Pion-mass dependence of $\gamma^{(*)}\pi \to \pi\pi$ and $\pi\pi \to \pi\pi$

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[MN, Hoferichter, Kubis: arXiv:2110.11372]

[MN, Hoferichter, Kubis, Ruiz de Elvira: Phys. Rev. Lett. **126** (2021) 102002]

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bctp

$\pi\pi \to \pi\pi$ P wave ($I = J = 1$)

phenomenology

ρ resonance $M_\rho = 763.7^{+1.7}_{-1.5}$ MeV, $\Gamma_\rho = 146.4^{+2.0}_{-2.2}$ MeV

[García-Martín et al.: Phys. Rev. Lett. **107** (2011)]

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lattice QCD

- computations at physical M_π available [ETMC: Phys. Lett. B **819** (2021)]
- $N_\mathrm{f} = 2 + 1$ often at unphysical M_π
  e.g. [CLS: Nucl. Phys. B **939** (2019)] 6 data sets @ $M_\pi \approx 200$–284 MeV
  - higher $M_\pi \leftrightarrow$ lower computational cost
  - multi-particle thresholds: 4π, 6π, ...
  - finite-volume error $\sim \exp(-M_\pi L)$

$\Rightarrow$ need for extrapolation in M_π

Inverse-amplitude method (IAM)

- T: $I = J = 1$ $\pi\pi \to \pi\pi$ partial wave
- ChPT expansion $T(s) = T_2(s) + T_4(s) + T_6(s) + \mathcal{O}(p^8)$
  - elastic unitarity only perturbatively
  - no resonances
- unitarize ChPT

$$T(s) \approx \frac{T_2(s)^2}{T_2(s) - T_4(s)} \qquad \text{NLO IAM}$$

$$T(s) \approx \frac{T_2(s)^2}{T_2(s) - T_4(s) + T_4^2(s)/T_2(s) - T_6(s)} \qquad \text{NNLO IAM}$$

[Truong: Phys. Rev. Lett. **67** (1991)]

[Dobado, Peláez: Phys. Rev. D **56** (1997)]

- straightforward description of M_π dependence

  [Hanhart, Peláez, Ríos: Phys. Rev. Lett. **100** (2008)]
- correct analytic structure: ρ pole on 2nd Riemann sheet

Fit to lattice data

previous IAM fits

- NLO: [Bolton, Briceño, Wilson: Phys. Lett. B **757** (2016)] , [Hu et al.: Phys. Rev. Lett. **117** (2016)] , [Molina, Ruiz de Elvira: JHEP **11** (2020)] , ...
- NNLO: only [Peláez, Ríos: Phys. Rev. D **82** (2010)]

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$I = J = 1$ partial waves in $SU(2)$ ChPT

work with M_π-independent pion decay constant in chiral limit, F

- T_2: only F and M_π
- T_4: one NLO LEC: $l_2^r - 2l_1^r$
- T_6: three NLO LECs: l_1^r, l_2^r, l_3^r & three NNLO LECs: r_a, r_b, r_c

[Gasser, Leutwyler: Annals Phys. **158** (1984)]

[Bijnens et al.: Phys. Lett. B **374** (1996)]

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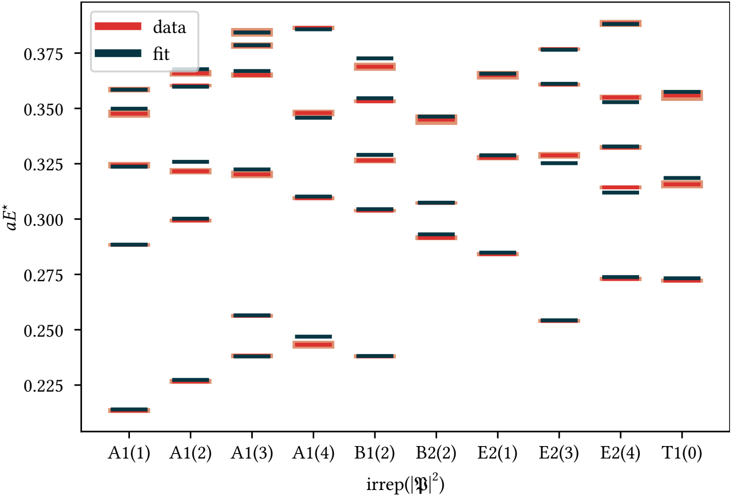
Lüscher's method

- lattice: $\pi\pi$ energy levels $E^\star_{\pi\pi}$
- quantization condition: $\delta(E^\star_{\pi\pi}) = \mathscr{Z}(E^\star_{\pi\pi})$
- fit energy levels directly

fit 5 data sets from [CLS: Nucl. Phys. B **939** (2019)] simultaneously
+ F_π from [CLS: Phys. Rev. D **95** (2017)]

NNLO IAM vs data @ $M_\pi \approx 223$ MeV



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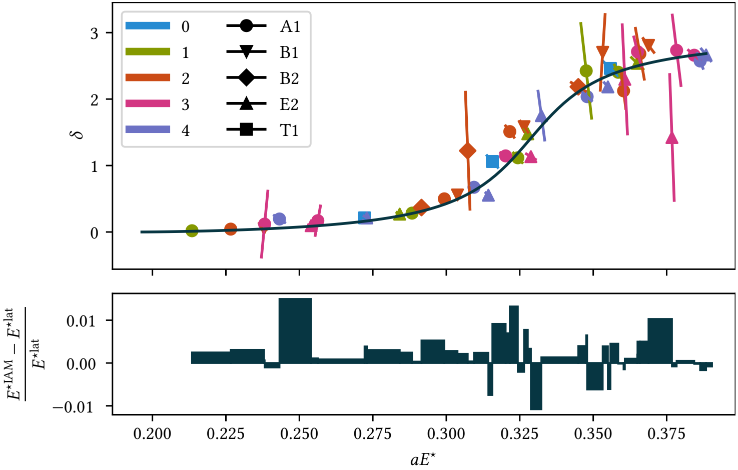


errors

- statistical error
- lattice spacing $a \pm \Delta a$
- truncation error
  - ChPT expansion in $\alpha = \frac{M_\pi^2}{M_\rho^2}$ and $\beta = \frac{s}{M_\rho^2}$ with $s = (E^\star)^2$
  - s dependence improved via IAM $\Rightarrow \beta$ not important
  - estimate truncation error of observable X as

$$\begin{aligned} \Delta X_{\mathrm{NLO}} &= \alpha |X_{\mathrm{NLO}}| \\ \Delta X_{\mathrm{NNLO}} &= \max\left\{\alpha^2 X_{\mathrm{NLO}}, \alpha |X_{\mathrm{NLO}} - X_{\mathrm{NNLO}}|\right\} \end{aligned}$$

[Epelbaum, Krebs, Meißner: Eur. Phys. J. A **51** (2015)]

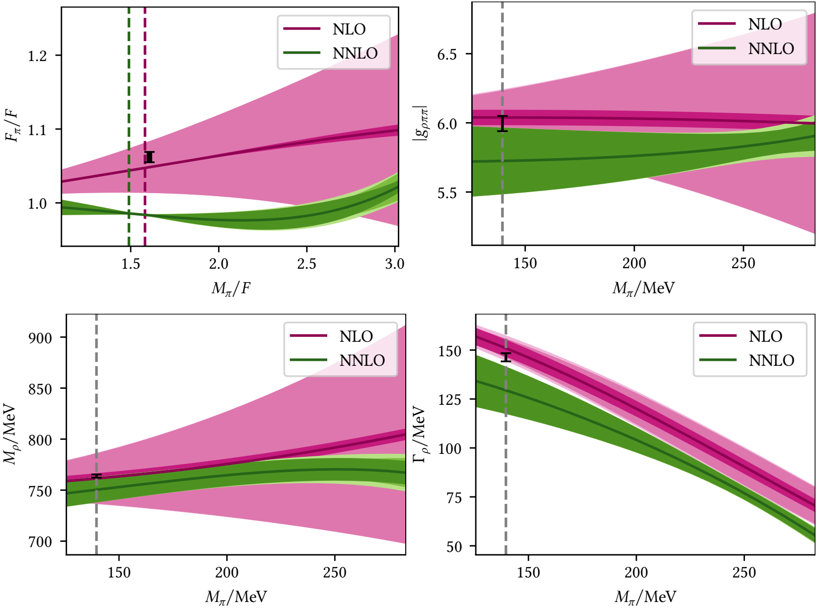
NLO: $\chi^2/\text{dof} = 216/(122-9) = 1.91$

	fit	Ref. [1]	FLAG
$(l_2^r - 2l_1^r) \times 10^3$	12.62(25)(0)	9.9(1.3)	19(17)
$l_4^r \times 10^3$	−2.6(1.1)(0.2)	6.2(1.3)	3.8(2.8)

NNLO: $\chi^2/\text{dof} = 165/(123-15) = 1.53$

	fit	Ref. [1]	Ref. [2]
$l_1^r \times 10^3$	−6.1(1.8)(0.1)	−4.03(63)	
$l_2^r \times 10^3$	2.58(90)(7)	1.87(21)	
$l_3^r \times 10^3$	0.776(65)(4)	0.8(3.8)	
$l_4^r \times 10^3$	−33(13)(0)	6.2(1.3)	
$r_a \times 10^6$	28(12)(1)		13
$r_b \times 10^6$	−4.8(2.6)(0.2)		−9.0
$r_c \times 10^6$	2.1(1.3)(0.1)		1.1
$r_F^r \times 10^3$	2.7(1.2)(0)		0

[1]: [Bijnens, Ecker: Ann. Rev. Nucl. Part. Sci. **64** (2014)]
[2]: [Bijnens et al.: Nucl. Phys. **B508** (1997) & Nucl. Phys. **B517** (1998) & JHEP **05** (1998)]



in black: values from [García-Martín et al.: Phys. Rev. Lett. **107** (2011)] (ρ) & FLAG (F_π/F)

$\gamma^{(*)}\pi \to \pi\pi$

amplitude

$$\mathscr{M}\left(\gamma^{(*)}(q)\pi^+(p) \to \pi^+(k)\pi^0(k')\right) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}\left(q^2\right) p^{\nu}k^{\alpha}k'^{\beta}\mathscr{F}\left(s, t, q^2\right)$$

$$s = (k + k')^2, \qquad t = (p - k)^2$$

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anomaly [Wess, Zumino: Phys. Lett. B **37** (1971)] [Adler et al.: Phys. Rev. D **4** (1971)]

$$\mathcal{F}(0,0,0) = eF_{3\pi}, \qquad F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3} = 32.23(10)\,\mathrm{GeV}^{-3}$$

tested experimentally only at 10 % level [Hoferichter, Kubis, Sakkas: Phys. Rev. D **86** (2012)]

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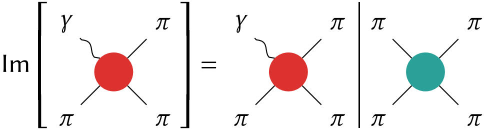
partial-wave expansion

$$\mathscr{F}\left(s,t,q^2\right) = \underbrace{f_1\left(s,q^2\right)}_{P\,\text{wave}} + \ldots$$

relation to $\pi\pi \to \pi\pi$

- elastic unitarity



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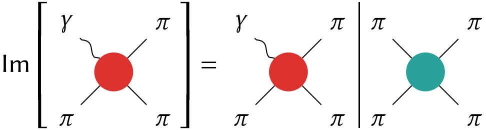

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$$\left|f_1\left(s,q^2\right)\right|^2 \sim \left[\frac{\partial\delta(E^\star)}{\partial E^\star} - \frac{\partial\mathscr{Z}(E^\star)}{\partial E^\star}\right]\left|f_1^{\text{finite volume}}\left(s,q^2\right)\right|^2$$

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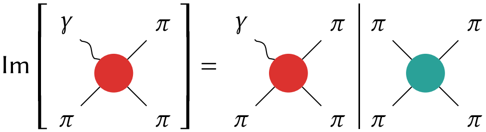

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lattice QCD data

- $M_\pi \approx 391$ MeV [HadSpec: Phys. Rev. D **87** (2013), Phys. Rev. Lett. **115** (2015)]
- $M_\pi \approx 317$ MeV [Alexandrou et al.: Phys. Rev. D **96** (2017), Phys. Rev. D **98** (2018)]

$\Rightarrow$ need to analyze both data sets simultaneously

Khuri-Treiman equations

$$f_1\left(s,q^2\right) = c_0\left(q^2\right) f_1^{(0)}\left(s,q^2\right) + c_1\left(q^2\right) f_1^{(1)}\left(s,q^2\right)$$

[Khuri, Treiman: Phys. Rev. **119** (1960)] [Niecknig, Kubis, Schneider: Eur. Phys. J. C **72** (2012)]

see also talks by D. Stamen, H. Akdag, T. Isken

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basis functions $f_1^{(k)}$

- describe $\pi\pi$ rescattering in all channels
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- completely fixed by IAM phase δ

subtraction functions c_k

describe q^2 dependence ($\gamma^* \to \pi\pi\pi$) e.g. via conformal variable ω

$$c_k\left(q^2\right) = \sum_{j=0}^{2-k} b_{kj}\omega\left(q^2\right)^j \qquad \text{strategy II}$$

$$c_k\left(q^2\right) = \frac{1}{1-\frac{q^2}{M_\omega^2}} \sum_{j=0}^{2-k} b_{kj}\omega\left(q^2\right)^j \qquad \text{strategy II}\mathcal{P}$$

M_π dependence:

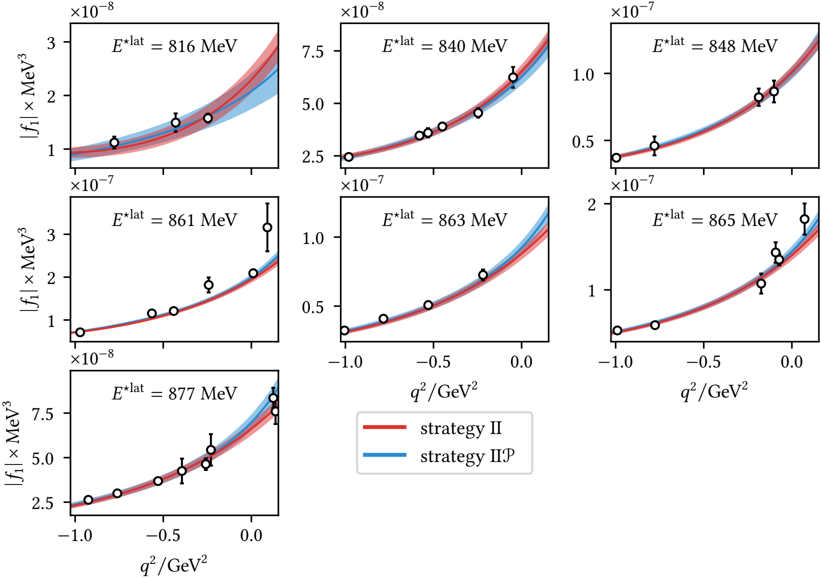
$$b_{kj}\left(M_\pi^2\right) = \alpha_{kj} + \beta_{kj}M_\pi^2, \qquad \alpha_{kj}, \beta_{kj} \in \mathbb{R}.$$

fit vs data @$M_\pi \approx 391$ MeV

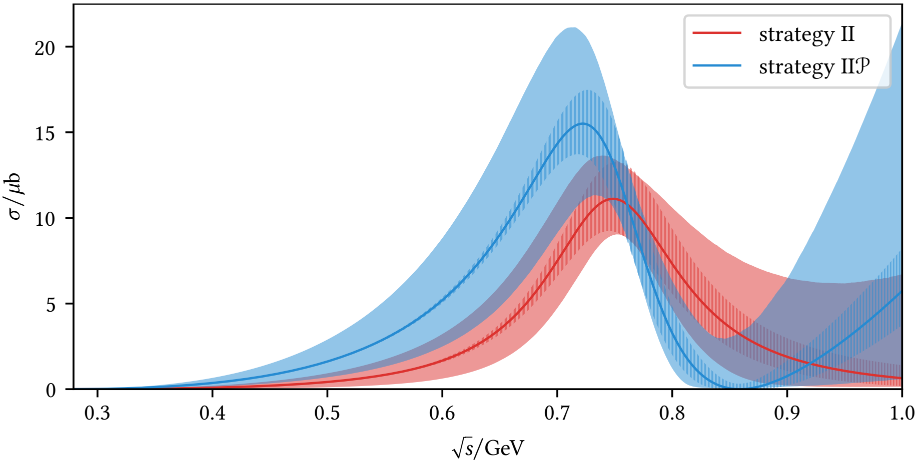
strategy	χ^2/dof	p value
II	$\frac{53.9}{37-5} = 1.69$	8.99×10^{-3}
II$\mathcal{P}$	$\frac{43.6}{37-5} = 1.36$	8.31×10^{-2}

- inclusion of ω pole improves fit

fit vs data @$M_\pi \approx 391$ MeV



@ $M_\pi = 139.57$ MeV



- to be compared with COMPASS: [Seyfried: Master's thesis, TU Munich, 2017]
- unphysical high-energy behavior: irrelevant in low-energy region

@ $M_\pi = 139.57$ MeV

$$F_{3\pi} = 38(16)(1)(11)\,\mathrm{GeV}^{-3}$$
$$\left|g_{\rho\gamma\pi}\right| = 0.60^{+0.12}_{-0.09}(3)(7)\,\mathrm{GeV}^{-1}$$

to be compared with

$$F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3} = 32.23(10)\,\mathrm{GeV}^{-3}$$
$$\left|g_{\rho\gamma\pi}\right| = 0.79(8)\,\mathrm{GeV}^{-1}$$

[PDG] [Klingl, Kaiser, Weise: Z. Phys. A **356** (1996)] [Hoferichter, Kubis, Zanke: Phys. Rev. D **96** (2017)]

errors dominated by:

- statistical error of $\left|f_1^{\text{finite volume}}\right|$
- systematic error of parametrization of c_k

⇒ can be improved by more precise data at more different M_π

conclusions

- stable NNLO IAM fits by now possible due to nice data
- IAM formalism extends to S wave: $f_0(500)$
- P-wave IAM as input for
  - HVP (talk by M. Hoferichter)
  - $\gamma^{(*)}\pi \to \pi\pi$
- Khuri-Treiman equations enable extraction of anomaly from lattice data
- state-of-the-art lattice data requires state-of-the-art models in analysis

Spares

IAM fits to CLS data @ $M_\pi = 139.57$ MeV

	NLO	NNLO	literature
M_ρ/MeV	761.4(5.1)(0.3)(24.7)	749(12)(1)(1)	$763.7^{+1.7}_{-1.5}$
Γ_ρ/MeV	150.9(4.4)(0.1)(4.9)	129(12)(1)(1)	$146.4^{+2.0}_{-2.2}$
$\mathrm{Re}(g_{\rho\pi\pi})$	5.994(54)(0)(194)	5.71(23)(2)(1)	$5.98^{+0.04}_{-0.07}$
$-\mathrm{Im}(g_{\rho\pi\pi})$	0.731(21)(0)(24)	0.46(14)(2)(1)	$0.56^{+0.07}_{-0.10}$
F/MeV	88.27(0.23)(0.04)(2.86)	93.5(2.3)(0.1)(0.2)	86.89(58)

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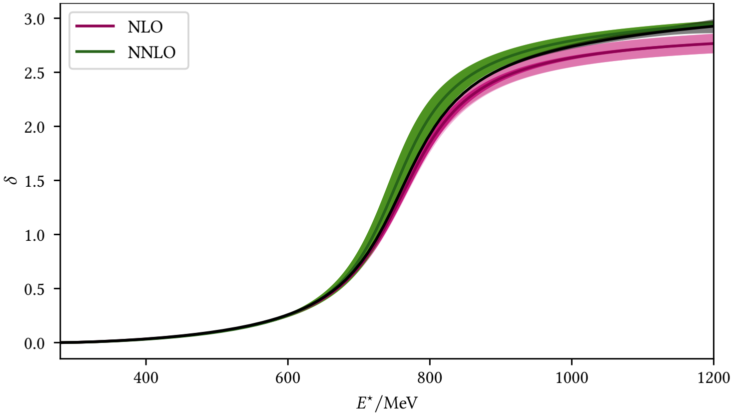

in black: phase from [Colangelo, Hoferichter, Stoffer: JHEP **02** (2019)]

- fit results in agreement with phenomenology
- NNLO phase works well even above the resonance peak