Dispersive approach to strong three-body decays of η and η' mesons

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presentation based on:

TI, Kubis, Schneider, and Stoffer, EPJC **77**, 489 (2017)

TI, Kubis, Kupść, and Stoffer, in preparation

Dispersion relations for strong three-body decays

dispersion theory based on **fundamental properties** of the S-matrix:

- analyticity (causality)
- unitarity (probability)
- crossing symm.

Khuri–Treiman (KT) eqs. for three-body decays: [*Khuri and Treiman (1960)*]

- coupled system of dispersion integrals **resumming** the final-state interaction (FSI) for considered two-body subsystems
- relies on low-order **partial-wave expansion & elastic unitarity**
- two-body T-matrix elements/scattering phase shifts as key input

powerful tool to **analyze $\pi\pi$ & $\pi\eta$ FSI effects** in three-body decays:

- correct analytic properties of KT eqs. in realm of validity
- **analytic continuation** into unphysical region possible

Simple three-body decays: $\omega/\phi \to 3\pi$

analyticity: $V(p_1;\lambda) \to \pi(p_2)\,\pi(p_3)\,\pi(p_4)$ **holomorphic** in kin. variables

$$s = (p_2+p_3)^2\,, \quad t = (p_2+p_4)^2\,, \quad u = (p_3+p_4)^2\,, \quad s+t+u = \text{const.}$$

decay amplitude for $V \to 3\pi$ given by [*Niecknig, Kubis, and Schneider (2012)*]

$$\mathcal{H}_\lambda(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\, n^\mu_\lambda(p_1)\, p_2^\nu\, p_3^\alpha\, p_4^\beta\, \mathcal{M}(s,t,u)$$

only **odd partial waves** (Bose symm.) $\Rightarrow$ decay driven by ***P*-waves**

reconstruction thm.: decomposition in single-variable amplitudes (SVAs)

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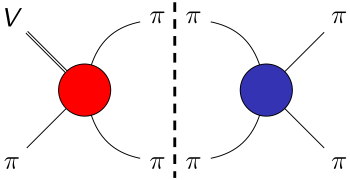
unitarity condition along right-hand cut (RHC):

$$\operatorname{disc} \mathcal{F}_1^1(s) = 2i\, \mathcal{F}_1^1(s)\, \sin\delta_1^1(s)\, \exp\left[-i\delta_1^1(s)\right]$$



$\Rightarrow$ **homogenous Omnès problem:** neglect crossed-channel interactions

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$$\mathcal{F}^1_1(s) = P(s)\,\Omega^1_1(s)\,, \qquad \Omega^1_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty \mathrm{d}x\,\frac{\delta^1_1(x)}{x\,(x-s)}\right\}$$

$\Rightarrow$ form-factor-like solution: **Omnès function** [*Omnès (1958)*]

fulfils **Watson's theorem:** $\arg\Omega^1_1(s) = \delta^1_1(s)$ [*Watson (1954)*]

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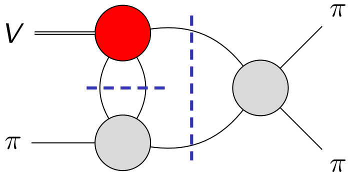

$\Rightarrow$ **inhomogenous Omnès problem:** $\hat{\mathcal{F}}_1^1(s)$ projections of $\mathcal{F}_1^1(t),\ \mathcal{F}_1^1(u)$

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$$\mathcal{F}_1^1(s) = \Omega_1^1(s)\left\{P_{N-1}(s) + \frac{s^N}{\pi}\int_{4M_\pi^2}^\infty \frac{\mathrm{d}x}{x^N}\,\frac{\hat{\mathcal{F}}_1^1(x)\,\sin\delta_1^1(x)}{|\Omega_1^1(x)|\,(x-s)}\right\}$$

$$\hat{\mathcal{F}}_1^1(s) = \frac{3}{2}\int_{-1}^1 \mathrm{d}z\,(1-z^2)\,\mathcal{F}_1^1(t(s,z))$$

$\Rightarrow$ **KT representation** for $\mathcal{F}_1^1$ & $\hat{\mathcal{F}}_1^1$: **scattering phase shift** is key input

Khuri–Treiman equations: solution strategy

- set of **coupled integral equations:**
  - $\mathcal{F}_1^1$ **dispersion relation (DR)** involving $\hat{\mathcal{F}}_1^1$
  - $\hat{\mathcal{F}}_1^1$ **angular projections** of crossed-channel SVAs $\mathcal{F}_1^1$
- **input:** P-wave $\pi\pi$ scattering phase shift δ_1^1
- problem linear in coeffs. of **subtraction polynomial:**
  - construct **basis solutions:** e.g. $P(s) = \alpha \quad \Rightarrow \mathcal{F}_1^1\big|_{\alpha=1}$ & $\hat{\mathcal{F}}_1^1\big|_{\alpha=1}$
  - system solved numerically by iteration or matrix inversion
- **afterwards:** subtraction constants determined by
  - fit to data **(experiments or lattice QCD)**
  - matching to **chiral effective field theories (EFTs)**

The η/η' sector: SM strong three-body decays

→ extension to BSM C-violating transitions: talk by H. Akdag

relatively **small phase space** only allows for one isospin-conserving (IC) and two isospin-breaking (IB) strong three-body channels:

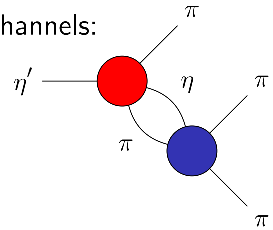
$\eta \to 3\pi$ $\qquad$ $\eta' \to \pi\pi\eta$ $\qquad$ $\eta' \to 3\pi$

- **charged/neutral modes** in all three decay channels
- two-body energy **below inel. $K\bar{K}$ thr.:** consider $\boldsymbol{\pi\pi/\pi\eta}$ **rescatt.**
- **Dalitz-plot distributions:** experimental measurements available
- chiral constraints at the soft-pion points: **Adler zeros**

$\boldsymbol{\pi\eta}$ **inelasticity in** $\boldsymbol{\eta' \to 3\pi}$ connects all three channels:

⇒ allowed to proceed via $\eta' \to \pi\pi\eta$ decay and $\pi\eta \to \pi\pi$ rescattering

⇒ consistent description of $\boldsymbol{\pi\pi/\pi\eta}$ **rescatt.** in all three channels needed



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KT analysis of $\eta' \rightarrow \pi\pi\eta$:

- **main (hadronic) decay channel:** $\mathrm{BR}(\eta' \rightarrow \pi\pi\eta) \approx 65\%$ [*PDG (2020)*]
- **chiral EFTs struggle** with inclusion of η' [*Escribano et al. (2011)*]
- **cusp effect** at $\pi^+\pi^-$ thr. in $\eta' \rightarrow \pi^0\pi^0\eta$ [*Kubis and Schneider (2009)*]

Decomposition of the $\eta' \to \pi\pi\eta$ decay amplitude

decay amplitude can be decomposed into **two SVAs**

$$\mathcal{M}(s,t,u) = \mathcal{F}_0^0(s) + \mathcal{G}_0^1(t) + \mathcal{G}_0^1(u)$$

$\Rightarrow$ $I = 0$ $\pi\pi$- & $I = 1$ $\pi\eta$-systems

***S*-wave contributions only:**

- $\pi\pi$-system: odd partial waves **forbidden by *C*-parity**
- $\pi\eta$-system: *P*-wave has **exotic quantum numbers**
- *D*- and higher partial waves safely neglected: **small phase space**

$\Rightarrow$ rely on *S*-wave $\delta_0^0(s)$ and $\delta_0^1(t)$ scattering phase shifts as input

DR4 subtraction scheme: $\mathcal{M}(s,t,u)$ describes **charged/neutral modes**

$$\eta' \to \pi^+\pi^-\eta / \pi^0\pi^0\eta$$

Dalitz-plot projections for $\eta' \to \pi\pi\eta$

phenomenological parameterisation used in experimental studies:

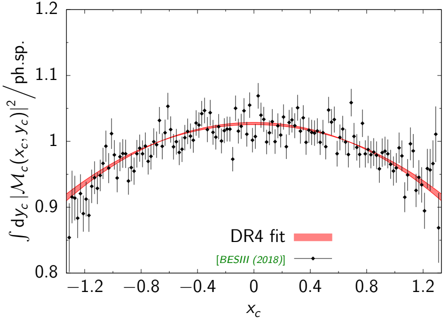
$$|\mathcal{M}(x,y)|^2 \sim 1 + ay + by^2 + \cancel{cx} + dx^2 + \ldots\,, \quad x \sim (t-u)\,, \quad y \sim -s$$

- Dalitz-plot **extremely flat:** $a, b, d \ll 1$
- $c \approx 0 \Rightarrow$ **no** $\boldsymbol{C}$**-violation**

[*VES (2007), GAMS-4π (2009), BESIII (2011), A2 (2018), BESIII (2018)*]

DR4 fit to exp. Dalitz-plot distributions $\Rightarrow \chi^2_{\text{tot}}/\text{ndof} \approx 1.01$

x-projection



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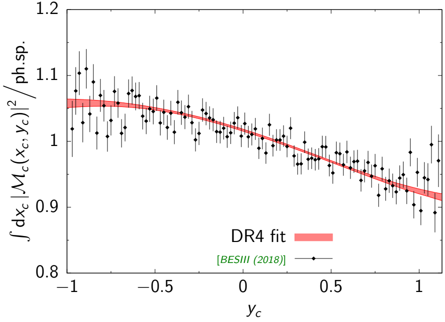
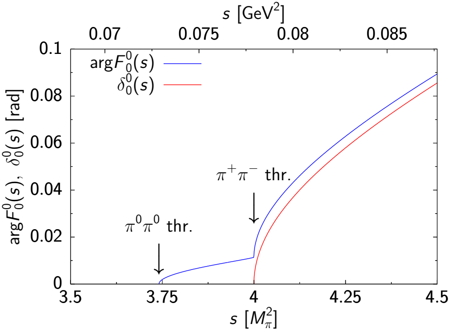
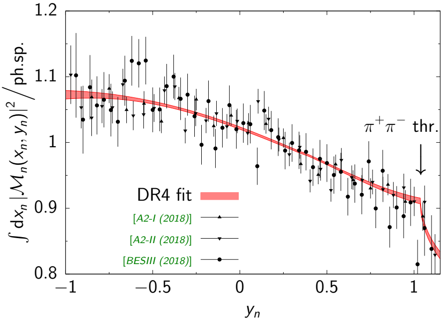


Isospin-breaking effects in $\eta' \to \pi^0\pi^0\eta$: the $\pi^+\pi^-$ cusp

elmag. pion-mass difference: thrs. in $\mathcal{M}(s,t,u)$ not at right places

- $\pi\pi$ phase shift relies on **isospin symmetry** [*Caprini et al. (2012)*]
- construct **effective $\pi^0\pi^0$ phase shift** based on NREFT approach [*Colangelo et al. (2009)*]





$\Rightarrow$ **correct analytic structure** near the $\pi\pi$-thresholds

A soft-pion theorem (SPT) for $\eta' \to \pi(p_\mu) + \pi\eta$

current algebra statement for $\mathcal{M}(s,t,u)$ in soft pion limit $p_\mu \to 0$:

- **two zeros** appear connected by crossing symm.

$$s_A = 0\,, \qquad t_A - u_A = \pm(M_{\eta'}^2 - M_\eta^2)$$

- consequence of chiral $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symm.: **Adler zeros**

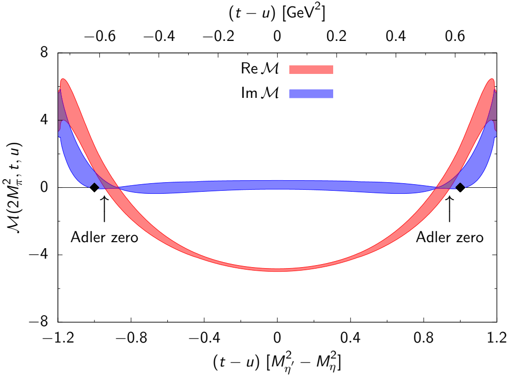
[*Adler (1965), Riazuddin and Oneda (1971)*]

study **critical line** $s = 2M_\pi^2$:

encounter zeros in $\mathcal{M}$ at

$$\frac{t-u}{M_{\eta'}^2 - M_\eta^2} \approx \pm 0.9$$

$\Rightarrow$ DR4 consistent with SPT



Dispersion relations for $\eta \to 3\pi$ decays

***G*-parity breaking decay:** driven by strong IB $\Delta I = 1$ operator

- Sutherland's thm.: elmag. contributions negligible

  [Bell and Sutherland (1966)]

- ChPT relates **amplitude normalization** to $(m_u - m_d)$

  [Osborn and Wallace (1970), Gasser and Leutwyler (1985), Bijnens and Ghorbani (2007)]

$\Rightarrow$ **analyzed in KT context** by various collaborations

[Kambor et al. (1996), Anisovich and Leutwyler (1996), Kampf et al. (2011), Guo et al. (2015), Albaladejo and Moussallam (2017), Colangelo et al. (2018), ...]

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reconstruction thm.: discontinuities of S- & P-waves considered

$$\mathcal{M}(s,t,u) = \mathcal{F}_0^0(s) - \tfrac{2}{3}\mathcal{F}_0^2(s) + (s-u)\,\mathcal{F}_1^1(t) + \mathcal{F}_0^2(t) + [t \leftrightarrow u]$$

isospin symmetry: relates the $\eta \to \pi^+\pi^-\pi^0/3\pi^0$ modes

$$\mathcal{M}_c(s,t,u) = \mathcal{M}(s,t,u)$$

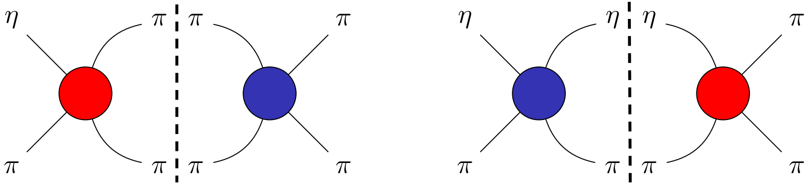
$$\mathcal{M}_n(s,t,u) = \mathcal{M}(s,t,u) + \mathcal{M}(t,u,s) + \mathcal{M}(u,s,t)$$

Modified unitarity condition: initial state $\pi\eta$ rescattering

unitarity condition along RHC:

$$\mathrm{disc}\,\mathcal{F}_\ell^I(s) = 2i\left\{\mathcal{F}_\ell^I(s) + \hat{\mathcal{F}}_\ell^I(s)\right\}\sin\Delta_\ell^I(s)\,\exp\left[-\,i\Delta_\ell^I(s)\right]$$

$\Rightarrow$ $\pi\eta \to \pi\pi$ scattering needs to know of $\pi\pi$ & $\pi\eta$ rescattering:



$$\Delta_0^0(s) \mapsto \delta_0^0(s) + \delta_0^1(s)\,, \qquad \Delta_1^1(s) \mapsto \delta_1^1(s)\,, \qquad \Delta_0^2(s) \mapsto \delta_0^2(s) + \delta_0^1(s)$$

$\Rightarrow$ generates $\boldsymbol{a_0}$**(980)-**$\boldsymbol{f_0}$**(980) mixing** in the $I=0$ $\pi\eta \to \pi\pi$ channel

[Albaladejo and Moussallam (2017)]

Dalitz-plot distributions for $\eta \to 3\pi$

phenomenological parameterisation used in experimental studies:

$$|\mathcal{M}_c(x_c, y_c)|^2 \sim 1 + a y_c + b y_c^2 + d x_c^2 + f x_c^2 y_c + g y_c^3 + \ldots$$

[*KLOE-2 (2016)*]

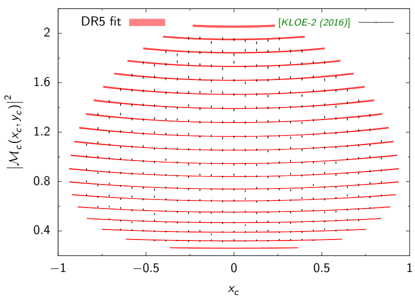
$$|\mathcal{M}_n(x_n, y_n)|^2 \sim 1 + 2\alpha z_n + 2\beta(3x_n^2 y_n - y_n^3) + \ldots, \quad z_n = x_n^2 + y_n^2$$

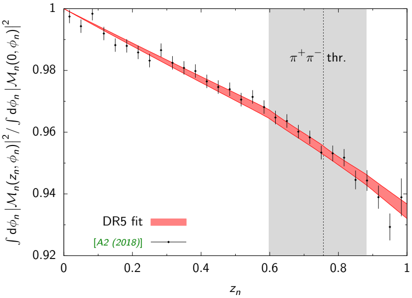
[*A2 (2018)*]

DR5 subtraction scheme: 5 real parameters need to be fixed

- the IB **normalization**
- 4 params. modify the **energy dep.** of $\mathcal{M}$

combined fit to exp. data & chiral constraints on $\mathcal{M}$: $\Rightarrow \chi^2_{\text{tot}}/\text{ndof} \approx 1.09$





Chiral constraints: Adler zero and extraction of Q

→ extraction of Q via $\eta \to 3\pi$: talk by E. Passemar [Colangelo et al. (2018)]

SPT for $\eta \to \pi(p_\mu) + \pi\pi$: **two crossing symm. Adler zeros** at

$$s_A = t_A = 0 \qquad \& \qquad s_A = u_A = 0$$

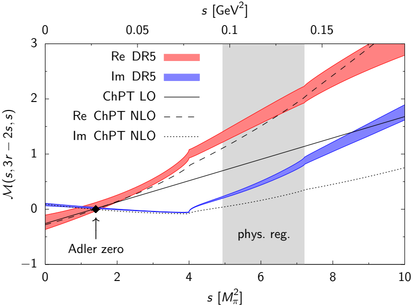
[Adler (1965), Osborn and Wallace (1970)]

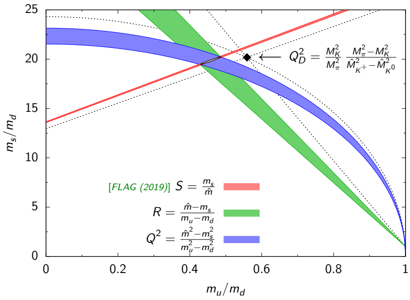
$\Rightarrow$ shifted to $s_A = t_A \approx 1.4 M_\pi^2$ for on-shell pions *[Gasser and Leutwyler (1985)]*

ChPT matching of IB normalization in vicinity of the Adler zero:

$$\Gamma(\eta \to 3\pi) \sim Q^{-4} \int \mathrm{d}x\, \mathrm{d}y\, |\mathcal{M}(x,y)|^2 \qquad \Rightarrow Q_{\mathrm{DR5}} = 22.3(8)$$

[Colangelo et al. (2018): $Q_{\mathrm{DR6}} = 22.1(7)$]





Dispersion relations for $\eta' \to 3\pi$ decays

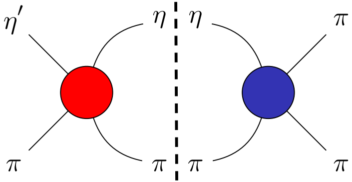
isospin decomposition of decay amplitude: identical to $\eta \to 3\pi$

but: higher decay mass

- richer dynamics $\Rightarrow$ imprints of **lowest resonances visible**
- **inelastic $\pi\eta$ threshold** lies within Dalitz plot

IB $\eta' \to 3\pi$ decay allowed to proceed via:

initial IC $\eta' \to \pi\pi\eta$ transition and subsequent IB $\pi\eta \to \pi\pi$ rescattering



$\Rightarrow$ elastic & inelastic channels perturbatively coupled (IB effect)

modified unitarity condition:

$$\mathrm{disc}\,\mathcal{F}_\ell^I(s) = 2i\,\theta_{\pi\pi}\big\{\mathcal{F}_\ell^I(s) + \hat{\mathcal{F}}_\ell^I(s)\big\}\,\sin\delta_\ell^I(s)\,\exp\big[-i\delta_\ell^I(s)\big] + 2i\,\theta_{\pi\eta}\,\mathcal{I}_\ell^I(s)$$

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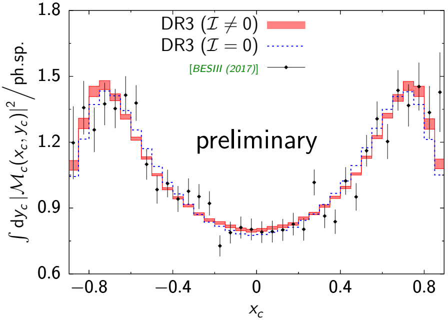
DR3 subtraction scheme: 3 complex parameters need to be fixed

$\Rightarrow$ **log-likelihood maximisation** for raw scatter data [*BESIII (2017)*]

x-projection



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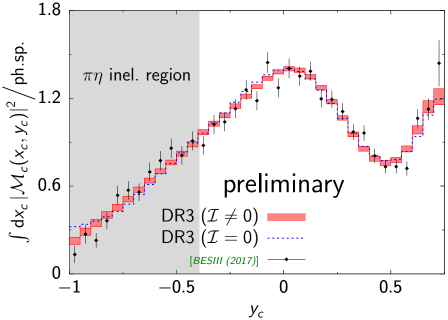


Summary

FSI of $\pi\pi$, $\pi\eta$, ... in three-body decays:

- can be **analyzed systematically** using the KT representation
- KT amplitudes possess **correct analytic properties**
- based on information of **universal two-body *T*-matrix elements**

strong three-body decays in η/η' sector:

- suitable description of **exp. Dalitz-plot distributions**
- analytic continuation into unphysical region: **Adler zeros**
- allows for **matching to chiral constraints**

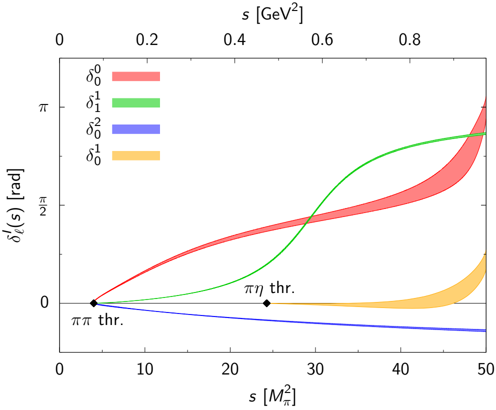
Spares

Scattering phase shifts

elastic unitarity for $\pi\pi/\pi\eta$ scattering demands along RHC

$$\mathrm{Im}\,\mathcal{T}^I_\ell(s) = \frac{\lambda^{1/2}_{PQ}(s)}{s}\left|\mathcal{T}^I_\ell(s)\right|^2 \quad \Rightarrow \quad \mathcal{T}^I_\ell(s) = \frac{s}{\lambda^{1/2}_{PQ}(s)}\,\sin\delta^I_\ell(s)\,\exp\left[i\delta^I_\ell(s)\right]$$

real-valued phase shifts: encode universal scattering information



- ***S*- & *P*-wave $\pi\pi$ phase shifts:** δ^0_0, δ^1_1, δ^2_0
  [*Ananthanarayan et al. (2001), Caprini et al. (2012)*]
- ***S*-wave $\pi\eta$ phase shift:** δ^1_0
  [*Albaladejo and Moussallam (2015), Lu and Moussallam (2020)*]

Khuri–Treiman equations for $\eta' \to \pi\pi\eta$

reconstruction thm.: discontinuities of S-waves only

$$\mathcal{M}(s,t,u) = \mathcal{F}_0^0(s) + \mathcal{G}_0^1(t) + \mathcal{G}_0^1(u)$$

dispersion integrals (DR4): four (real) subtraction constants $\alpha, \beta, \gamma, \varepsilon$

$$\mathcal{F}_0^0(s) = \Omega_0^0(s)\left\{\alpha + \beta s + \gamma s^2 + \frac{s^3}{\pi}\int_{s_{\text{thr}}}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\hat{\mathcal{F}}_0^0(x)\sin\delta_0^0(x)}{|\Omega_0^0(x)|\,(x-s)}\right\}$$

$$\mathcal{G}_0^1(t) = \Omega_0^1(t)\left\{\varepsilon t + \frac{t^3}{\pi}\int_{t_{\text{thr}}}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\hat{\mathcal{G}}_0^1(x)\sin\delta_0^1(x)}{|\Omega_0^1(x)|\,(x-t)}\right\}$$

with scatt. thresholds $s_{\text{thr}} = 4M_\pi^2$ and $t_{\text{thr}} = u_{\text{thr}} = (M_\eta + M_\pi)^2$

angular projections:

$$\hat{\mathcal{F}}_0^0(s) = \int_{-1}^{1}\mathrm{d}z_s\,\mathcal{G}_0^1\big(t(s,z_s)\big)$$

$$\hat{\mathcal{G}}_0^1(t) = \frac{1}{2}\int_{-1}^{1}\mathrm{d}z_t\left\{\mathcal{F}_0^0\big(s(t,z_t)\big) + \mathcal{G}_0^1\big(u(t,z_t)\big)\right\}$$

Khuri–Treiman equations for $\eta \to 3\pi$

dispersion integrals (DR5): $\pi\pi/\pi\eta$ rescatt. contained in Δ_ℓ^I

$$\mathcal{F}_0^0(s) = \Omega_0^0(s)\left\{\alpha + \beta s + \gamma s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^\infty \frac{dx}{x^3}\frac{\hat{\mathcal{F}}_0^0(x)\,\sin\Delta_0^0(x)}{|\Omega_0^0(x)|\,(x-s)}\right\}$$

$$\mathcal{F}_1^1(s) = \Omega_1^1(s)\left\{\varepsilon + \frac{s}{\pi}\int_{4M_\pi^2}^\infty \frac{dx}{x}\frac{\hat{\mathcal{F}}_1^1(x)\,\sin\Delta_1^1(x)}{|\Omega_1^1(x)|\,(x-s)}\right\}$$

$$\mathcal{F}_0^2(s) = \Omega_0^2(s)\left\{\zeta s + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{dx}{x^2}\frac{\hat{\mathcal{F}}_0^2(x)\,\sin\Delta_0^2(x)}{|\Omega_0^2(x)|\,(x-s)}\right\}$$

angular projections:

$$\hat{\mathcal{F}}_0^0(s) = \frac{2}{9}\left\{3\langle\mathcal{F}_0^0\rangle + 9(s-s_0)\langle\mathcal{F}_1^1\rangle + 3\kappa\,\langle z\,\mathcal{F}_1^1\rangle + 10\langle\mathcal{F}_0^2\rangle\right\}$$

$$\hat{\mathcal{F}}_1^1(s) = \frac{1}{2\kappa}\left\{6\langle z\,\mathcal{F}_0^0\rangle + 9(s-s_0)\langle z\,\mathcal{F}_1^1\rangle + 3\kappa\,\langle z^2\,\mathcal{F}_1^1\rangle - 10\langle z\,\mathcal{F}_0^2\rangle\right\}$$

$$\hat{\mathcal{F}}_0^2(s) = \frac{1}{6}\left\{6\langle\mathcal{F}_0^0\rangle - 9(s-s_0)\langle\mathcal{F}_1^1\rangle - 3\kappa\,\langle z\,\mathcal{F}_1^1\rangle + 2\langle\mathcal{F}_0^2\rangle\right\}$$