New insights into the nucleon form factors and proton radius from dispersive analysis

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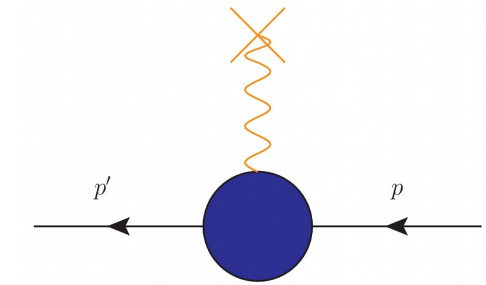
Nucleon Form Factors

- Definition

$$\langle p'|j_\mu^{\mathrm{em}}|p\rangle = \bar{u}(p')\left[F_1(t)\gamma_\mu + i\frac{F_2(t)}{2m}\sigma_{\mu\nu}q^\nu\right]u(p)\,,$$



$$t = q^2 = (p'-p)^2 \equiv -Q^2,\ \ t>0 \text{ for time-like, } t<0 \text{ for space-like}$$

- Normalization $F_1^p(0) = 1\,, F_1^n(0) = 0\,,\ F_2^p(0) = \kappa_p\,, F_2^n(0) = \kappa_n\,.$

- Isospin decomposition

$$F_i^{\text{isoscalar}} = \frac{1}{2}(F_i^p + F_i^n)\,, F_i^{\text{isovector}} = \frac{1}{2}(F_i^p - F_i^n)\,, i=1,2$$

- Sachs NFFs

$$G_E(t) = F_1(t) - \tau F_2(t)\,, G_M(t) = F_1(t) + F_2(t)$$

$$\tau = -t/(4m^2)$$

Proton charge radius

- Definition $r_p^2 \equiv -6G'_E(0)$
- Measurements
  - Leptonic hydrogen Lamb shift

$$(\Delta E_{LS})_{\text{measured}} = \Delta E_1 + \Delta E_2 C(r_p^2) + \mathcal{O}(m_r\alpha^6),$$
$$C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

  - Lepton-proton Scattering (Rosenbluth separation)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma_{\text{Mott}}}{d\Omega}\frac{1}{1+\tau}\left({G_E}^2 + \frac{\tau}{\varepsilon}G_M^2\right)(1+\delta_{\text{radia.}}) + \mathcal{O}(\alpha^2)$$

Why Dispersion Theory?

- Difficulties on NFFs
  - Unknown expression $\longrightarrow$ parametrization-dependent
  - Data at $Q^2 = 0$ is unachievable $\longrightarrow$ extrapolation needed
- Dispersion theoretical NFFs $$F(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon}\, dt'$$
  - Unitarity and analyticity guaranteed,
  - Works well in the whole t-region, $(\sim 10^{-4}\text{-}10\mathrm{s}\ \mathrm{GeV}^2)$ experimentally
  - Theoretical constraints of asymptotic behavior of NFFs can be added consistently,
  - Connects to data from various processes. (πN-scattering, $\cdots$)

Dispersion Relations of NFFs

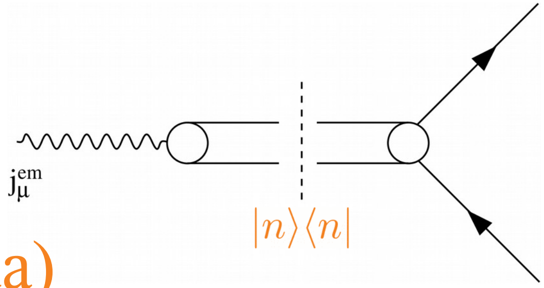
- Definition

$$F(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon}\, dt'$$

Starting from the lowest two-pion cuts $(t_0 = (2M_\pi)^2)$

- Low-mass part: Spectral Decomposition
  - Crossing symmetry $\langle N(p')|j_\mu^{\mathrm{em}}|N(p)\rangle \longleftrightarrow \langle N(p)\bar{N}(\bar{p})|j_\mu^{\mathrm{em}}|0\rangle$
  - Spectral decomposition G. F. Chew, *et al.* PhysRev110, 265(1958)

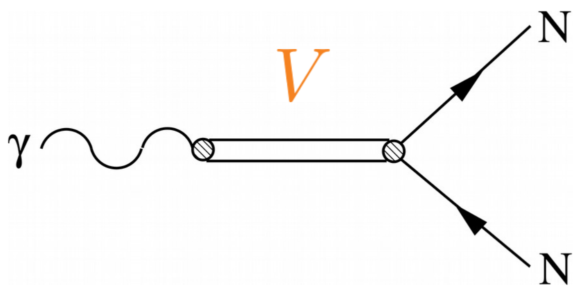
$$\mathrm{Im}\langle N(p)\bar{N}(\bar{p})|j_\mu^{\mathrm{em}}|0\rangle \sim \sum_n \langle N(p)\bar{N}(\bar{p})|n\rangle\langle n|j_\mu^{\mathrm{em}}|0\rangle$$



- Intermediate mass states (continua)
  - Isoscalar $3\pi, 5\pi, K\bar{K}, \pi\rho, \cdots$ – Isovector $2\pi, 4\pi, \cdots$

Dispersion Relations of NFFs

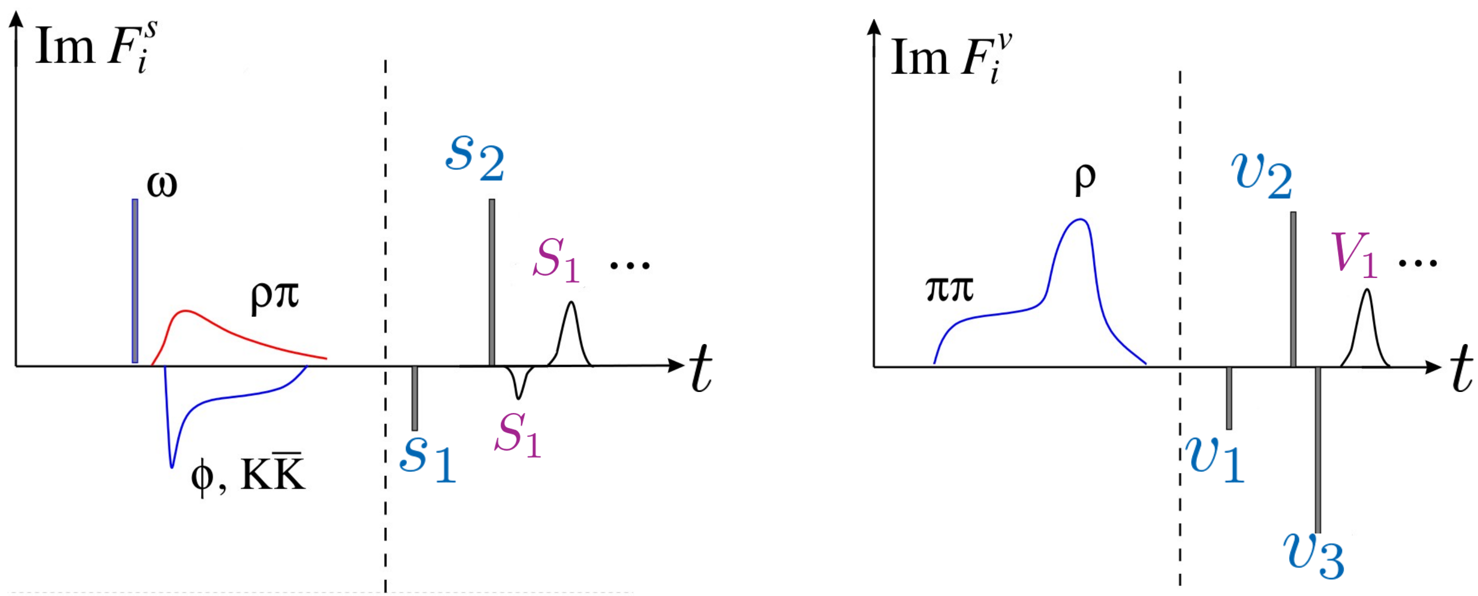
- High-mass part: Vector meson dominance



Effective poles

- Narrow state $s_1, s_2, \cdots, v_1, v_2, \cdots$
- Broad state $S_1, S_2, \cdots, V_1, \cdots$

- Sketch of the spectral functions



Theoretical constraints

- Normalization (4)

- Neutron charge radius squared (1)

A. A. Filin, *et al.* PhysRevLett124, 082501(2020)

$$\langle r_n^2 \rangle = -0.105^{+0.005}_{-0.006}\ \mathrm{fm}^2$$

- Superconvergence relations from pQCD (6)

$$\int_{t_0}^{\infty} \mathrm{Im}\, F_i(t)\, t^n dt = 0\,, \quad i = 1, 2$$

with $n = 0$ for F_1, $n = 0, 1$ for F_2

Data basis

Region	Observables	Coll.	$\|t\|(GeV^2)$	number
Spacelike($t<0$)	$d\sigma/d\Omega$	MAMI	0.00384–0.977	1422
		PRad	0.000215–0.058	71
	$\mu_p G_E^p/G_M^p$	JLab	1.18–8.49	16
	G_E^n	world	0.14–1.47	25
	G_M^n	world	0.071–10.0	23
Timelike($t>0$)	$\|G_{\rm eff}^p\|$	world	3.52–20.25	153
	$\|G_{\rm eff}^n\|$	world	3.53–9.49	27
	$\|G_E^p/G_M^p\|$	BaBar	3.52–9.0	6
	$d\sigma/d\Omega$	BESIII	3.52–3.80	10

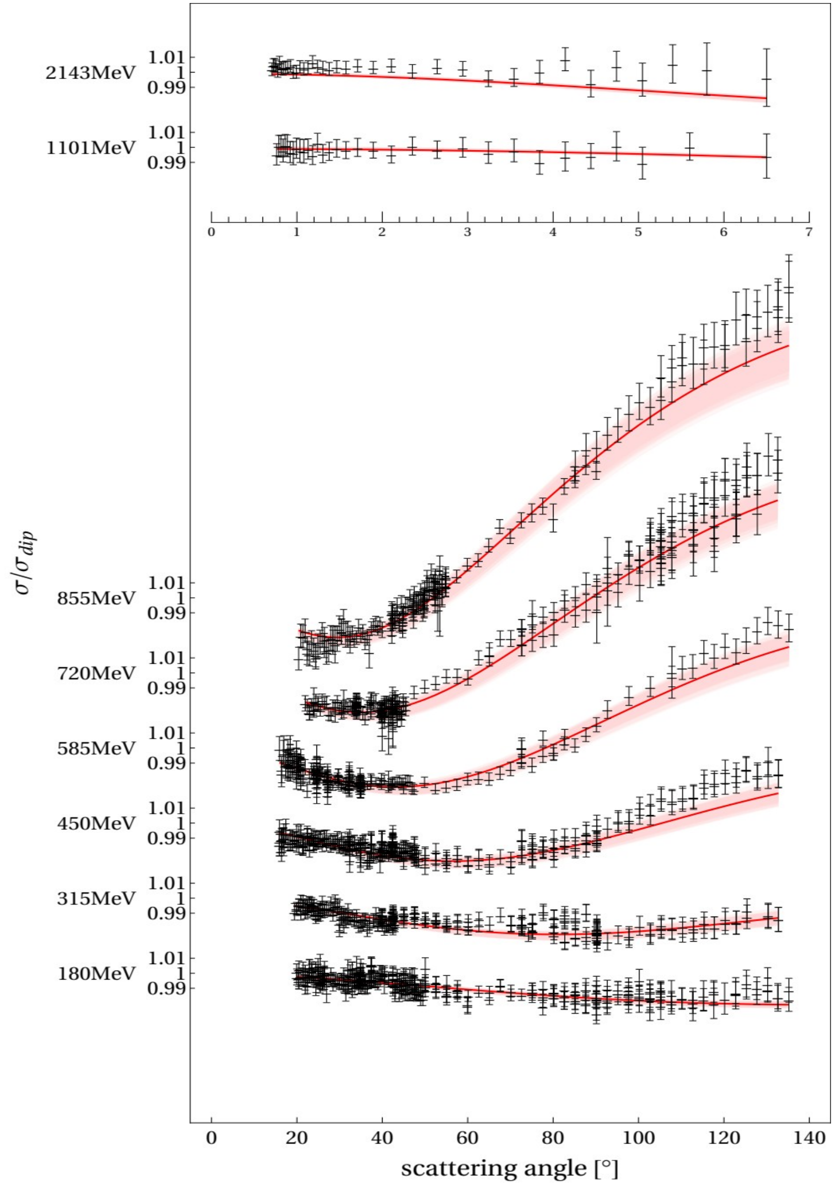
- Number of data/free parameters

$$1753 \Longrightarrow 4+3(N_s+N_v)+4(N_S+N_V)-11+31+2$$

- Fitting strategy: $$\chi^2_{\rm total} \equiv \sum_{i \text{ in } \{\text{data basis}\}} \frac{\chi_i^2}{\text{data points}}$$

Space-like results

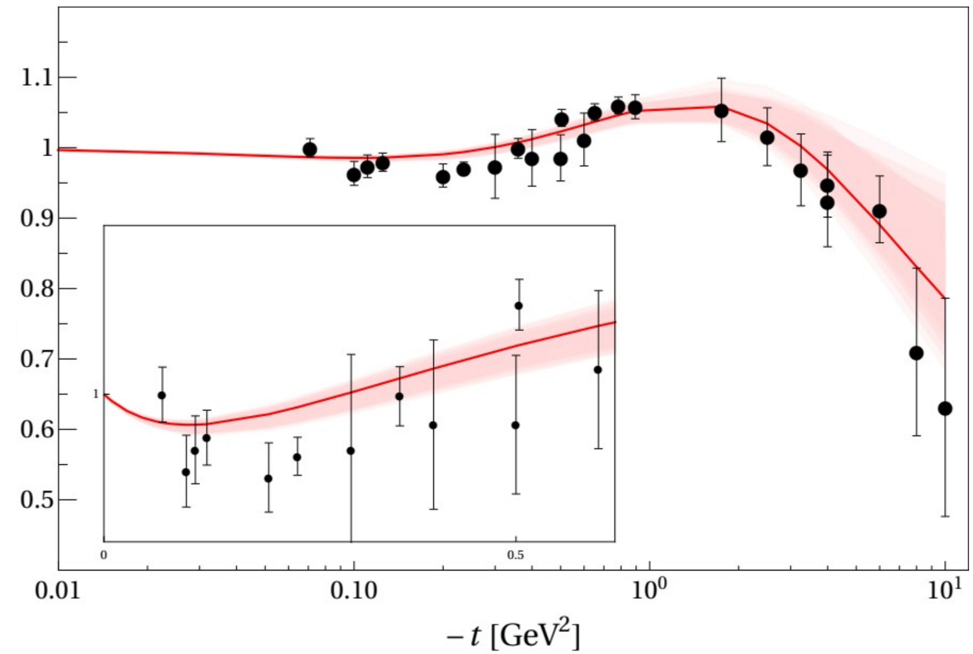
- Best configuration '3s+5v+3S+3V'
  - 74 parameters
  - $\omega, \phi, s_1, s_2, s_3, S_1, S_2, S_3 + K\bar{K} + \rho\pi$
  - $v_1, v_2, v_3, v_4, v_5, V_1, V_2, V_3 + \pi\pi$
  - $\chi^2/\mathrm{dof} = 1.223$



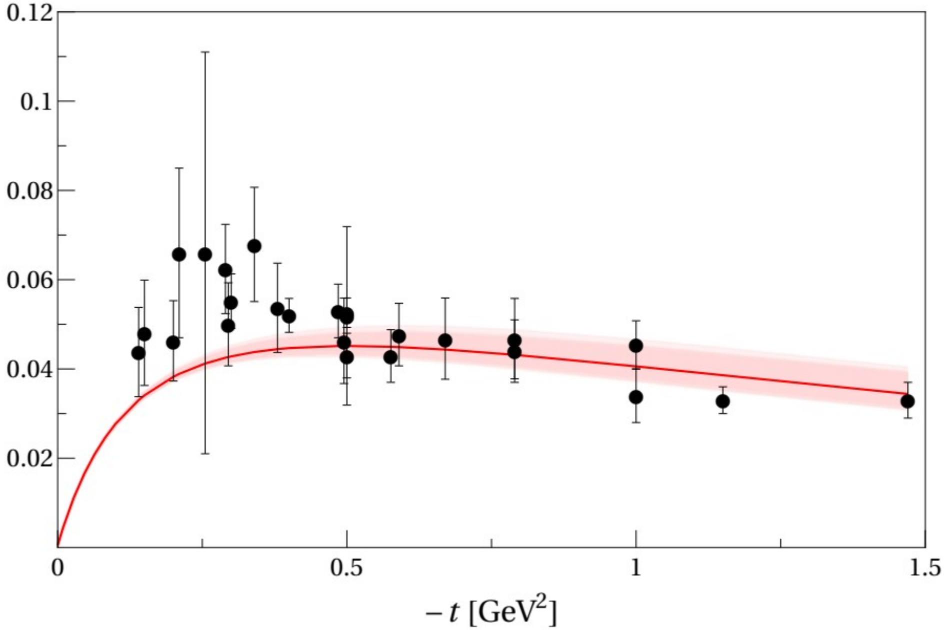
Line best fit
Band error from bootstrap sampling.

Space-like results

$G_M^n/(\mu_n G_{\rm dip})$

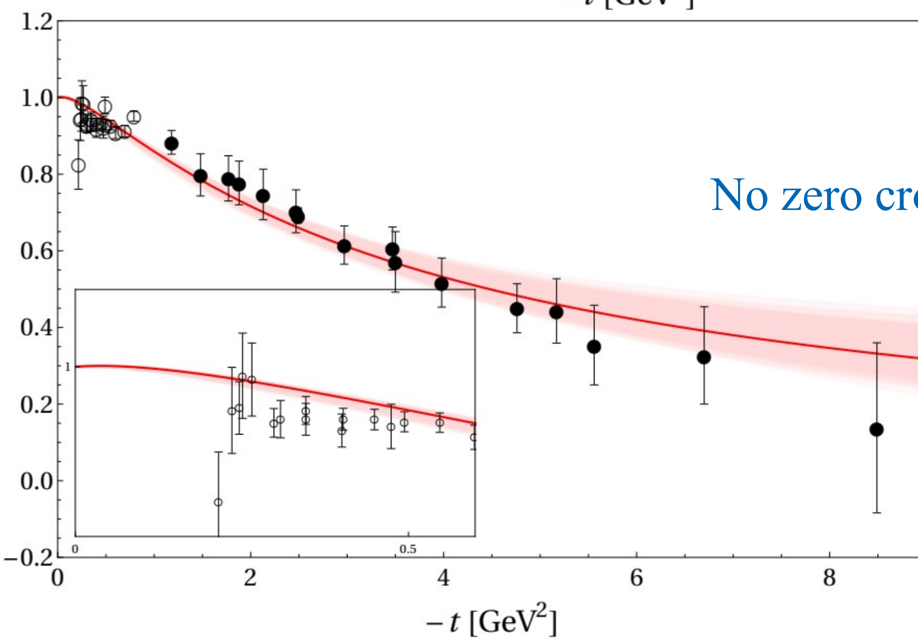


G_E^n



$\mu_p G_E^p / G_M^p$

Empty symbols not fitted



No zero crossing

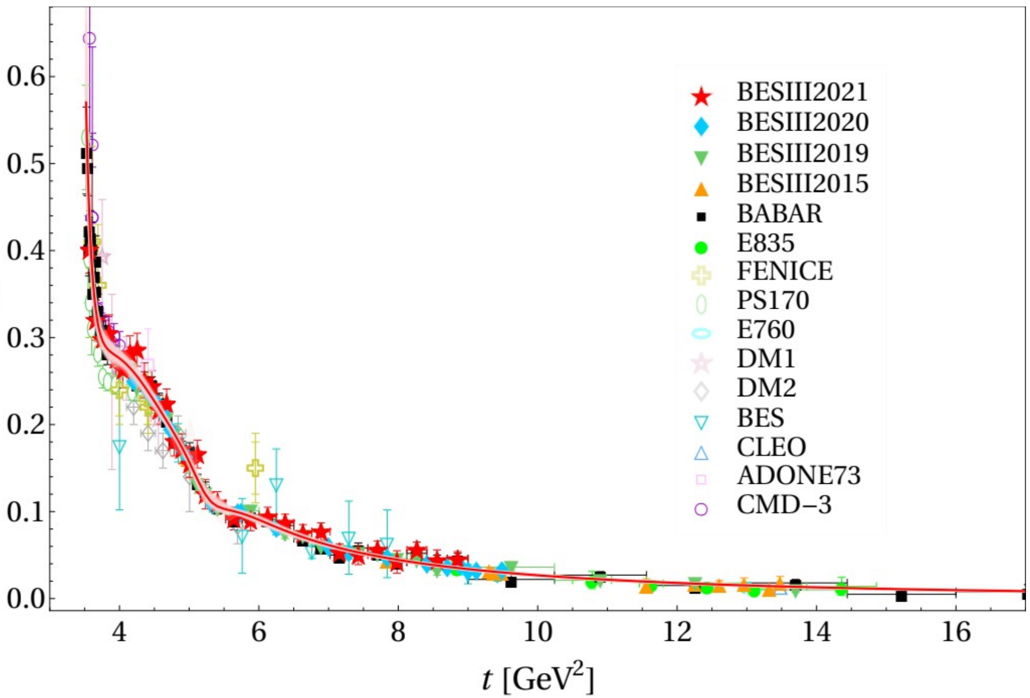
Line best fit

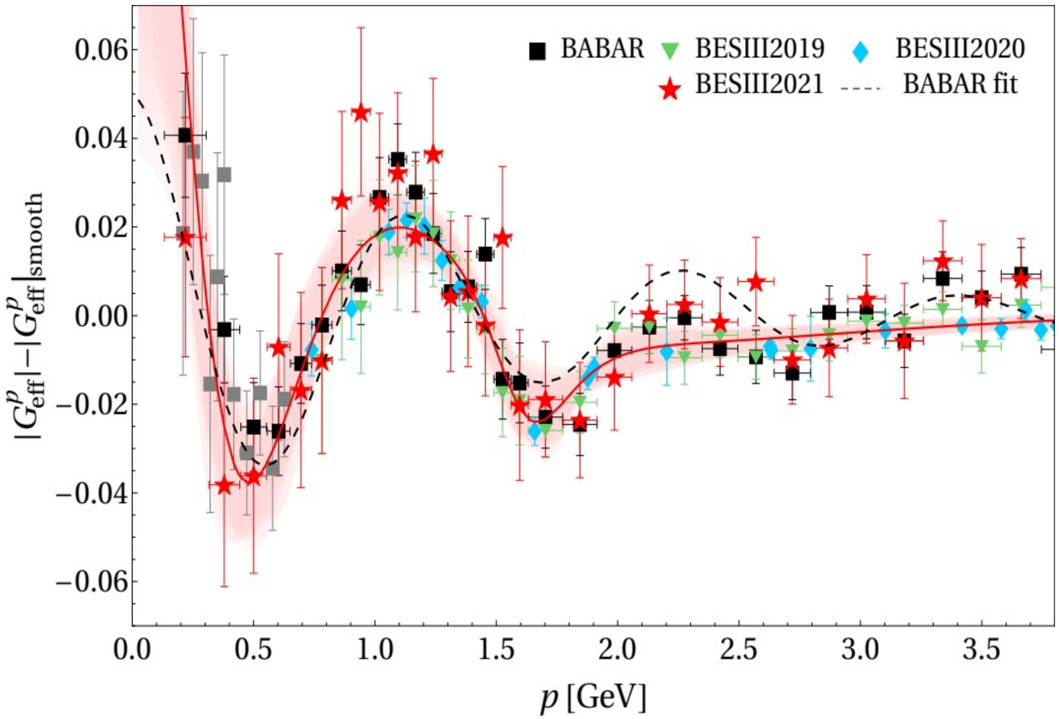
Band error from bootstrap sampling.

Time-like results

- Effective FFs $|G_{\rm eff}^p|$

$$|G_{\rm eff}| \equiv \sqrt{\frac{|G_E|^2+\xi|G_M|^2}{1+\xi}} \quad \xi = t/(2m^2)$$





$$|G_{\rm eff}^p|_{\rm smooth} = \frac{7.7}{(1+t/14.8)(1-t/0.71)^2}$$

Empty symbols not fitted

Line best fit

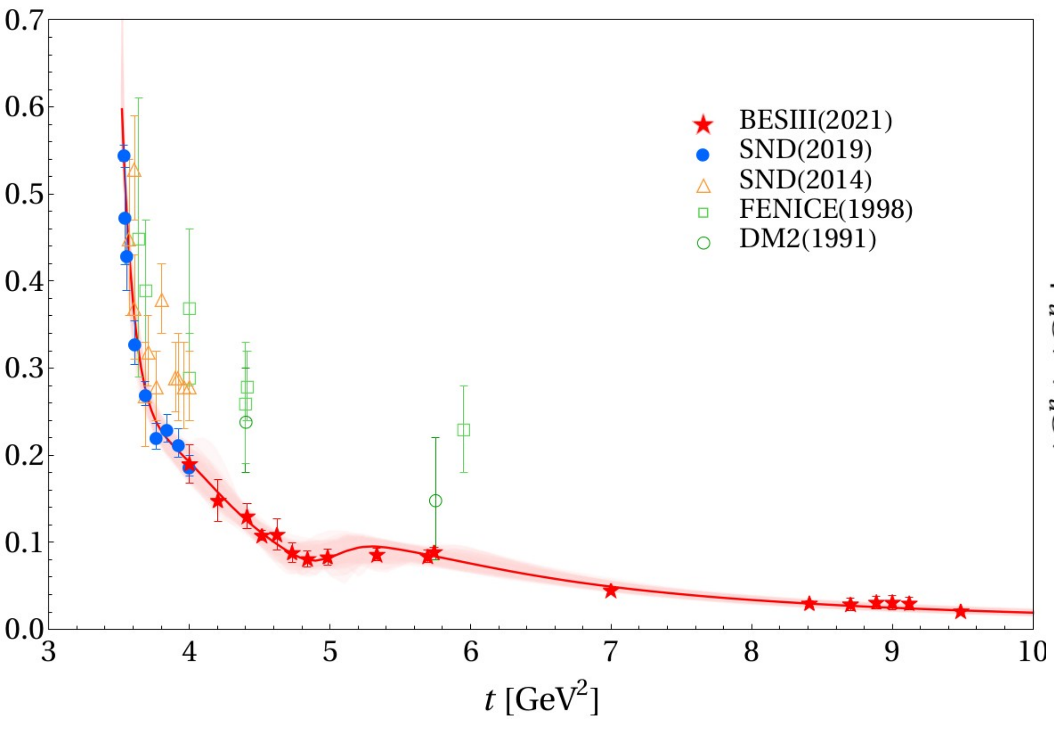
Band error from bootstrap sampling.

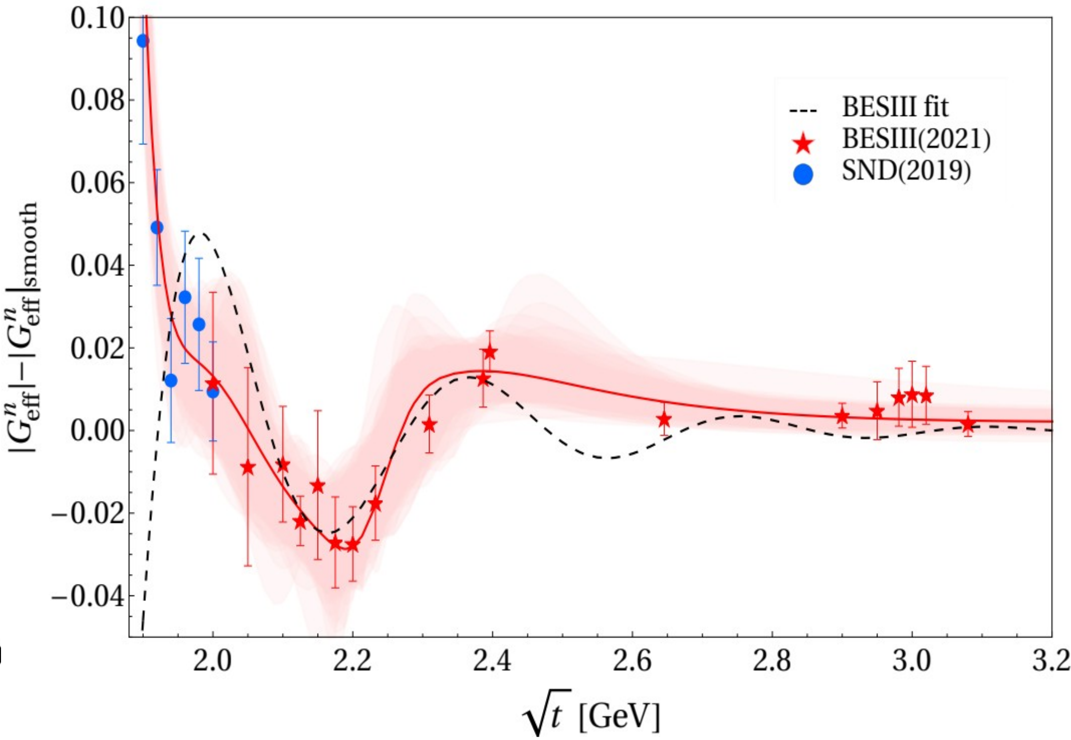
BABAR fit formula

$$F_p \equiv |G_{\rm eff}^p| - |G_{\rm eff}^p|_{\rm smooth} = A\exp(-Bp)\cos(Cp+D)$$

Time-like results

- Effective FFs $|G_{\rm eff}^n|$





Empty symbols not fitted

Line best fit

Band error from bootstrap sampling.

$$|G_{\rm eff}^n|_{\rm smooth} = \frac{4.87}{(1+t/14.8)(1-t/0.71)^2}$$

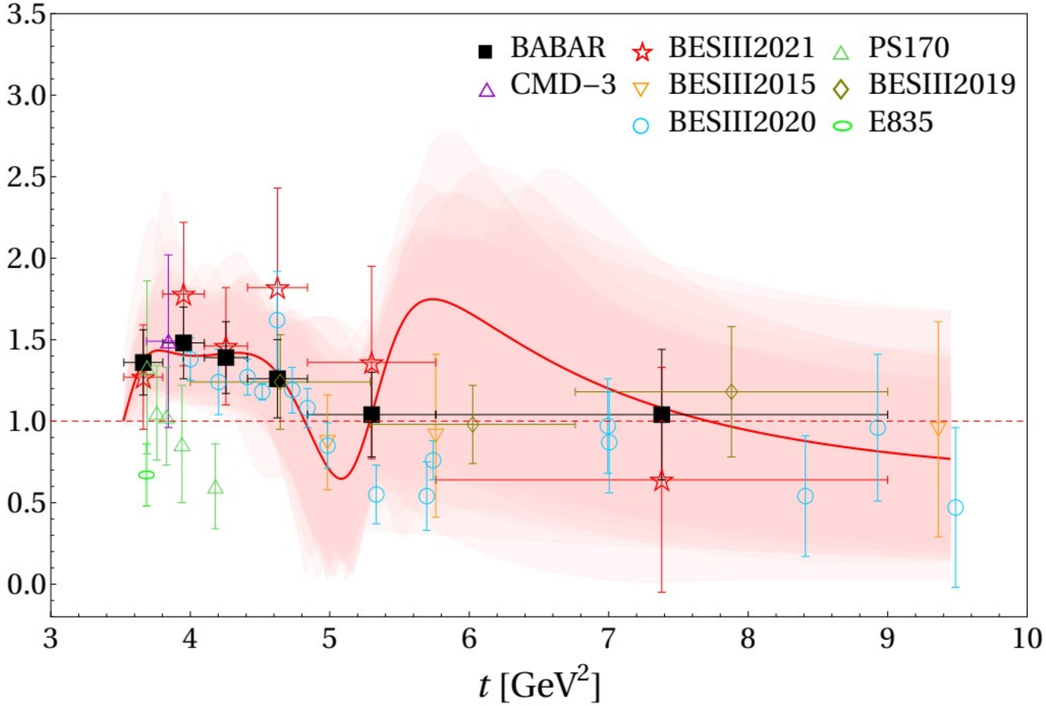
BESIII fit formula

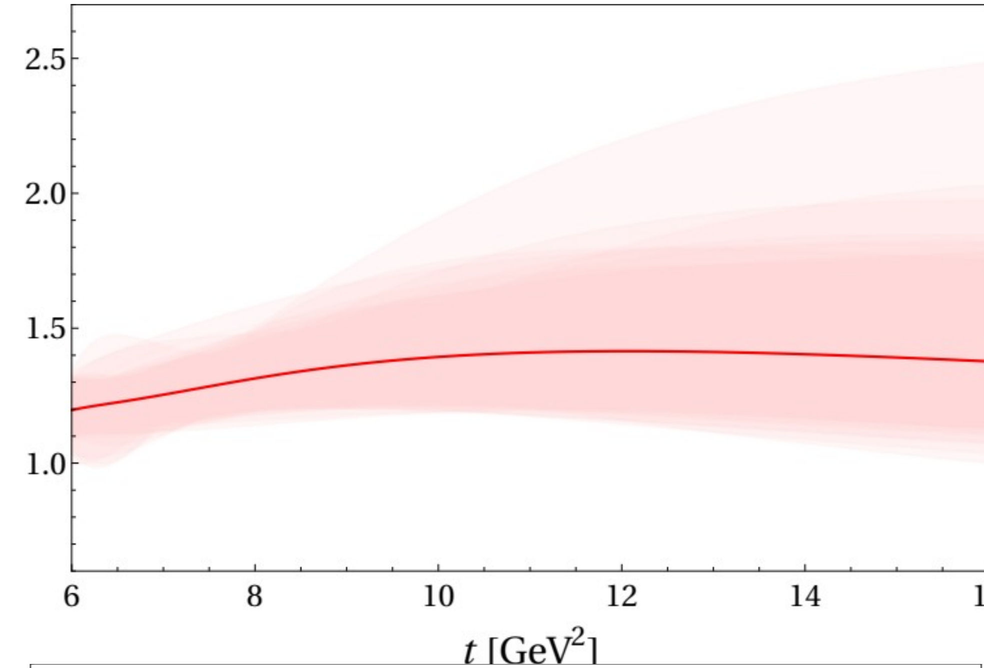
$$F_n \equiv |G_{\rm eff}^n| - |G_{\rm eff}^n|_{\rm smooth} = A\exp(-Bp)\cos(Cp+D)$$

Time-like results

$|G_{\text{eff}}^p / G_{\text{eff}}^n|$

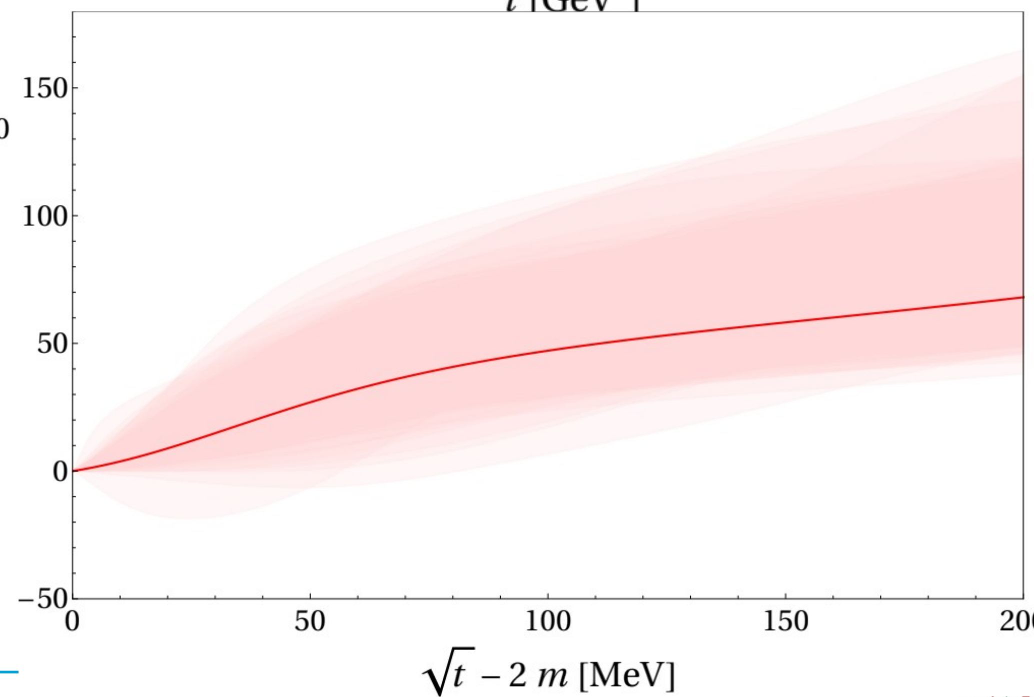
$|G_E^p / G_M^p|$





Empty symbols not fitted

$\mathrm{Arg}(G_E^p / G_M^p)[\mathrm{deg}]$



Line best fit

Band error from bootstrap sampling.

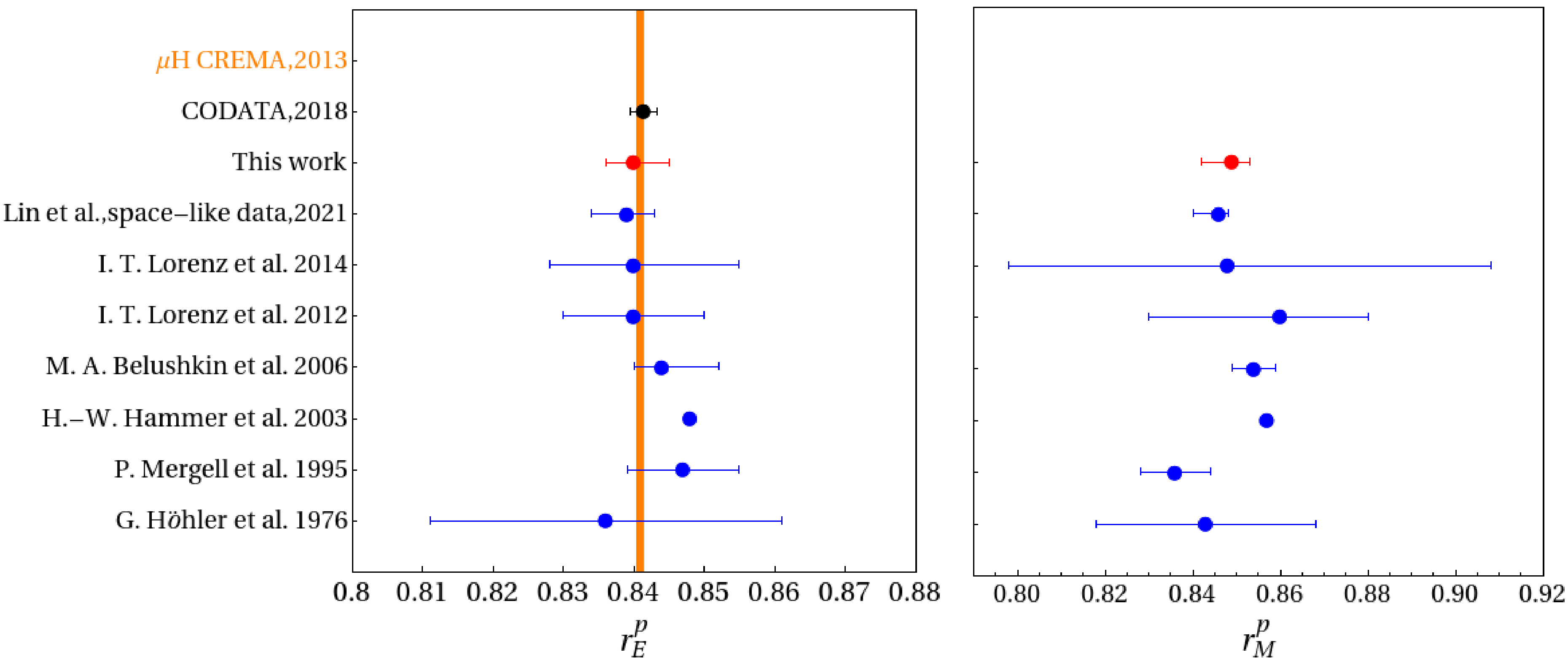
Proton charge radius

- Our determination

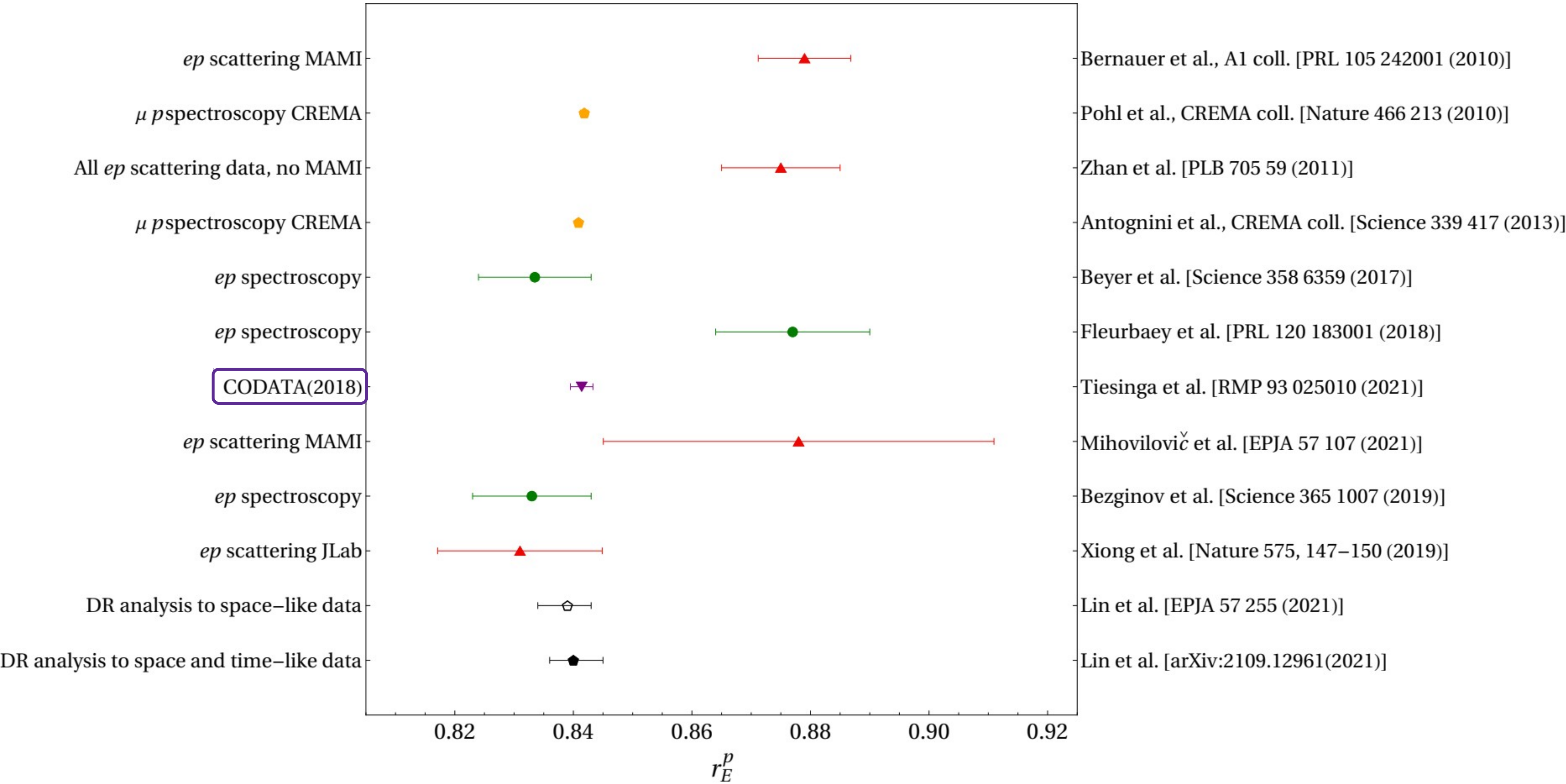
"Systematical" error from variation of the spectral functions
"Statistical" error form bootstrap

$$r_E^p = 0.840^{+0.003+0.002}_{-0.002-0.002}\ \text{fm},\ r_M^p = 0.849^{+0.003+0.001}_{-0.003-0.004}\ \text{fm}$$

- Comparing to existing DR determinations



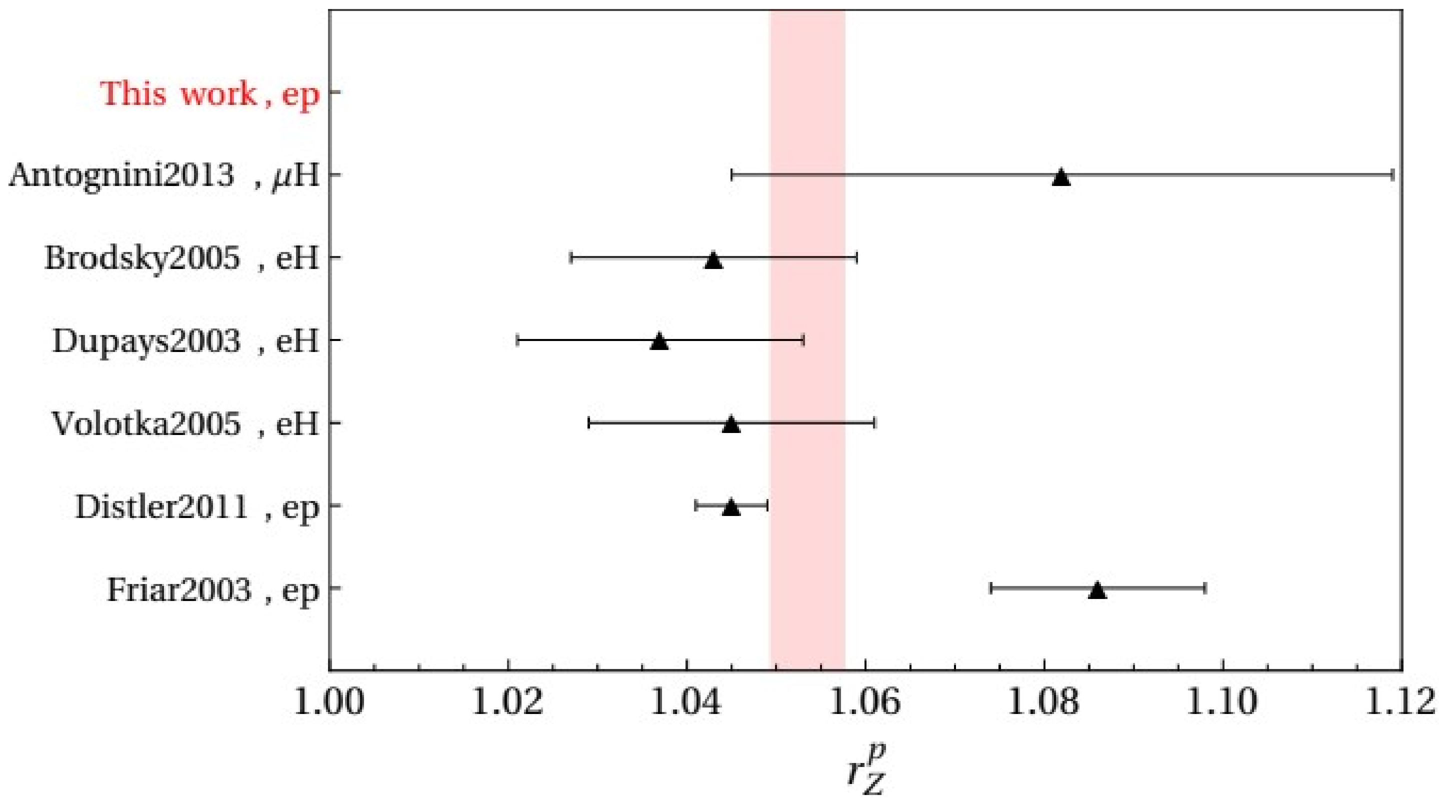
Comparing to recent measurements



Zemach radius and third Zemach moment

- Our determination

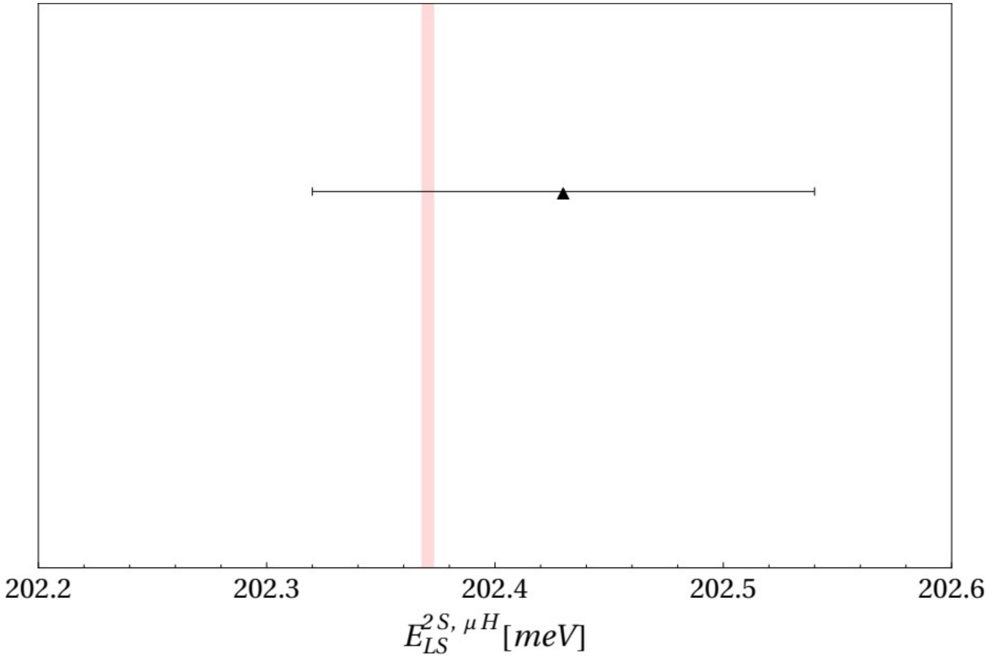
$$r_z = 1.054^{+0.003+0.000}_{-0.002-0.001}\ \text{fm},$$
$$\langle r^3\rangle_{(2)} = 2.310^{+0.022+0.014}_{-0.018-0.015}\ \text{fm}^3.$$

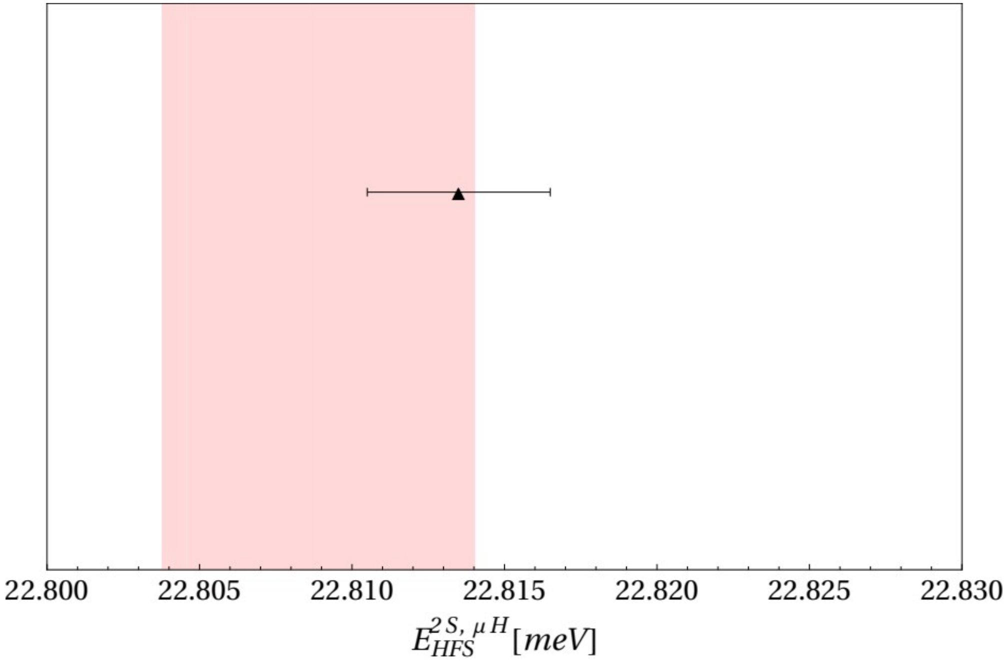


Comparison to spectroscopy observables

$$\Delta E_{\mathrm{LS}} = 206.0336(15) - 5.2275(10)\langle r^2 \rangle_p + 0.0347\langle r^3 \rangle_{(2)}$$
$$\Delta E_{\mathrm{HFS}} = 22.9843(30) - 0.1621(10) r_z$$

A. Antognini, *et al.* AnnalsPhys, 331 127(2013)





Bands from μ-H meaurement, **Black** triangles are our determination.
A. Antognini, *et al.* Science, 339 417(2013)

Summary

- NFFs is extracted from the latest experimental data over the full range of momentum transfer by using dispersion theory for the first time.
  - Spacelike data 0.000215-0.977 GeV^2
  - Timelike data 3.52-20.25 GeV^2
- DR analysis on NFFs data provide robust and consistent proton radius over decades, agrees with the small one.
- The DR NFFs is also consistent with μ-H spectroscopy observables.
- Several issues on the behavior of NFFs are addressed,
  - Disfavor the zero crossing at intermediate spacelike momentum transfer.
  - $|G^p_{\rm eff}/G^n_{\rm eff}|$ and $\mathrm{Arg}(G^p_E/G^p_M)$ in the time-like region are predicted.

Thank you very much for your attention!

Parametrization of NFFs

- Our spectral functions of NFFs read

$$
\begin{aligned}
\operatorname{Im} F_i^s(t) &= \operatorname{Im} F_i^{(s,K\bar{K})}(t) + \operatorname{Im} F_i^{(s,\rho\pi)}(t) + \sum_{V=\omega,\phi,s_1,\ldots} \pi a_i^V \delta(M_V^2 - t) \\
&+ \sum_{V=r_{s1},\ldots} \operatorname{Im} F_i^{(s,V_i)}(t)\,, \\
\operatorname{Im} F_i^v(t) &= \operatorname{Im} F_i^{(v,2\pi)}(t) + \sum_{V=v_1,\ldots} \pi a_i^V \delta(M_V^2 - t) \\
&+ \sum_{V=r_{v1},\ldots} \operatorname{Im} F_i^{(v,V_i)}(t)\,,
\end{aligned}
$$

- Then NFFs can be written as

$$
F_i(t) = \frac{a_i^V}{M_V^2 - t} \quad \text{for } \operatorname{Im} F_i^{(s,K\bar{K})}(t),\ \operatorname{Im} F_i^{(s,\rho\pi)}(t),\ \pi a_i^V \delta(M_V^2 - t)
$$

$$
F_i(t) = \frac{a_i^V}{M_V^2 - t - iM_V\Gamma_V} \quad \text{for } \operatorname{Im} F_i^{(s,V_{r_s})}(t),\ \operatorname{Im} F_i^{(v,V_{r_v})}(t)
$$

Fitting procedure

- Definition of χ^2

$$\chi_1^2 = \sum_i \sum_k \frac{(n_k C_i - C(t_i, \theta_i, \vec{p}))^2}{(\sigma_i + \nu_i)^2}$$

$$\chi_2^2 = \sum_{i,j} \sum_k (n_k C_i - C(t_i, \theta_i, \vec{p}))[V^{-1}]_{ij}(n_k C_j - C(t_j, \theta_j, \vec{p}))$$

$$V_{ij} = \sigma_i \sigma_j \delta_{ij} + \nu_i \nu_j$$

- Theoretical constraints are imposed in a soft way (implemented as additive terms to the total χ^2)

$$\chi^2_{\text{add.}} = p\,[x - \langle x \rangle]^2 \, \exp\left(p\,[x - \langle x \rangle]^2\right)$$

Two-pion continuum

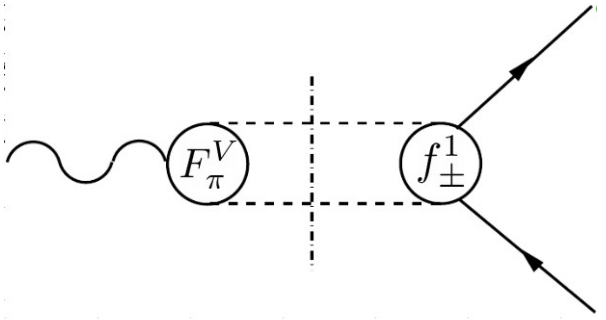
- Two-pion contribution to isovector spectral functions

M. Hoferichter, *et al.* EPJA52, 331(2016)

$$\mathrm{Im}\ G_E^v(t) = \frac{q_t^3}{m\sqrt{t}}\, F_\pi^V(t)^*\, f_+^1(t)\, \theta\big(t - 4M_\pi^2\big)\ ,$$

$$\mathrm{Im}\ G_M^v(t) = \frac{q_t^3}{\sqrt{2t}}\, F_\pi^V(t)^*\, f_-^1(t)\, \theta\big(t - 4M_\pi^2\big)\ ,$$

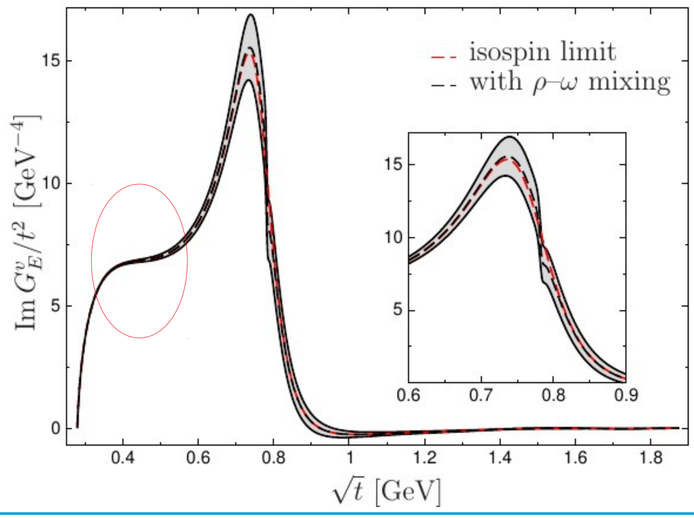
$$q_t = \sqrt{t/4 - M_\pi^2}$$



  - Pion FFs F_π : from $\pi\pi$ scattering phase shift
  - P-wave $\pi\pi \to N\bar{N}$ $f_\pm^1$: from analytic continuation of πN data

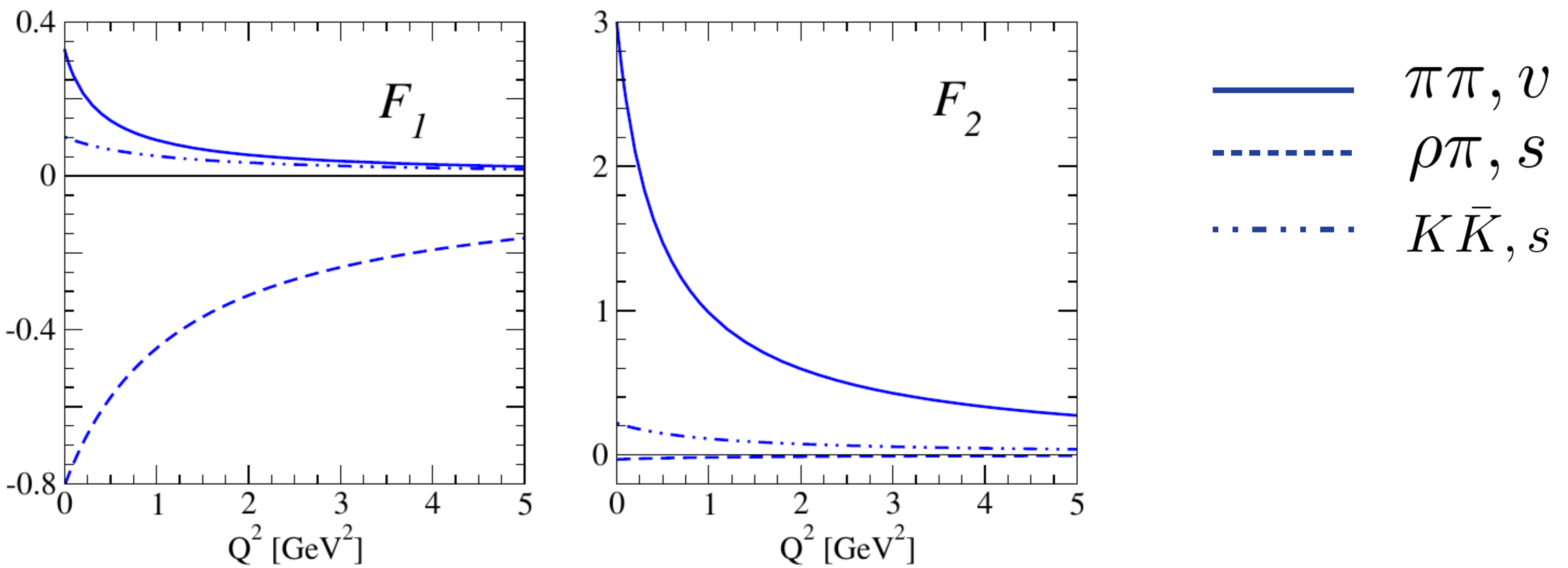
- Substantially different with the single ρ meson approximation

A visible enhancement on the left shoulder of ρ is found. $t_c = 3.98 m_\pi^2$ very close to threshold



Parametrization of NFFs

- Continuum contributions (not fitted to data)
  - $\pi\pi$ based on precise analysis of pion-nucleon scattering
    M. Hoferichter, *et al.* EPJA52, 331(2016)
  - $K\bar{K}$ from an analytic continuation of kaon-nucleon scattering data
    H.-W. Hammer, *et al.* PRC60, 045205(1999)
  - $\rho\pi$ from investigation of the Bonn-Jülich N-N interaction model
    U.-G.Meißner, *et al.* PLB633, 507(2006)



Two-photon-exchange correction

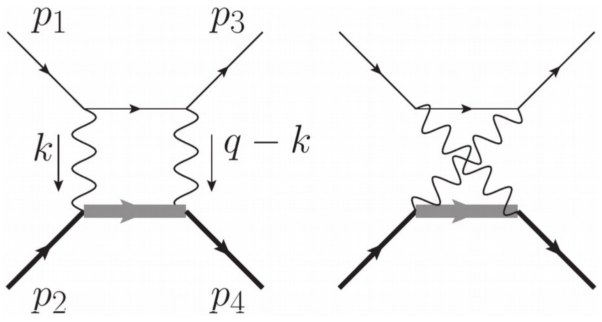
- Leading TPE contribution I. T. Lorenz, *et al.* PRD91, 014023(2015)

$$\frac{d\sigma_{\rm corr}}{d\Omega} = \frac{d\sigma_{1\gamma}}{d\Omega}(1+\delta_{2\gamma}+\ldots)\ ,\ \delta_{2\gamma} \underbrace{\approx}_{\mathcal{O}(\alpha)} \frac{2{\rm Re}(\mathcal{M}_{1\gamma}^\dagger \mathcal{M}_{2\gamma})}{|\mathcal{M}_{1\gamma}|^2}$$

- One-photon amplitude $\mathcal{M}_{1\gamma} = -\frac{e^2}{q^2}\bar{u}_e(p_3)\gamma_\mu u_e(p_1)\bar{u}_N(p_4)\Gamma^\nu u_N(p_2)$
- Two-photon amplitude $\mathcal{M}_{2\gamma}^{\rm box} = -ie^4\int \frac{d^4k}{(2\pi)^4} L_{\mu\nu}^{\rm box}(H_N^{\mu\nu}+H_\Delta^{\mu\nu})D(k)D(q-k)$

$$L_{\mu\nu}^{\rm box} = \bar{u}_e(p_3)\gamma_\mu S_F(p_1-k,m_e)\gamma_\nu u_e(p_1)$$

$$H_N^{\mu\nu} = \bar{u}_N(p_4)\Gamma^\mu(q-k)S_F(p_2+k,m_N)\Gamma^\nu(k)u_N(p_2)$$
$$H_\Delta^{\mu\nu} = \bar{u}_N(p_4)(p_4)\Gamma^{\mu\alpha}_{\gamma\Delta\to N}(p_2+k,q-k)S_{\alpha\beta}$$
$$\times(p_2+k)\Gamma^{\beta\nu}_{\gamma N\to\Delta}(p_2+k,k)u_N(p_2),$$



Values of the proton charge radius

- Historical DR determination

Ref.	r_E^p [fm]	r_M^p [fm]
G. Höhler et al. 1976	0.836 ± 0.025	0.843 ± 0.025
P. Mergell et al. 1995	0.847 ± 0.008	0.836 ± 0.008
H.-W. Hammer et al. 2003	0.848^{*}	0.857^{*}
M. A. Belushkin et al. 2006	$0.844^{+0.008}_{-0.004}$	0.854 ± 0.005
I. T. Lorenz et al. 2012	0.84 ± 0.01	$0.86^{+0.02}_{-0.03}$
I. T. Lorenz et al. 2014	$0.840^{+0.015}_{-0.012}$	$0.848^{+0.06}_{-0.05}$
Lin et al. 2021	$0.838^{+0.005+0.004}_{-0.004-0.003}$	$0.847 \pm 0.004 \pm 0.004$

Here * means that no error analysis has been performed.

Parameters of best fit

- Best fit

V_s	m_V	Γ	a_1^V	a_2^V	V_v	m_V	Γ	a_1^V	a_2^V
ω	0.783	0	0.701	0.338	v_1	1.050	0	0.782	-0.132
ϕ	1.019	0	-0.526	-0.997	v_2	1.323	0	-4.873	-0.645
s_1	1.031	0	0.422	-2.827	v_3	1.368	0	3.518	-0.987
s_2	1.120	0	0.122	3.655	v_4	1.462	0	2.243	-3.813
s_3	1.827	0	0.955	-1.122	v_5	1.532	0	-1.422	3.668
r_{s1}	1.903	0.973	-2.653	-1.753	r_{v1}	2.256	0.239	2.552	-1.217
r_{s2}	1.914	0.541	-3.069	2.017	r_{v2}	2.253	0.245	-1.947	0.551
r_{s3}	1.879	0.895	4.953	0.501	r_{v3}	2.220	0.362	-0.985	1.061

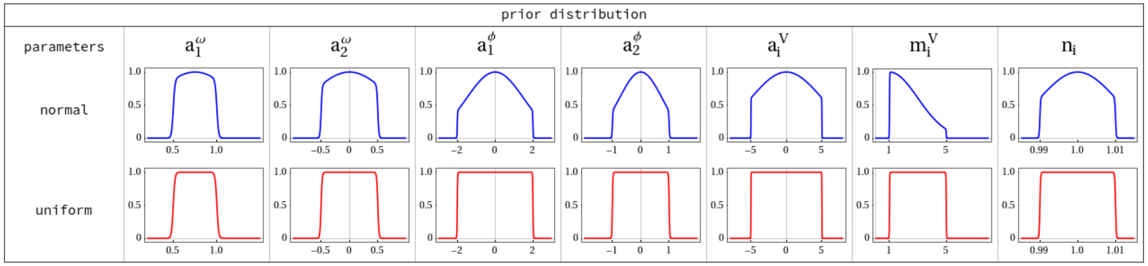
Bootstrap vs Bayesian in PRad fits

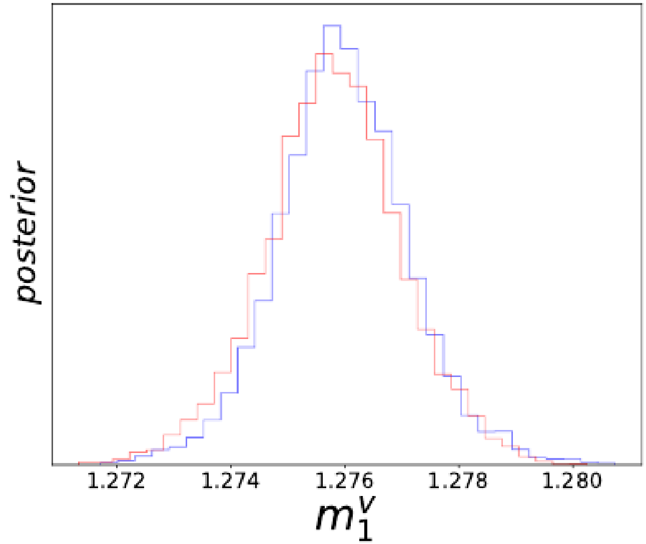
- Bayesian theorem

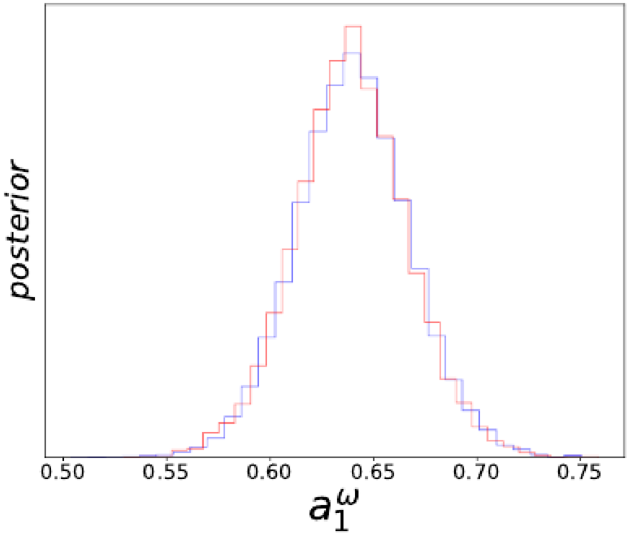
$$\color{red}{P(\text{parameters}|\text{data})} = \frac{\color{blue}{P(\text{parameters})}\color{green}{P(\text{data}|\text{parameters})}}{P(\text{data})}$$

$$\color{red}{\text{posterior}} \sim \color{blue}{\text{prior}} \times \color{green}{\text{likelihood}}$$

- Bayesian analysis



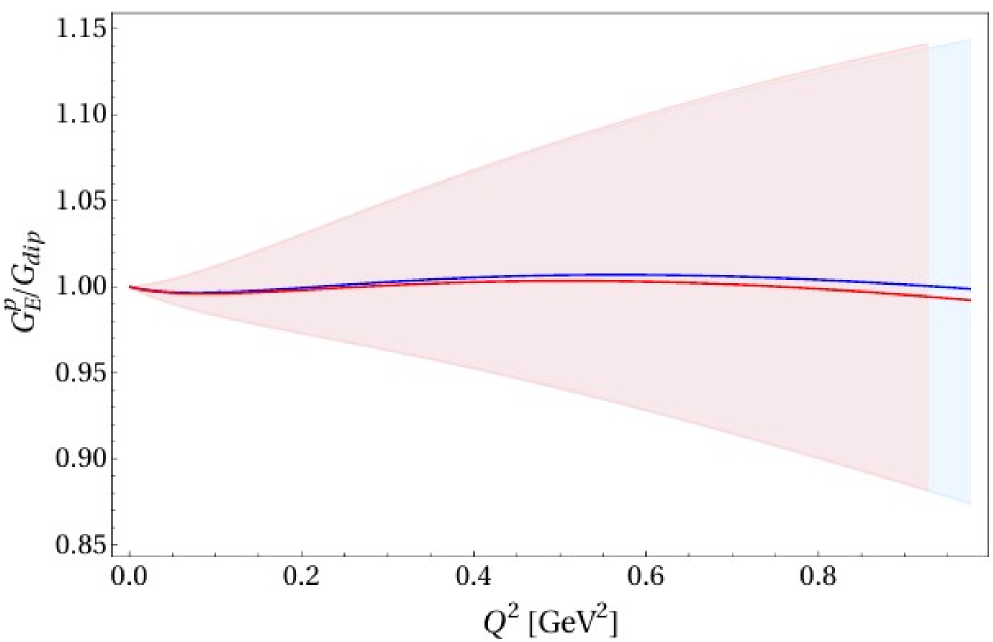


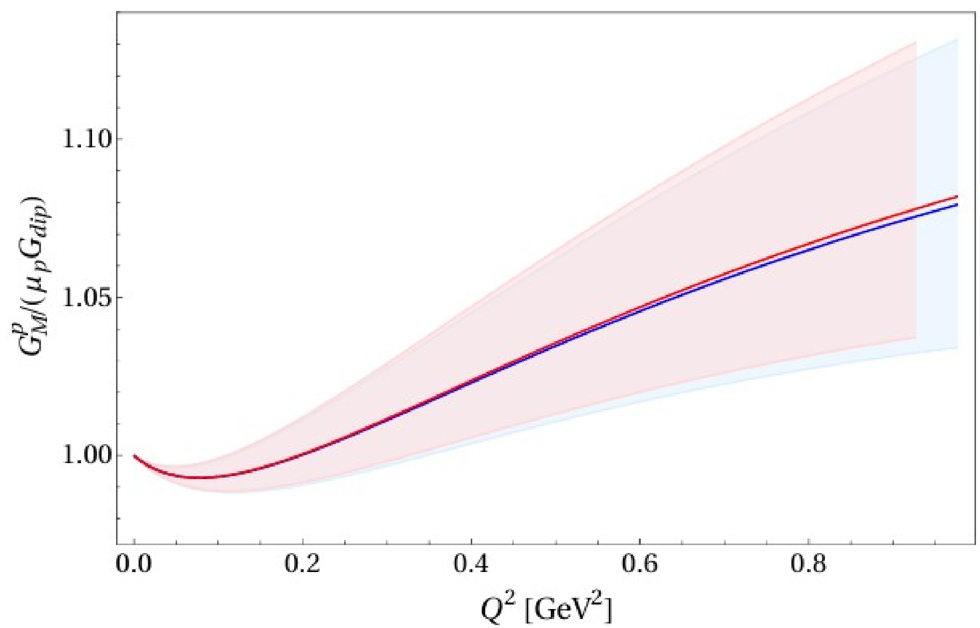


Bootstrap vs Bayesian in PRad fits

- Comparing with bootstrap sampling

Method	r_E^p [fm]	r_M^p [fm]
Bayesian normal	0.828 ± 0.011	0.843 ± 0.004
Bayesian uniform	0.828 ± 0.011	0.843 ± 0.004
Bootstrap	0.828 ± 0.012	0.843 ± 0.005





From “puzzle” to precision

- Some discussions

H.-W. Hammer, *et al.* SciBull65, 257(2020)
C. Peset, *et al.* 2106.00695

$$(\Delta E_L)_{\text{measured}} = E_1 + E_2 C(r_p^2) + \mathcal{O}(m_r \alpha^6), \quad C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

δ_{TPE} encoded in coefficients E_1, E_2, and $C \sim \mathcal{O}(m_l)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma_{\text{Mott}}}{d\Omega} \frac{1}{1+\tau_p} \left(G_E^2 + \frac{\tau_p}{\varepsilon} G_M^2\right)(1+\delta_{\text{TPE}}) + \mathcal{O}(\alpha^2)$$

$\langle r_p^2 \rangle$, δ_{TPE}, higher moments($\langle r_p^n \rangle$) and polarizabilities

interwinded together when goes to the higher order corrections.

Only when the theory and experiment are at the same order of accuracy

can the same $\langle r_p^2 \rangle$ be obtained.

Precision is the issue that really matters in the proton charge radius problem.