Three-pion scattering in the Chiral Perturbation Theory

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Beijing, China (virtually)
November 17, 2021

Swedish Research Council

Introduction

In low-energy region, we cannot study perturbatively the interactions of hadrons directly from QCD
↪ alternative approaches → Chiral perturbation theory (ChPT)
Weinberg, Phys.A 96, (1979), *Gasser and Leutwyler*, Ann.Ph.158 (1984)

Many observables are known in ChPT to a high loop order
↪ only recently it has become of interest to calculate the six-pion amplitude at low energies after it has been estimated using lattice QCD

Blanton et al., PRL 124 (2020), JHEP 10 (2021),
Fischer et al., EPJC 81 (2021), *Hansen et al.*, PRL 126 (2021),
Brett et al., PRD 104 (2021)

The six-pion amplitude at tree level was first done using current algebra methods
e.g. *Osborn*, Lett.N.Cim.2 (1969)

It has been redone with Lagrangian methods many times, not known to one-loop order
e.g. *Low et al.*, JHEP 11 (2019), *Bijnens et al.*, JHEP 11 (2019)

We have therefore calculated at NLO the six-pion amplitude,
(as well as the four-pion amplitude, pion mass and decay constant)
↪ within ChPT generalization to the O$(N+1)$/O(N) massive nonlinear sigma model
↪ two(-quark)-flavour ChPT equivalent to O(4)/O(3)

The relation to the measurement of the lattice is nontrivial to implement given
↪ complexity of the three-body finite volume calculations
↪ subtraction of the two-body rescatterings involved

Hansen et al., PRD 90 (2014), PRD 92 (2015), *Hammer et al.*, JHEP 09 (2017), JHEP 10 (2017),
Mai et al., EPJA 53 (2017), PRL 122 (2019), *Romero-López et al.*, JHEP 02 (2021),
Blanton et al., PRD 102 (2020)

Massive $\mathrm{O}(N+1)/\mathrm{O}(N)$ nonlinear sigma model extended beyond the LO

$$\mathcal{L} = \frac{F^2}{2}\,\partial_\mu \Phi^\mathsf{T} \partial^\mu \Phi + F^2 \chi^\mathsf{T} \Phi$$
$$+ l_1 \left(\partial_\mu \Phi^\mathsf{T} \partial^\mu \Phi\right)\left(\partial_\nu \Phi^\mathsf{T} \partial^\nu \Phi\right) + l_2 \left(\partial_\mu \Phi^\mathsf{T} \partial_\nu \Phi\right)\left(\partial^\mu \Phi^\mathsf{T} \partial^\nu \Phi\right) + l_3 \left(\chi^\mathsf{T} \Phi\right)^2 + l_4 \partial_\mu \chi^\mathsf{T} \partial^\mu \Phi$$

Φ: real vector of $N+1$ components, $\Phi^\mathsf{T}\Phi = 1$
$\chi^\mathsf{T} = \left(M^2, \vec{0}\right)$

F, M: bare pion decay constant and mass

↪ calculate the four-pion and six-pion amplitudes at NLO

External fields can be added as in *Gasser and Leutwyler*, Ann.Ph.158 (1984)

The coefficients (low-energy constants) l_i are free parameters in the theory
↪ both UV divergent and finite parts

$$l_i = (c\mu)^{d-4} \left(\frac{1}{16\pi^2} \frac{1}{d-4}\, \gamma_i + l_i^\mathrm{r} \right), \quad \log c = -\frac{1}{2}\left(1 - \gamma_\mathsf{E} + \log 4\pi\right)$$

Massive O$(N+1)$/O(N) nonlinear sigma model extended beyond the LO

$$\mathcal{L} = \frac{F^2}{2}\,\partial_\mu\Phi^\mathsf{T}\partial^\mu\Phi + F^2\chi^\mathsf{T}\Phi$$
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$$l_i = l_i^\mathrm{r} - \frac{1}{16\pi^2}\,\frac{\gamma_i}{2}\left(\frac{2}{4-d} - \gamma_\mathsf{E} + \log 4\pi - \log\mu^2 + 1\right), \qquad l_i^r = \frac{1}{16\pi^2}\,\frac{\gamma_i}{2}\left(\bar{l}_i + \ln\frac{M_\pi^2}{\mu^2}\right)$$

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From studying the pion mass, decay constant and the four-pion amplitude

$$\gamma_3 = 1 - \frac{N}{2} \qquad \gamma_1 = \frac{N}{2} - \frac{7}{6}$$
$$\gamma_4 = N - 1 \qquad \gamma_2 = \frac{2}{3}$$

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$$\Phi_1 = \left(\sqrt{1-\varphi},\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

Gasser and Leutwyler, Ann.Ph.158 (1984)

$$\Phi_2 = \frac{1}{\sqrt{1+\varphi}}\left(1,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

simple variation

$$\Phi_3 = \left(1-\frac{1}{2}\varphi,\, \sqrt{1-\frac{1}{4}\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

ESB term only gives mass terms of ϕ_is

$$\Phi_4 = \left(\cos\sqrt{\varphi},\, \frac{1}{\sqrt{\varphi}}\sin\sqrt{\varphi}\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

follows the general prescription from *Coleman, Wess and Zumino*, PR 177 (1969)

$$\Phi_5 = \frac{1}{1+\frac{1}{4}\varphi}\left(1-\frac{1}{4}\varphi,\, \frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T}$$

Weinberg, PR 166 (1968)

$\varphi \equiv \frac{\boldsymbol{\phi}^\mathsf{T}\boldsymbol{\phi}}{F^2}$, with $\boldsymbol{\phi}^\mathsf{T} = (\phi_1, \ldots, \phi_N)$ a real vector of N components (flavours)

$\hookrightarrow$ few examples of the whole class of parametrizations

$$\Phi = \left(\sqrt{1-\varphi\, f^2(\varphi)},\, f(\varphi)\,\frac{\boldsymbol{\phi}^\mathsf{T}}{F}\right)^\mathsf{T},$$

with $f(x)$ any analytical function satisfying $f(0) = 1$

Four-pion amplitude

On-shell amplitude in general

p_i, $i = 1, \ldots, 4$ pion incoming four-momenta, $\sum p_i = 0$

i_i flavours

Invariance under rotation in the isospin space and crossing symmetry implies

$$A_{4\pi}(p_1, i_1, p_2, i_2, p_3, i_3, i_4)$$
$$= \delta_{i_1 i_2}\delta_{i_3 i_4} A(p_1, p_2, p_3) + \delta_{i_1 i_3}\delta_{i_2 i_4} A(p_3, p_1, p_2) + \delta_{i_2 i_3}\delta_{i_1 i_4} A(p_2, p_3, p_1)$$

Mandelstam variables

$s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2$, $s + t + u = 4M^2$

$\hookrightarrow$ subamplitude $A(p_1, p_2, p_3) = A(s, t, u)$

Up-to-and-including $\mathcal{O}(p^4)$, order by order

$$A(s,t,u) = A^{(2)}(s,t,u) + A^{(4)}(s,t,u)$$

Four-pion amplitude

Leading order

The leading-order $\mathcal{O}(p^2)$ amplitude stems from a single diagram

$\hookrightarrow$ schematically $A_{4\pi}^{(2)} = \mathcal{M}_{\mathsf{LO}}^{(2)}\big|_{\text{on-shell}}$

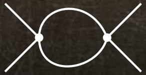
Related LO subamplitude (with LO relations $M \to M_\pi$ and $F \to F_\pi$)

$$A^{(2)}(s,t,u) = \frac{1}{F_\pi^2}\left(s - M_\pi^2\right)$$

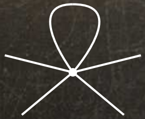
Four-pion amplitude

Next-to-leading order

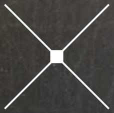
At NLO, one-loop diagrams (two topologies of 4 one-loop diagrams in total) and a counterterm



(a) $3\times$



(b) $1\times$



(c) $1\times$

$+$ NLO field renormalization, and mass and decay-constant redefinitions applied to the LO graph
$\hookrightarrow$ schematically $A_{4\pi}^{(4)} = \mathcal{M}_{\text{1-loop}} + \mathcal{M}_{\text{CT}} + 4(Z^{1/2}-1)\mathcal{M}_{\text{LO}}^{(2)} + \mathcal{M}_{\text{LO}}^{(4)}$

The Z factor is related to the pion self-energy Σ

$$\frac{1}{Z} = 1 - \frac{\partial\Sigma(p^2)}{\partial p^2}\bigg|_{p^2=M_\pi^2}$$

Standard relations $M_\pi^2 = M^2 - \overline{\Sigma}$, $F_\pi = F(1+\delta F)$ give the substitutions at the given order

$$M^2 \to M_\pi^2 + \overline{\Sigma}\,, \qquad \overline{\Sigma} = \frac{M_\pi^4}{F_\pi^2}\left[2l_3^{\mathrm{r}} + \frac{1}{2}(N-2)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

$$\frac{1}{F^2} \to \frac{1}{F_\pi^2}(1+2\delta F)\,, \quad \delta F = \frac{M_\pi^2}{F_\pi^2}\left[l_4^{\mathrm{r}} - \frac{1}{2}(N-1)L\right] + \mathcal{O}\left(\frac{1}{F_\pi^4}\right)$$

Parametrization-independent and UV-finite result

$$
\begin{aligned}
F_\pi^4 A^{(4)}(s,t,u) = {} & (t-u)^2\left(-\frac{5}{36}\,\kappa-\frac{1}{6}\,L+\frac{1}{2}\,l_2^{\mathrm{r}}\right)\\
& + M_\pi^2 s\left[\left(N-\frac{29}{9}\right)\kappa+\left(N-\frac{11}{3}\right)L-8l_1^{\mathrm{r}}+2l_4^{\mathrm{r}}\right]\\
& + s^2\left[\left(\frac{11}{12}-\frac{N}{2}\right)\kappa+\left(1-\frac{N}{2}\right)L+2l_1^{\mathrm{r}}+\frac{1}{2}\,l_2^{\mathrm{r}}\right]\\
& + M_\pi^4\left[\left(\frac{20}{9}-\frac{N}{2}\right)\kappa+\left(\frac{8}{3}-\frac{N}{2}\right)L+8\,l_1^{\mathrm{r}}+2l_3^{\mathrm{r}}-2l_4^{\mathrm{r}}\right]\\
& + \bar{J}(s)\left[\left(\frac{N}{2}-1\right)s^2+(3-N)M_\pi^2 s+\left(\frac{N}{2}-2\right)M_\pi^4\right]\\
& + \left\{\frac{1}{6}\,\bar{J}(t)\left[2t^2-10M_\pi^2 t-4M_\pi^2 s+st+14M_\pi^4\right]+(t\leftrightarrow u)\right\}
\end{aligned}
$$

Above we used

$$
\kappa=\frac{1}{16\pi^2}\,,\quad L\equiv\kappa\log\frac{M_\pi^2}{\mu^2}\,,\quad \bar{J}(q^2)\equiv\kappa\left(2+\sigma\log\frac{\sigma-1}{\sigma+1}\right),\quad \sigma=\sqrt{1-\frac{4M_\pi^2}{q^2}}
$$

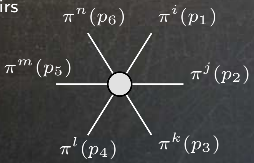
Form as given in *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997), generalized to $N\neq 3$
↪ somewhat different from the form given in *Gasser and Leutwyler*, Ann.Ph.158 (1984)
  ↪ equivalent to a given order but different off-shell extrapolations

The expressions agree with the known results
↪ for $N=3$, *Gasser and Leutwyler*, Ann.Ph.158 (1984)
↪ on the N dependence, e.g. *Dobado and Morales*, PRD 52 (1995),
Bijnens and Carloni, NPB 827 (2010), NPB 843 (2011)

Six-pion amplitude

4 pions $\rightarrow$ 3 channels/permutations/ways to distribute 4 pions in 2 pairs

6 pions $\rightarrow$ 10 ways in 2 groups of three (P_{10})
$\hookrightarrow$ 15 ways in 3 pairs (P_{15})



The full six-pion amplitude at $\mathcal{O}(p^4)$

$$A_{6\pi} = A_{6\pi}^{(4\pi)} + A_{6\pi}^{(6\pi)}$$

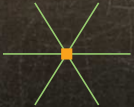
$A_{6\pi}^{(4\pi)}$ can be written in terms of the four-pion amplitude and $A_{6\pi}^{(6\pi)}$ is the remainder

(a) $1\times$

(b) $10\times$

Six-pion amplitude

Feynman diagrams of relevant topologies



(a) 1×

(b) 1×

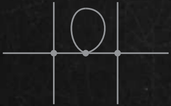
(c) 15×

(d) 20×

(e) 20×

(f) 60×

(g) 10×



(h) 10×

(i) 15×

Six-pion amplitude
The pole part

$$A_{6\pi}^{(4\pi)} \equiv \sum_{P_{10},o} A_{4\pi}(p_i, i_i, p_j, i_j, p_k, i_k, o) \frac{(-1)}{p_{ijk}^2 - M_\pi^2} A_{4\pi}(p_l, i_l, p_m, i_m, p_n, i_n, o)$$

$\hookrightarrow$ residue at the pole unique, off-shell extrapolation away from $p_{ijk}^2 \equiv (p_i + p_j + p_k)^2 = M_\pi^2$ not
$A_{4\pi}(p_i, i_i, p_j, i_j, p_k, i_k, o)$ is the four-pion amplitude with one leg off-shell

$$\begin{aligned} &A_{4\pi}(p_i, i_i, p_j, i_j, p_k, i_k, o) \\ &= \delta_{i_i i_j}\delta_{i_k o} A(p_i, p_j, p_k) + \delta_{i_i i_k}\delta_{i_j o} A(p_k, p_i, p_j) + \delta_{i_j i_k}\delta_{i_i o} A(p_j, p_k, p_i) \end{aligned}$$

The (four-pion) subamplitude $A(p_i, p_j, p_k) = A(s, t, u)$ is defined as usual
$\hookrightarrow$ $s = (p_i + p_j)^2$, $t = (p_i + p_k)^2$ and $u = (p_j + p_k)^2$, although now $s + t + u = 3M_\pi^2 + p_{ijk}^2$

We have chosen a particular form for the off-shell four-pion subamplitude $A(s, t, u)$
$\hookrightarrow$ other off-shell extrapolations are possible and will lead to a different $A_{6\pi}^{(6\pi)}$
$\hookrightarrow$ independent of the parametrization used
- $\hookrightarrow$ also $A_{4\pi}$ and, consequently, the respective parts $A_{6\pi}^{(4\pi)}$ and $A_{6\pi}^{(6\pi)}$ by definition

$\hookrightarrow$ the way how the contributions from the one-particle irreducible and reducible diagrams are distributed within the final result is parametrization dependent

$$A_{6\pi}^{(6\pi)} \equiv \sum_{P_{15}} \delta_{i_i i_j} \delta_{i_k i_l} \delta_{i_m i_n} A(p_i, p_j, p_k, p_l, p_m, p_n)$$

The (six-pion) subamplitude $A(p_1, p_2, p_3, p_4, p_5, p_6)$

↪ no poles, only cuts (however, the imaginary part of the triangle integrals can contain poles)

↪ function of three pairs of momenta

- ↪ fully symmetric under the interchange of any of the pairs
- ↪ symmetric for the interchange within a pair

The six-pion subamplitude respecting orders in the expansion

$$A(p_1, p_2, p_3, p_4, p_5, p_6) = A^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) + A^{(4)}(p_1, p_2, p_3, p_4, p_5, p_6)$$

Six-pion amplitude

Leading order

(a) 1× (b) 10×

The full six-pion amplitude at $\mathcal{O}(p^4)$

$$A_{6\pi} = A_{6\pi}^{(4\pi)} + A_{6\pi}^{(6\pi)}$$

↪ (a) only contributes to $A_{6\pi}^{(6\pi)}$

↪ (b) contributes to both the pole and non-pole parts $A_{6\pi}^{(4\pi)}$ and $A_{6\pi}^{(6\pi)}$

At LO a simple expression

$$A^{(2)}(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{1}{F_\pi^4}\left(2p_1 \cdot p_2 + 2p_3 \cdot p_4 + 2p_5 \cdot p_6 + 3M_\pi^2\right)$$

↪ dependence on momenta is the only one at this order compatible with the symmetries

Six-pion amplitude
Next-to-leading order

The main new result is the next-order six-pion subamplitude
$\hookrightarrow$ split it up into numerous parts:

$$F_\pi^6 A^{(4)}(p_1, p_2, \ldots, p_6) = A_{C_3} + A^{(1)}_{C_{21}} + A^{(2)}_{C_{21}} + A_{C_{11}} + A^{(1)}_{C} + A^{(2)}_{C} + A^{(3)}_{C} + A^{(1)}_{J} + A^{(2)}_{J} + A_\pi + A_L + A_l$$

(suppressed the arguments $(p_1, p_2, \ldots, p_6) \equiv (p_1, p_2, p_3, p_4, p_5, p_6)$)

$\hookrightarrow$ each of the terms has the required symmetries under interchange of momenta

Large number of kinematic invariants $\rightarrow$ reduction to master integrals (scalar triangle integrals) leads to an enormous expression

$\hookrightarrow$ we have chosen a redundant basis of integrals that have good symmetry properties

Results are rather lengthy, but can be written in a relatively compact way
$\hookrightarrow$ see paper PRD 104 (2021) 054046, arXiv:2107.06291

We choose a symmetric $3 \to 3$ scattering configuration given by

$$p_1 = \left(E_p, p, 0, 0\right) \qquad p_4 = \left(-E_p, 0, 0, p\right)$$

$$p_2 = \left(E_p, -\frac{1}{2}p, \frac{\sqrt{3}}{2}p, 0\right) \qquad p_5 = \left(-E_p, \frac{\sqrt{3}}{2}p, 0, -\frac{1}{2}p\right)$$

$$p_3 = \left(E_p, -\frac{1}{2}p, -\frac{\sqrt{3}}{2}p, 0\right) \qquad p_6 = \left(-E_p, -\frac{\sqrt{3}}{2}p, 0, -\frac{1}{2}p\right)$$

We use following numerical inputs:

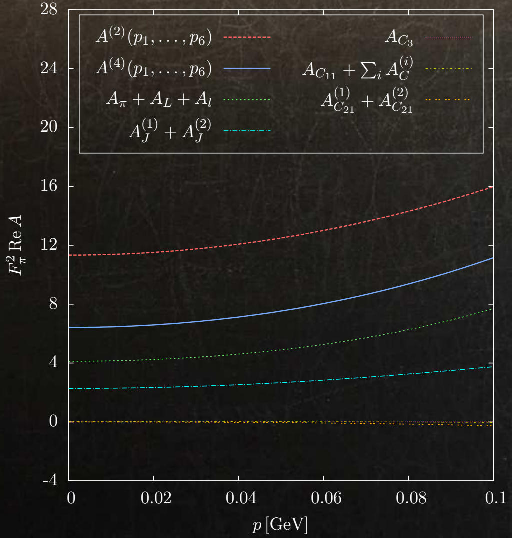
$$M_\pi = 0.139570\,\text{GeV} \qquad \bar{l}_1 = -0.4$$

$$F_\pi = 0.0927\,\text{GeV} \qquad \bar{l}_2 = 4.3$$

$$\mu = 0.77\,\text{GeV} \qquad \bar{l}_3 = 3.41$$

$$N = 3 \qquad \bar{l}_4 = 4.51$$

Bijnens, Ecker, ARNPS 64 (2014)
Colangelo, Gasser, Leutwyler, NPB 603 (2001)
Aoki et al., EPJC 77 (2017)



$F_\pi^2 \mathrm{Re} A$			
$A_{6\pi}^{(4\pi)}$ (LO)	-319.00	$A^{(2)}(p_1,\ldots,p_6)$	15.99
$A_{6\pi}^{(4\pi)}$ (NLO)	-28.54	$A^{(4)}(p_1,\ldots,p_6)$	11.16
$F_\pi^2 \times \mathrm{Re} A / F_\pi^6$			
A_{C_3}	0.002	$A_J^{(1)}$	1.917
$A_{C_{21}}^{(1)}$	-0.948	$A_J^{(2)}$	1.835
$A_{C_{21}}^{(2)}$	0.682	A_π	-2.488
$A_{C_{11}}$	0.090	A_L	8.985
$A_C^{(1)}$	-0.026	A_l	1.209
$A_C^{(2)}$	0.890		
$A_C^{(3)}$	-0.984		

Real parts of the amplitudes for $p = 0.1$ GeV

In the limit $p \to 0$, we find the following analytical expressions:

$$F_\pi^2 A^{(2)}(p_1, p_2, \ldots, p_6)\big|_{p\to 0} = 5\,\frac{M_\pi^2}{F_\pi^2} \approx 11.33$$

$$F_\pi^2\,\mathrm{Re}\,A^{(4)}(p_1, p_2, \ldots, p_6)\big|_{p\to 0}$$
$$= \frac{M_\pi^4}{F_\pi^4}\Bigg\{ \underbrace{\frac{1}{18}(-2-225N)\kappa}_{A_\pi} + \underbrace{\frac{1}{6}(-14-75N)L}_{A_L} + \underbrace{(16l_1^r + 56l_2^r + 6l_3^r + 20l_4^r)}_{A_l}$$
$$+ \underbrace{(-44+30N)\kappa}_{A_J^{(1)}} + \underbrace{(24)\kappa}_{A_J^{(2)}} + \frac{1}{2}\kappa\Big[\underbrace{-(30-9N)}_{A_C^{(1)}} + \underbrace{(20)}_{A_C^{(2)}} + \underbrace{(-16)}_{A_C^{(3)}}\Big]\Bigg\} \approx 6.416$$

Summary

We calculated the pion mass, decay constant, the four-pion and six-pion amplitude to NLO in the massive O(N) nonlinear sigma model

$\hookrightarrow$ relevant NLO Lagrangian constructed in analogy with two(-quark)-flavour ChPT Lagrangian
$\hookrightarrow$ our results agree with previous results for $N = 3$ and general-N behaviour

Our main result is the six-pion amplitude
$\hookrightarrow$ split in one-particle reducible and irreducible parts

The reducible part employs the off-shell four-pion amplitude generalizing (beyond $N = 3$) the amplitude given by *Bijnens et al.*, PLB 374 (1996), NPB 508 (1997)

The irreducible part can be divided in a large number of subparts
$\hookrightarrow$ each subpart satisfies the expected permutation symmetries
$\hookrightarrow$ the choice of triangle loop integrals with high symmetry allows for a fairly compact expression
$\hookrightarrow$ NLO correction is sizable but not very large

Outlook

Work in progress
$\hookrightarrow$ combine our results with the methods for extracting three-body scattering from finite volume in lattice QCD

Might be of interest for the amplitude community

More details in PRD 104 (2021) 054046, arXiv:2107.06291