sqrt{s_p} \simeq (1.35 \pm 0.05) - i(0.24 \pm 0.04)$ GeV

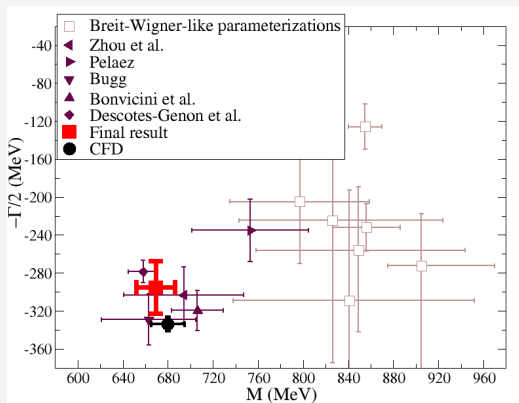
The κ resonance

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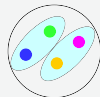
- $K_0^*(700)$ Padé \rightarrow triggered the change of name from $K_0^*(800)$.

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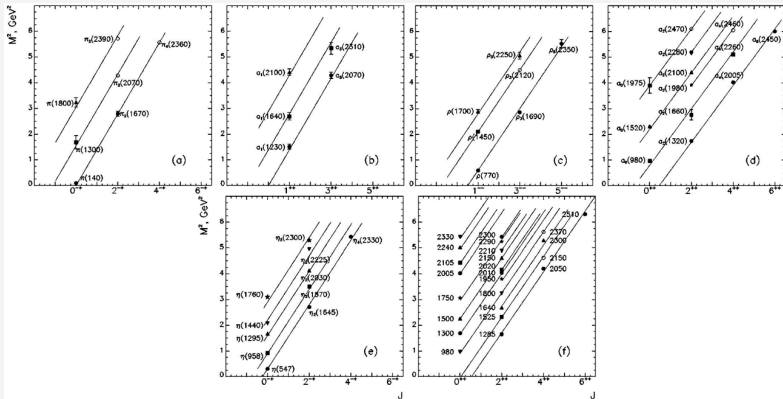


Regge Theory



Eur.Phys.J. C77

- For ordinary resonances: All hadrons are classified in linear (J, M^2) trajectories. PhysRevD.62.05150
- σ and κ -mesons are not included in these plots.



- The contribution of a single pole to a partial wave is

$$t(J, s) = t_{background} + \frac{\beta(s)}{J - \alpha(s)} \approx \frac{\beta(s)}{J - \alpha(s)}$$

- $\alpha(s)$ is the position of the pole, whereas $\beta(s)$ is the residue.
- Unitarity condition on the real axis implies

$$Im\alpha(s) = \rho(s)\beta(s)$$

- The analytical properties of $\beta(s)$ implies

$$\beta(s) = \frac{\hat{s}^{\alpha(s)}}{\Gamma(\alpha(s) + 3/2)} \gamma(s)$$

- Following coupled integral eqs.

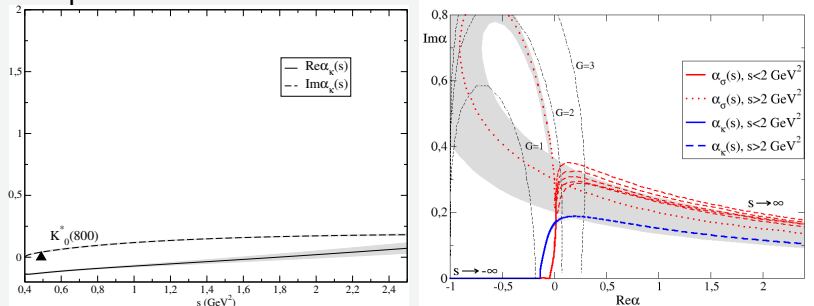
$$\text{Re } \alpha(s) = \alpha_0 + \alpha' s + \frac{s}{\pi} PV \int_{m_+^2}^{\infty} ds' \frac{\text{Im } \alpha(s')}{s'(s' - s)}$$

$$\text{Im } \alpha(s) = \frac{\rho(s) b_0 \hat{s}^{\alpha_0 + \alpha' s}}{|\Gamma(\alpha(s) + \frac{3}{2})|} \exp\left(-\alpha' s [1 - \log(\alpha' s_0)]\right) + \frac{s}{\pi} PV \int_{m_+^2}^{\infty} ds' \frac{\text{Im } \alpha(s') \log \frac{\hat{s}}{s'} + \arg \Gamma(\alpha(s') + \frac{3}{2})}{s'(s' - s)},$$

$$\beta(s) = \frac{b_0 \hat{s}^{\alpha_0 + \alpha' s}}{\Gamma(\alpha(s) + \frac{3}{2})} \exp\left(-\alpha' s [1 - \log(\alpha' s_0)]\right) + \frac{s}{\pi} \int_{m_+^2}^{\infty} ds' \frac{\text{Im } \alpha(s') \log \frac{\hat{s}}{s'} + \arg \Gamma(\alpha(s') + \frac{3}{2})}{s'(s' - s)},$$

- 3 Constants fixed \leftrightarrow fitting pole position and residue

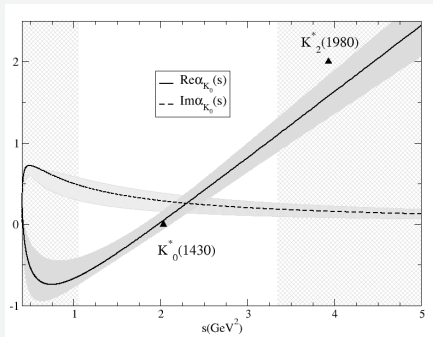
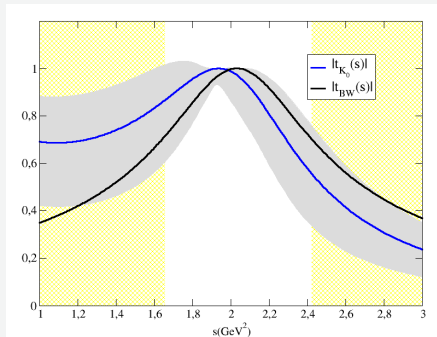
- Slope \rightarrow almost 10 times smaller



- Striking similarity with Yukawa potentials at low energy:
 $V(r) = Ga \times \exp(r/a)/r$.
- Similar order of magnitude for range: $a_{\pi\pi} = 0.5 \text{ GeV}^{-1}$ and $a_{\pi K} = 0.32 \text{ GeV}^{-1}$.
- We obtain that $a_{\pi\pi}/a_{\pi K} \approx \mu_{\pi K}/\mu_{\pi\pi}$.

$K_0^*(1430)$ resonance

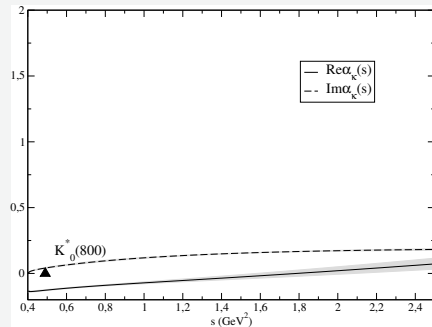
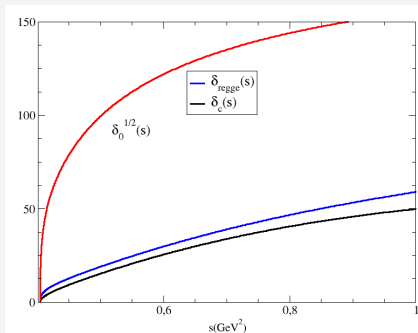
Eur.Phys.J. C77



- The result obtained with our method is compatible near the pole.
- It is almost linear.
- Intercept $\alpha_0 = -1.15^{+0.23}_{-0.15}$, and Slope $\alpha' = 0.81 \pm 0.1 \text{ GeV}^{-2}$.

K resonance

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- Imposing a linear Regge trajectory \rightarrow huge deviation from data.
- Trajectory very far from real, slope 6 times smaller than usual.
- Intercept $\alpha_0 = -0.28 \pm 0.02$, slope $\alpha' = 0.16 \pm 0.03 \text{GeV}^{-2}$.

Future project: New HDR

- It's been shown that symmetric variables under s, t, u exchanges offer the biggest convergence in the complex plane.
- Maximum energy in the real axis \rightarrow 1.7 GeV.
- It offers two possibilities:
 - 1- Select between incompatible data sets above 1.4 GeV.
 - 2- Determine if the $f_0(1370), f_0(1500)$ appear in this process \rightarrow glueball related .