Theoretical Aspects on bound states, virtual states and resonances from Friedrichs model

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Motivation

- Hadronic states: Mesons, $q\bar{q}$, Baryons, qqq, ...
- The intermediate states in the scatterings: Resonance, virtual state(anti-bound), bound states.
- The intermediate state could be: $|q\bar{q}\rangle + |\text{two hadrons}\rangle \cdots$?
  E.g. $D\bar{D}^* \to \chi_{c1} \to D\bar{D}^*$.
- Pure composite states: dynamically generated. How to express using the component states?
- Can we define the compositeness and elementariness for a state?
- Dynamically generated states: How is it generated from interaction?
- To study these theoretical problems, look at a solvable model is instructive: Friedrichs model.

The simplest Friedrichs model[Friedrichs, Commun. Pure Appl. Math.,1(1948),361, See O. Civitaresea, M. Gadella, Phys.Rep.396,41 for review]

Different models in the same spirit: Lee model, Anderson model, Jaynes Cummings, ...

$$H = H_0 + V$$

- Free Hamiltonian:bare discrete state $|1\rangle$, a continuum state $|\omega\rangle$, (set threshold=0 for simplicity)

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| \mathrm{d}\omega$$

- Interaction:

$$V = \lambda \int_0^\infty [f(\omega)|\omega\rangle\langle 1| + f^*(\omega)|1\rangle\langle\omega|]\mathrm{d}\omega$$

- Orthonormal condition: $\langle 1|1\rangle = 1$, $\langle 1|\omega\rangle = 0$, and $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$

  Completeness: $|1\rangle\langle 1| + \int_0^\infty d\omega|\omega\rangle\langle\omega| = 1$

This model is exactly solvable.

Eigenvalue equation:

$$H|\Psi(E)\rangle = (H_0 + V)|\Psi\rangle = E|\Psi(E)\rangle.$$

Solutions:

- Continuum: Eigenvalue $E > 0$, real
  Solution: define inverse resolvent

$$\eta^{\pm}(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{E - \omega \pm i\epsilon} \mathrm{d}\omega$$

$$|\Psi_{\pm}(E)\rangle \quad = \quad |E\rangle + \lambda \frac{f^*(E)}{\eta^{\pm}(E)} \left[|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{E - \omega \pm i\epsilon} |\omega\rangle \mathrm{d}\omega \right]$$

- S-matrix:

$$S(E, E') = \delta(E - E') \left(1 - 2\pi i \frac{\lambda f(E) f^*(E)}{\eta^+(E)} \right).$$

- Discrete states:The zero point of $\eta(E)$ corresponds to eigenvalues of the full Hamiltonian — discrete states.

Discrete state solutions:Bound states

$$\eta^I(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{E - \omega} \mathrm{d}\omega = 0$$

- Bound states: solutions on the first sheet real axis below the threshold.

$$|z_B\rangle = N_B \Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_B - \omega} |\omega\rangle \mathrm{d}\omega \Big)$$

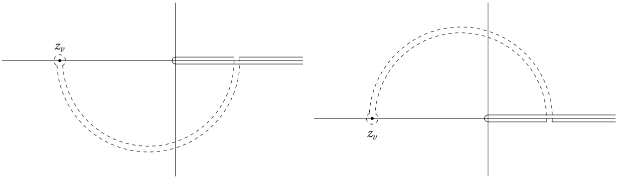
  where $N_B = (\eta'(z_B))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B - \omega)^2})^{-1/2}$, such that $\langle z_B | z_B \rangle = 1$.
- Elementariness: $Z = N_B^2$;
  Compositeness: $X = N_B^2 \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B - \omega)^2}$.
- Eg. If $\omega_0 < 0$, there could be a bound state. In the weak coupling limit, it $\to |1\rangle$,
- Eg. there could also be dynamically generated bound state when the coupling is strong.

Discrete state solutions: Virtual states

- Virtual states: Solutions on the second sheet real axis below the threshold.

  $$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda\int_0^{\infty}\frac{f(\omega)}{[z_v-\omega]_{\pm}}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^{\pm}| = \langle z_v^{\mp}|\,,$$

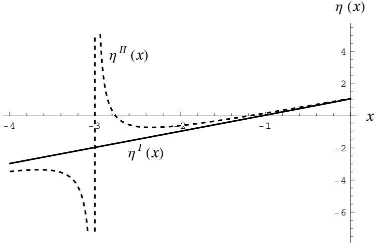
  where
  $N_v^{-} = N_v^{+*} = (\eta'^{+}(z_v))^{-1/2} = (1+\lambda^2\int d\omega\frac{|f(\omega)|^2}{[(z_v-\omega)_+]^2})^{-1/2}$,
  such that $\langle\tilde{z}_v^{\pm}|z_v^{\pm}\rangle = 1$. No probability explanation.
- Elementariness & compositeness not well-defined.



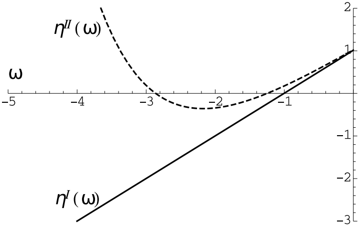
Discrete state solutions:Virtual states

- When $\omega_0 < 0$, a bound state generated from $|1\rangle$ is always accompanied with a virtual state for weak coupling, $\rightarrow |1\rangle$.
- Virtual states from the singularity of the form factor, $(|z_v\rangle \not\rightarrow |1\rangle$, at $\lambda \rightarrow 0)$

$$\eta^I = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega, \quad (G(\omega) \equiv |f(\omega)|^2)$$
$$\eta^{II}(\omega) = \eta^I(\omega) + 2\pi i \lambda^2 \, G^{II}(\omega) = \eta^I(\omega) - 2\lambda^2 \pi i \, G(\omega),$$



$G(\omega) \sim \frac{\sqrt{\omega}}{\omega+3}$

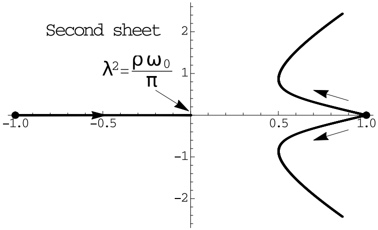


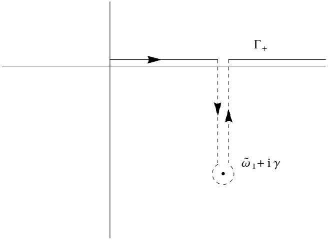
$G(\omega) \sim \sqrt{\omega} e^{-\omega/\Lambda}$,

Discrete state solutions: Resonance

- Resonant states: $\omega_0 >$ threshold, the discrete state becomes a pair of solutions z_R, z_R^*, on the second sheet of the complex plane. $\hat{H}|z_R\rangle = z_R|z_R\rangle$

$$|z_R\rangle = N_R\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R-\omega]_+}|\omega\rangle\Big),$$
$$|z_R^*\rangle = N_R^*\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^*-\omega]_-}|\omega\rangle\Big),$$





Discrete state solutions: Resonance

Resonant states:

- Normalization: $\langle z_R | z_R \rangle = 0$, naïve argument, $z_R^* \neq z_R$,

$$\langle z_R | \hat{H} | z_R \rangle = z_R \langle z_R | z_R \rangle = z_R^* \langle z_R | z_R \rangle = 0$$

  $|z_R\rangle$ is not in the Hilbert space — need rigged Hilbert space description.

- Left eigenstates:$\langle \tilde{z}_R | \hat{H} = \langle \tilde{z}_R | z_R$

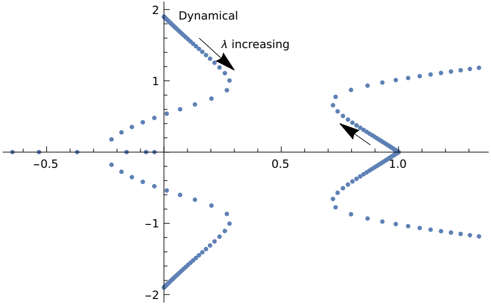
$$\langle \tilde{z}_R | = \langle z_R^* | = N_R \Big(\langle 1 | + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R - \omega]_+} \langle \omega | \Big),$$
$$\langle \tilde{z}_R^* | = \langle z_R | = N_R^* \Big(\langle 1 | + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^* - \omega]_-} \langle \omega | \Big).$$

  N_R is a complex normalization parameter, $N_R = (\eta'^{+}(z_R))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{[(z_R - \omega)_+]^2})^{-1/2}$ such that $\langle \tilde{z}_R | z_R \rangle = 1$, [Sekihara,Hyodo,Jido,PTEP 2015 (2015) 063D04]

- Other physical proposal of "elementariness" and "compositeness": [Guo,Oller,PRD93,096001].

Discrete state solutions: Dynamically generated Resonance

Dynamical resonance generated from the singularity of the form factor.



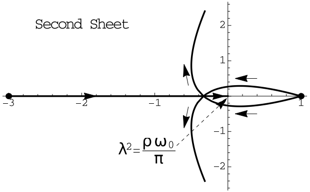
$$G(\omega) \sim \frac{\sqrt{\omega}}{\omega^2+a^2},\ a = 1.9,\ \omega_0 = 1$$

- $G = \sqrt{\omega}e^{-\omega^2/a^2}$ case : Similar situation could happen.
- A caveat to using form factor put by hand to suppress the high E contribution: The form factor may play an important role in generating the dynamical state.

Other interesting things

- Higher order poles: [A. Mondragon and E Hernandez,J.Phy.A26(1993),5595;A. Bohm et.al.JMP38(1997),6072; I. E. Antoniou et.al.,JMP39(1997),2459; E. Hernández et.al., Int.J.Theo.Phys.,42(2003), 2167]

  Hamiltonian:
  can not be diagonalized exactly,
  $\rightarrow$ Jordan form



- Completeness relation: redefine the continuum states to including the resonances into the completeness relation [T. Petrosky et..al. Phys.A173(1991),175;ZX,Zhou,PRD94(2016)076006]

GENERALIZATION: [ZYZ&ZX,JMP.58(2017),062110;JMP58(2017), 072102]

Real world: interaction between $|0; JM\rangle$ and $|\mathbf{p}_1\mathbf{p}_2, S\rangle$

- Partial wave decomposition: $|\mathbf{p}_1\mathbf{p}_2\rangle \to |p, JM, lS\rangle \sim |\omega, l\rangle$

$$H = M_0|0\rangle\langle 0| + \sum_l \int \mathrm{d}\omega\, \omega|\omega, l\rangle\langle\omega, l| + \sum_l \int d\omega\, g_l(\omega)|0\rangle\langle\omega, l| + h.c.$$

- Include more discrete states.
- Include interaction among continua: in general not solvable anymore.
- Separable interaction potential like in [E. Hernández et.al, PRC29(1984),722;Aceti et.al., PRD86,(2012),014012;Sekihara, PTEP(2015)063D04;Weinberg,PR131(1963),441;...]: solvable.

$$H = \sum_{i=1}^{D} M_i|i\rangle\langle i| + \sum_{i=1}^{C}\int_{a_i}^{\infty} \mathrm{d}\omega\, \omega|\omega; i\rangle\langle\omega; i|$$

$$+ \sum_{i,j=1}^{C} v_{ij}\Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i(\omega)|\omega; i\rangle\Big)\Big(\int_{a_j}^{\infty} \mathrm{d}\omega f_j^*(\omega)\langle\omega; j|\Big)$$

$$+ \sum_{j=1}^{D}\sum_{i=1}^{C}\Big[u_{ji}^*|j\rangle\Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i^*(\omega)\langle\omega; i|\Big) + u_{ji}\Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i(\omega)|\omega; i\rangle\Big)\langle j|\Big]$$

Dynamically generated states

Study the near threshold behavior of the dynamically generated states.

- No discrete bare states $\to$ dynamically generated discrete state — Bound state (molecular state), resonances, or virtual state.
- Hamiltonian:

$$H = \int_a \mathrm{d}\omega\, \omega |\omega\rangle\langle\omega| \pm \lambda^2 \int_a \mathrm{d}\omega \int_a \mathrm{d}\omega' f(\omega) f^*(\omega') |\omega\rangle\langle\omega'| \quad (1)$$

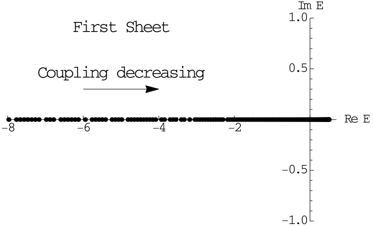
- Form factor
  $f(\omega) = (\omega - a)^{(l+1/2)/2} \exp\{-(\omega - a)/(2\Lambda)\}$.
- Discrete state pole position:

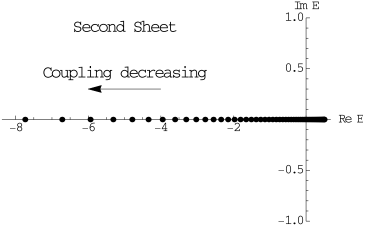
$$\mathbf{M}_\pm(E) = \det\, M_\pm = 1 \pm \lambda^2 G(E) = 1 \pm \lambda^2 \int_a \mathrm{d}\omega \frac{|f(\omega)|^2}{\omega - E} = 0$$

- $-$ sign: attractive.

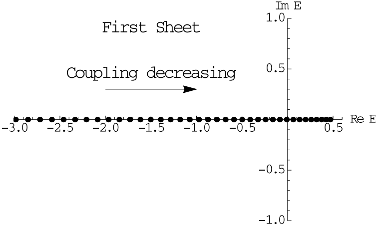
Eg: dynamically generated states, attractive potential

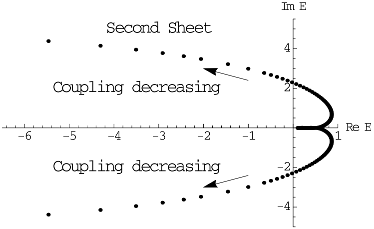
- S-wave: Strong coupling, a bound state → Weak coupling, a virtual state





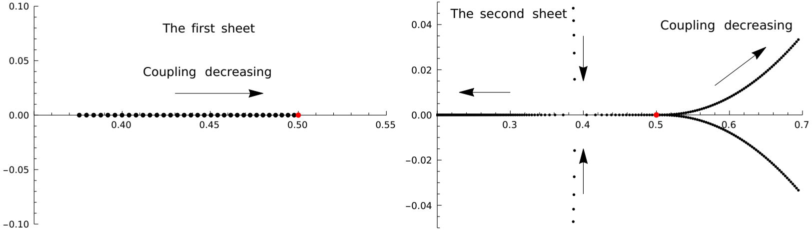
- P-wave: Strong coupling, A bound state
  and a virtual state → Weak coupling, a pair of resonance poles.





Eg: D-wave dynamically generated states

Attractive coupling:



- The resonance poles merge at the threshold, and one becomes a virtual state, the other becomes a bound states.

Near threshold poles for attractive potential: when coupling is becoming stronger

- $l \geq 1$: a virtual state and a bound state appear together.
- $l = 0$, one bound/virtual state near the threshold.
- Explained using the effective range expansion in Hanhart et. al. PLB739(2014)375 and also using Jost function in Hyodo,PRC90:055208(2014);

Dynamical v.s. Elementary

Elementary: originated from the bare discrete state
Dynamical: generated by interaction

- S-wave bound state:
  Dynamical state: have no acompanied virtual state.
  Elementary state: always accompanied with a virtual state pole at weak coupling
  — Pole counting rule [D. Morgan, NPA543(1992),632;Ou Zhang,C. Meng,H.Q. Zheng Phys.Lett.B680(2009),453]
- Higher partial wave , no such a difference: The dynamically generated state if appears from the threshold (resonance pole merging), it must acompanied with a virtual state.
- In weak coupling limit: The dynamically generated states do not go to bare states, but towards the singular point of the form factor.

Relativistic Friedrichs-Lee model

[Antoniou,et.al., JMP39(1998),2995; ZYZ&ZX,EPJC80(2020),1191]

- Creation and annihilation operators:
  Discrete bare state: $a^\dagger_{\vec{k}}|0\rangle = |\vec{k}\rangle$;
  Continuum: treat the two-particle states together,
  [JMP4(1963),490, Macfarlane; Nuo.Cim,34,1289,McKerrell]
  $B^\dagger_{\vec{p}m[wj]ls}|0\rangle = |\vec{p}m[wj]ls\rangle$

$$[B_{\vec{p}'m'[w'j']l's'}, B^\dagger_{\vec{p}m[wj]ls}] = \delta^{(3)}(\vec{p}-\vec{p}')\frac{\delta(q-q')}{q^2}\delta_{mm'}\delta_{ss'}\delta_{ll'}\delta_{jj'}$$

$$= \beta^{-1}\delta^{(3)}(\vec{p}-\vec{p}')\delta(E-E')\delta_{mm'}\delta_{ss'}\delta_{ll'}\delta_{jj'}, \quad \beta(E) = \frac{qq_1^0q_2^0E}{w^2}$$

- Hamiltonian:

$$P_0 = \int d^3\mathbf{k}\beta(E)dE\,E\,B^\dagger(E,\mathbf{k})B(E,\mathbf{k}) + \int d^3\mathbf{k}\,\omega(\mathbf{k})\,a^\dagger(\mathbf{k})a(\mathbf{k})$$
$$+ \int d^3\mathbf{k}\,\beta(E)dE\,\alpha(E,\mathbf{k})\left(a(\mathbf{k}) + a^\dagger(-\mathbf{k})\right)\left(B^\dagger(E,\mathbf{k}) + B(E,-\mathbf{k})\right)$$

$$\omega(\mathbf{k}) = \sqrt{m^2+\mathbf{k}^2}, \quad \alpha(E,\mathbf{k}) = \alpha^*(E,-\mathbf{k})$$

α: interaction form factor between the discret state and the continuum.

Relativisitc Friedrichs-Lee model

Find $b^\dagger$ s.t. $[H, b^\dagger(E)] = Eb^\dagger(E)$

- Continuum: $E > E_{th}$

$$b^\dagger_{in}(E,\mathbf{p}) = B^\dagger(E,\mathbf{p}) - \frac{2\omega(\mathbf{p})\alpha(k(E,\mathbf{p}))}{\eta_+(E,\mathbf{p})}\Big[\int_{M_{th}} dE'\beta(E')\alpha(k(E',\mathbf{p}))\Big[\frac{B^\dagger(E',\mathbf{p})}{(E'-E-i0)} - \frac{B(E',-\mathbf{p})}{(E'+E+i0)}\Big] - \frac{1}{2\omega(\mathbf{p})}\Big((\omega(\mathbf{p})+E)a^\dagger(\mathbf{p}) - (\omega(\mathbf{p})-E)a(-\mathbf{p})\Big)\Big],$$

$$\eta_\pm(s) = s - \omega_0^2 - \int_{s_{th}} ds'\frac{\rho(s')}{s-s'\pm i0}, \quad \rho = 2\omega_0\frac{k\varepsilon_1\varepsilon_2}{W}|\alpha(k)|^2$$

- S-matrix

$$S(E,\mathbf{p};E',\mathbf{p}') = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(E-E')\Big(1-2\pi i\frac{\rho(s)}{\eta_+(s)}\Big).$$

- Discrete state: at the solution of $\eta(z)=0$

$$b^\dagger(E_0,\mathbf{p}) = N\Big[\frac{(\omega(\mathbf{p})+E_0)}{\sqrt{2\omega(\mathbf{p})}}a^\dagger(\mathbf{p}) - \frac{(\omega(\mathbf{p})-E_0)}{\sqrt{2\omega(\mathbf{p})}}a(-\mathbf{p}) - \sqrt{2\omega(\mathbf{p})}\int_{M_{th}} dE'\beta(E')\Big[\frac{\alpha(k(E',\mathbf{p}))}{E'-E_0}B^\dagger(E',\mathbf{p}) - \frac{\alpha(k(E',\mathbf{p}))}{E'+E_0}B(E',-\mathbf{p})\Big]\Big],$$

For bound state $N = \frac{1}{\sqrt{2E_0}}\Big[1 + 2\omega(\mathbf{p})\int_{M_{th}} dE'\beta(E')\frac{2E'|\alpha(k(E',\mathbf{p}))|^2}{(E'+E_0)^2(E'-E_0)^2}\Big]^{-1/2}$

Applications: Friedrichs-QPC scheme

Friedrichs-QPC scheme

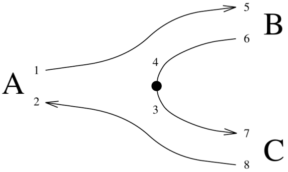
To study to hadron spectrum using nonrelativistic Friedrichs model: Solve $\eta(E) = 0$.

$$\eta = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega$$

- Coupling vertex between the discrete state and continuum $f(\omega)$: dynamically given.
- The interactions can be estimated using differnent models: we will use the QPC (3P0) model.

$$\langle BC|T|A\rangle = \delta^3(\vec{P}_f - \vec{P}_i) M^{ABC}$$

$$T = -3\gamma \sum_m \langle 1m1-m|00\rangle \int d^3\vec{p_3}\, d^3\vec{p_4}\, \delta^3(\vec{p_3}+\vec{p_4}) \times \mathcal{Y}_1^m\left(\frac{\vec{p_3}-\vec{p_4}}{2}\right)\chi^{34}_{1-m}\phi^{34}_0\omega^{34}_0 b_3^\dagger(\vec{p_3}) d_4^\dagger(\vec{p_4}).$$



γ: the strength of creating a quark-antiquark pair.
[Blundell,Godfrey,PRD53(1996),3700]

- The bare mass and wave functions of A, B, C are GI's results. [Godfrey & Isgur, PRD32,189(1985)].

$2P$ Charmonium-like states [Zhou& ZX,PRD96(2017),054031]

$L = 1,\ S = 1,\ J^{PC} = 0, 1, 2^{++}, \chi_{c0,1,2}$; $L = 1,\ S = 0,\ J^{PC} = 1^{+-}$, $h_c(2P)$.

Current status:

- 2^3P_2 is well established: $X(3930)$, [Belle,PRL96,082003;BaBar,PRD81,092003]
- 2^3P_1 channel: $X(3872)$ [Belle,PRL91,262001] molecular state or $c\bar{c}$? Mixture of molecule and $c\bar{c}$, and which is the dominant component?
- 2^1P_1 channel: The $h_c(2P)$ state still has not been seen by experiments.
- 2^3P_0 channel: $X(3915)$ [Belle,PRL104,092001;BaBar,PRD86,072002] (0^{++} or 2^{++}?[Zhou et.al., PRL115,022001]), $\chi_{c0}(3860)$ [Belle, PRD95,112003] ?
  Confusions [Guo,Meissner,PRD,86,091501; Olsen,PRD91,057501].

Our scheme

Friedrichs model + QPC model.

- Bare states: the masses and wave functions from the GI.
- Bare discretes $c\bar{c}$ states:
  $\chi_{c0}(2P)$ at 3917 MeV, $\chi_{c1}(2P)$ at 3953 MeV,
  $\chi_{c2}(2P)$ at 3979 MeV. $h_c(2P)$ at 3956MeV.
- OZI allowed continuum states: $D\bar{D}$, $D\bar{D}^*$, $D^*\bar{D}^*$ threshold, upto D-wave.

  Channels:
  $\chi_{c0}(2P)$: $D\bar{D}$ (S-wave), $D^*\bar{D}^*$ (S-wave, D-wave).
  $\chi_{c1}(2P)$: $D\bar{D}^*$(S,D-wave), $D^*\bar{D}^*$ (D-wave)
  $\chi_{c2}(2P)$: $D\bar{D}$ (D-wave), $D\bar{D}^*$ (D-wave), $D^*\bar{D}^*$ (S,D-wave)
  $h_c(2P)$: $D\bar{D}^*$ (S,D-wave), $D^*\bar{D}^*$ (S,D-wave).
- Parameterize the interactions between the bare states and the continua using the QPC model— only one free parameter γ.

Numerical results

Table: Comparison of the experimental masses and the total widths (in MeV) [PDG2016] with our results.

$n^{2s+1}L_J$	M_{expt}	Γ_{expt}	M_{BW}	Γ_{BW}	pole	GI
2^3P_2	3927.2 ± 2.6	24 ± 6	3910	12	3908-5i	3979
2^3P_1	3942 ± 9	37^{+27}_{-17}			3917-45i	3953
	3871.69 ± 0.17	< 1.2	3871	0	3871-0i	
2^3P_0	3862^{+66}_{-45}	201^{+180}_{-110}	3860	25	3861-11i	3917
2^1P_1			3890	26	3890-22i	3956

Numerical results

- Narrow 2^3P_2 state $\rightarrow$ well-established χ_{c2}.
- 2^3P_0 state: around 3860, narrow width $\sim 22MeV$. Belle: $M \sim 3862$, $\Gamma \sim 201^{+180}_{-110}$ MeV.
- Other predictions with small width, [Barnes et.al., PRD72,054026; Eichten et.al, PRD69,094019]

X(3872)

(2^3P_1) : $X(3872)$ & χ_{c1}

- Dynamical generated bound state $\sim 3871 \to X(3872)$
- Sensitive to γ parameter: decrease γ, $X(3872)$ pole $\to$ second sheet virtual state pole.
- Bare state pole $\to$ about 3917 MeV, a large width — may be related to $X(3940)$.
- $X(3872)$

$$\frac{\text{elementariness}}{\text{compositeness}} \sim 1 : 2.7.$$

  A large portion of continuum state $D\bar{D}^*$ — more molecular component than the $c\bar{c}$ component.
- This information helps us in understanding its decay. [Z.Y.Zhou,ZX,PRD97(2018),034011;PRD100(2019),094025]
- This method can also be used to discuss the X_b — bottomnium counterpart for $X(3872)$ [Z.Y.Zhou,ZX,PRD99 (2019) 3, 034005].

Application: Two-pole structure

- $X(3872)$ as an accompanying pole: dynamically generated by interaction between χ_{c1} and continuum $D\bar{D}^*$,$D^*\bar{D}^*$.
- This mechanism may be a general phenomenon in hadron spectra.
- Lightest scalars, two nonets:

  Non-$q\bar{q}$: $f_0(500)$, $K_0^*(700)$, $a_0(980)$ $f_0(980)$;

  $q\bar{q}$: $f_0(1370)$, $K_0^*(1430)$, $a_0(1450)$, $f_0(1500)$ or $f_0(1710)$.
- Non-$q\bar{q}$ with heavy c quark: $D_0^*(2300)$, $D_{s0}^*(2317)$.
- These non-$q\bar{q}$ candidates may be generated by this mechanism. We also consider the corresponding states with b quarks.
- For simplicity, only consider one discrete bare state $+$ one continuum.

Relativistic Friedrichs-Lee-QPC scheme

To discuss both the heavy and the light hadron states:

- The relativistic Freidrichs model: adding negative frequency modes. Dispersion integral $E \to s$.

$$\eta(s) = s - \omega_0^2 - \int_{s_{th}} ds' \frac{\rho(s')}{s - s'}, \quad \rho = 2\omega_0 \frac{k\varepsilon_1\varepsilon_2}{W} |\alpha(k)|^2$$

  Solve $\eta(z) = 0$, find poles of S-matrix: resonance, bound state, virtual state.

- Relativistic QPC: including the boost effect into the state definition[Beveran et.al. PRD27(1983),1527; Fuda,PRC86(2012), EPJC80(2020),1191;ZYZ&ZX, 055205;ZYZ&ZX, EPJC,81(2021),551].

When $\gamma = 4.3$ GeV, Single channel approximation: general appearance of two-pole structures

"discrete"	"continuum"	GI mass	Input	poles	experiment states	PDG values [15]
$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}(1^3P_0)$	$(\pi\pi)_{I=0}$	1.09	1.3	$\sqrt{s_{r1}} = 1.34 - 0.29i$	$f_0(1370)$	$1.35^{\pm 0.15} - 0.2^{\pm 0.05}i$
				$\sqrt{s_{r2}} = 0.39 - 0.26i$	$f_0(500)$	$0.475^{\pm 0.075} - 0.275^{\pm 0.075}i$
$u\bar{s}(1^3P_0)$	$(\pi K)_{I=\frac{1}{2}}$	1.23	1.42	$\sqrt{s_{r1}} = 1.41 - 0.17i$	$K_0^*(1430)$	$1.425^{\pm 0.05} - 0.135^{\pm 0.04}i$
				$\sqrt{s_{r2}} = 0.66 - 0.34i$	$K_0^*(700)$	$0.68^{\pm 0.05} - 0.30^{\pm 0.04}i$
$s\bar{s}(1^3P_0)$	$K\bar{K}$	1.35	1.68	$\sqrt{s_{r1}} = 1.71 - 0.16i$	$f_0(1710)$	$1.704^{\pm 0.012} - 0.062^{\pm 0.009}i$
				$\sqrt{s_b} = 0.98, \sqrt{s_v} = 0.19$	$f_0(980)$	$0.99^{\pm 0.02} - 0.028^{\pm 0.023}i$
$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}(1^3P_0)$	$\pi\eta$	1.09	1.3	$\sqrt{s_{r1}} = 1.26 - 0.14i$	$a_0(1450)$	$1.474^{\pm 0.019} - 0.133^{\pm 0.007}i$
				$\sqrt{s_{r2}} = 0.70 - 0.42i$	$a_0(980)$	$0.98^{\pm 0.02} - 0.038^{\pm 0.012}i$
$c\bar{u}(1^3P_0)$	$D\pi$	2.4	2.4	$\sqrt{s_{r1}} = 2.58 - 0.24i$	$D_0^*(2300)$	$2.30^{\pm 0.019} - 0.137^{\pm 0.02}i$
				$\sqrt{s_{r2}} = 2.08 - 0.10i$		
$c\bar{s}(1^3P_0)$	DK	2.48	2.48	$\sqrt{s_{r1}} = 2.80 - 0.23i$		
				$\sqrt{s_b} = 2.24, \sqrt{s_v} = 1.8$	$D_{s0}^*(2317)$	$2.317^{\pm 0.0005} - 0.0038^{\pm 0.0038}i$
$b\bar{u}(1^3P_0)$	$\bar{B}\pi$	5.76	5.76	$\sqrt{s_{r1}} = 6.01 - 0.21i$		
				$\sqrt{s_{r2}} = 5.56 - 0.07i$		
$b\bar{s}(1^3P_0)$	$\bar{B}K$	5.83	5.83	$\sqrt{s_{r1}} = 6.23 - 0.17i$		
				$\sqrt{s_b} = 5.66, \sqrt{s_v} = 5.3$		
$c\bar{c}(2^3P_1)$	$D\bar{D}^*$	3.95	3.95	$\sqrt{s_{r1}} = 4.01 - 0.049i$	$X(3940)$	
				$\sqrt{s_b} = 3.785$	$X(3872)$	$3.87169^{\pm 0.00017}$

Two-pole structures

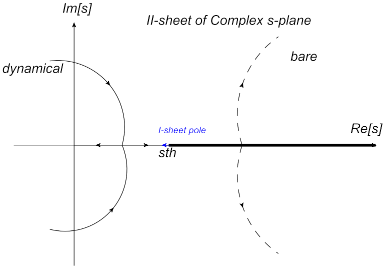
Two pole structure, a general phenomenon:

> Coupling a seed $q\bar{q}$ state with the nearest open flavor states in S-wave — another new dynamical state ("dynamical pole").

Other models:[Törnqvist,PRL49(1982),624,Z.Phys.68(1995),647; E. van Beveren et.al.,Z.Phys.C30,615,PRD.27,1527; Boglione,Pennington, PRD65,114010; Kalashnikova,PRD72,034010; Ortega,et.al., PRD81,054023; Wolkanowski,et.al.,PRD93,014002; NPB909(2016)418 ...]

When the coupling γ is turned on

- The seed will move into the second sheet — a pair of resonance poles ("bare pole").
- The dynamical pole comes from faraway on the second sheet towards the real axis: Resonance or virtual state or /and bound state poles.



$c\bar{u}$ SEED, $b\bar{u}$ SEED

- $c\bar{u}$ seed couples to $D\pi$: $D_0^*(2300)$, two broad poles

$$\gamma = 4.3: \quad \sqrt{s_1} = 2.08 - i0.10; \quad \sqrt{s_2} = 2.58 - i0.24$$

$$\gamma = 3: \quad \sqrt{s_1} = 2.21 - i0.28; \quad \sqrt{s_2} = 2.39 - i0.18$$

- Two-poles From Unitarized χPT: $D_0^*(2300)$, two poles

  PLB582(2004),39,EEK et.al; PLB641(2006),278, FK.Guo, et. al.; PLB,767(2017),465, MA,et.al.:

$$\sqrt{s_1} = 2.105 - i0.102; \quad \sqrt{s_2} = 2.451 - i0.134$$

  PRD92(2015),094008,ZH.Guo et.al.:

$$\sqrt{s_1} = 2.114 - i0.111; \quad \sqrt{s_2} = 2.473 - i0.140$$

- $b\bar{u}$ couples to $\bar{B}\pi$:

$$\gamma = 4.3: \sqrt{s_1} = 5.556 - i0.07; \quad \sqrt{s_2} = 6.01 - i0.21$$

$$\gamma = 3.0: \quad \sqrt{s_1} = 5.62 - i0.13; \quad \sqrt{s_2} = 5.85 - i0.26$$

  Unitarized χPT:

$$\sqrt{s_1} = 5.537 - i0.116; \quad \sqrt{s_2} = 5.840 - i0.025$$

$c\bar{s}$, $b\bar{s}$ SEEDS

- $c\bar{s}$ couples to DK: $D^*_{s0}(2317)$, dynamically generated;

$$\gamma = 4.3 : \sqrt{s_b} = 2.24, \quad \sqrt{s_v} = 1.8, \quad \sqrt{s_{r1}} = 2.80 - 0.23i$$

$$\gamma = 3.0 : \sqrt{s_b} = 2.32, \quad \sqrt{s_v} = 1.9, \quad \sqrt{s_{r1}} = 2.68 - 0.26i$$

- $b\bar{s}$ couples to $\bar{B}K$:

$$\gamma = 4.3 : \quad \sqrt{s_b} = 5.66, \quad \sqrt{s_v} = 5.3, \quad \sqrt{s_{r1}} = 6.23 - 0.17i$$

$$\gamma = 3.0 : \quad \sqrt{s_b} = 5.72, \quad \sqrt{s_v} = 5.4, \quad \sqrt{s_{r1}} = 6.11 - 0.22i$$

Conclusion

- As an rigourously solvable model, Friedrichs model helps us in understanding the resonances, virtual states, and bound states
- Understand why resonances, virtual states are not normalizable as usual, and compositeness and elementariness not well-defined.
- How dynamical state is generated from the interaction between the discrete state and the continuum.
- Given the interaction vertices, it can still be used in the discussion of the real hadronic states.
- Two pole structure — Two states dynamically related — May be a general phenomenon.

Thank you !