Auxiliary mass flow and Feynman integrals

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Liu, Ma, Wang, *Phys. Lett. B* **779**, (2018) 353-357
Liu, Ma, Tao, Zhang, *Chin. Phys. C* **45**, (2021) 013115
Liu, Ma, arXiv: 2107.01864 (2021)

Outline

- Introduction
- Auxiliary mass flow
- Summary

Introduction

Physics at Large Hadron Collider

- improving requirements on theoretical predictions
- new NNLO and N3LO cross sections
- bottleneck problems
  - integrals reduction
  - master integrals calculation

Introduction

Methods

- Integrals reduction
  - integration by parts [Chetyrkin , Tkachov, *1981*] [Laporta, 2000]
  - syzygy equation [Gluza , Kajda , Kosower , 2011] [Schabinger , 2012] [Larsen, Zhang, 2016]
  - finite field interpolation [Manteuffel , Schabinger , 2015] [Peraro, 2016]
  - intersection numbers [Mastrolia , Mizera , 2019] [Frellesvig , et al, 2019]
  - block triangular systems [Liu, Ma, 2019] [Guan, Liu, Ma, 2020]
- Master integrals calculation
  - differential equation and its canonical form [Kotikov, 1991] [Henn, 2013]
  - Mellin Barnes representation [Smirnov, 1999]
  - sector decomposition [Binoth, Heinrich, 2000]
  - difference equation [Tarasov, 1996] [Laporta, 2000] [Lee, 2009]
  - numerical differential equation [Caffo, et al, 2008] [Czakon, 2008] [Lee, et al, 2017]
  - auxiliary mass flow [Liu, Ma, Wang, 2018]

General phase-space and loop integral

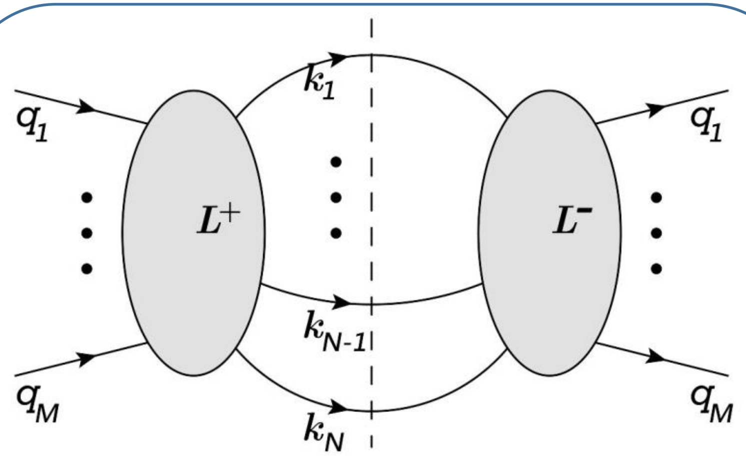
- form

$$F(\vec{\nu};\vec{s}) \equiv \int \mathrm{dPS}_N \prod_\alpha \frac{1}{(\mathcal{D}^{\mathrm{t}}_\alpha)^{\nu^{\mathrm{t}}_\alpha}}$$

$$\int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_\beta \frac{1}{(\mathcal{D}^+_\beta + \mathrm{i}0^+)^{\nu^+_\beta}}$$

$$\int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_\gamma \frac{1}{(\mathcal{D}^-_\gamma - \mathrm{i}0^+)^{\nu^-_\gamma}}$$

$$\prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu^\pm_{ij}},$$



$$Q = \sum_{i=1}^{M} q_i = \sum_{i=1}^{N} k_i$$

- $\vec{\nu} \equiv (\nu^{\mathrm{t}}_1, \nu^{\mathrm{t}}_2, \cdots, \nu^+_1, \nu^+_2, \cdots, \nu^-_1, \nu^-_2, \cdots, \nu^\pm_{11}, \nu^\pm_{12}, \nu^\pm_{21}, \cdots)$
- $\vec{s}$: kinematical invariants (including Q^2)
- phase-space

$$\mathrm{dPS}_N \equiv (2\pi)^D \delta^D(Q - \sum_{i=1}^{N} k_i) \prod_{i=1}^{N} \frac{\mathrm{d}^D k_i}{(2\pi)^D} (2\pi)\delta(\mathcal{D}^{\mathrm{c}}_i)\Theta(k_i^0 - m_i)$$

Auxiliary mass flow

Examples: VRR of $\gamma^* \to t\bar{t} + X$

- notation: $\mathrm{V}^{L^+}\mathrm{R}^{N-1}\mathrm{V}^{L^-}$ to distinguish sub-processes
  - (a) is the most complicated diagram
  - (b) is (a)'s sub-diagram, take it as an example
- the phase-space integrals

$$\hat{F}(\{\nu_1^{\mathrm{t}}, \nu_1^+, \nu_2^+\}; s) = \int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}\,\nu_1^{\mathrm{t}}}} \int \frac{\mathrm{d}^D l_1^+}{(2\pi)^D} \frac{1}{(\mathcal{D}_1^+ + \mathrm{i}0^+)^{\nu_1^+}(\mathcal{D}_2^+ + \mathrm{i}0^+)^{\nu_2^+}}$$

- inverse propagators

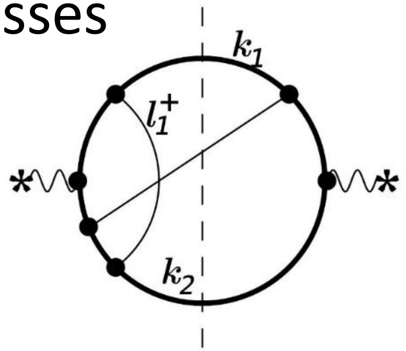
$$\mathcal{D}_1^{\mathrm{t}} = (Q - k_2)^2 - m_t^2,$$
$$\mathcal{D}_1^+ = (k_1 + l_1^+)^2 - m_t^2,$$
$$\mathcal{D}_2^+ = (k_2 - l_1^+)^2 - m_t^2,$$

- take

$$s = Q^2\,, \quad x = \frac{4m_t^2}{s}$$

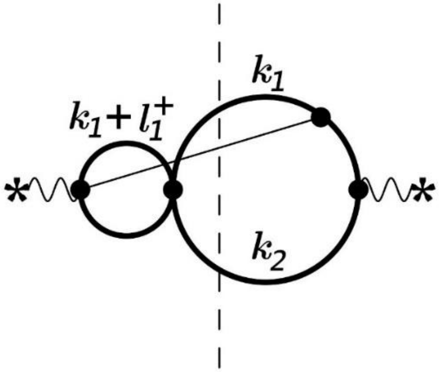
- dimensionless integrals

$$\hat{F}(\{\nu_1^{\mathrm{t}}, \nu_1^+, \nu_2^+\}; x) = s^{3-\frac{3}{2}D+\nu_1^{\mathrm{t}}+\nu_1^++\nu_2^+} \hat{F}(\{\nu_1^{\mathrm{t}}, \nu_1^+, \nu_2^+\}; s)$$



(a)

a typical Feynman diagrams



(b)

Auxiliary mass flow

Differential equations (DEs)

- Reverse unitarity relation

Anastasiou, Melnikov, *Nucl. Phys. B* **646** (2002), 220-256

$$(2\pi)\delta(\mathcal{D}_i^c) = \frac{\mathrm{i}}{\mathcal{D}_i^c + \mathrm{i}0^+} + \frac{-\mathrm{i}}{\mathcal{D}_i^c - \mathrm{i}0^+}$$

$\mathcal{D}_i^c = k_i^2 - m_i^2$

  - map phase-space integrals onto pure loop integrals

- IBP reduction

Chetyrkin, Tkachov, *Nucl. Phys. B* **192**, (1981) 159-204

- set up DEs w.r.t x among MIs ($\vec{I}(x)$)

$$\frac{\partial}{\partial x}\vec{I}(x) = M(x)\vec{I}(x)$$

Kotikov, *Phys. Lett. B* **254**, (1991) 158-164
Gehrmann, Remiddi, *Nucl. Phys. B* **580**, (2000) 485-518

  - DEs w.r.t. kinematical invariants
    - auxiliary mass flow (AMF) : DEs w.r.t. auxiliary mass

- Boundary conditions of DEs
  - At singularity: analyze regions
  - At regular point: auxiliary mass flow

Auxiliary mass flow

Add auxiliary mass to inverse propagators

- general form

$$\vec{\eta} \equiv (\eta_1^{\mathrm{t}}, \eta_2^{\mathrm{t}}, \cdots, \eta_1^{+}, \eta_2^{+}, \cdots, \eta_1^{-}, \eta_2^{-}, \cdots)$$

$$F(\vec{\nu}; \vec{s}, \vec{\eta}) \equiv \int \mathrm{dPS}_N \prod_{\alpha} \frac{1}{(\mathcal{D}_\alpha^{\mathrm{t}} + \eta_\alpha^{\mathrm{t}})^{\nu_\alpha^{\mathrm{t}}}} \int \prod_{i=1}^{L^+} \frac{\mathrm{d}^D l_i^+}{(2\pi)^D} \prod_{\beta} \frac{1}{(\mathcal{D}_\beta^+ + \mathrm{i}\eta_\beta^+)^{\nu_\beta^+}} \int \prod_{j=1}^{L^-} \frac{\mathrm{d}^D l_j^-}{(2\pi)^D} \prod_{\gamma} \frac{1}{(\mathcal{D}_\gamma^- - \mathrm{i}\eta_\gamma^-)^{\nu_\gamma^-}} \prod_{i,j} (l_i^+ \cdot l_j^-)^{-\nu_{ij}^{\pm}}$$

- direction choice of $\vec{\eta} \to 0$
  - Rule of Feynman prescription for Feynman propagators

    take $\eta_\beta^+ \to 0^+$ and $\eta_\gamma^- \to 0^+$
  - $\eta_\alpha^t \to 0^+$ or $\eta_\alpha^t \to 0^-$ are both fine

    take $\eta_\alpha^t \to 0^+$ for convenient
- choice of finite $\vec{\eta}$
  - either related to each others or completely independent
- our choice: either 0^+ or η
  - if $\{\mathcal{D}_\beta^+\}$ or $\{\mathcal{D}_\gamma^-\}$ depend on $\vec{s}$, choose $\eta_\alpha^t \to 0^+$ and $\eta_\beta^+ = \eta_\gamma^- = \eta$
  - else, choose $\eta_\alpha^t = \eta$

Auxiliary mass flow

VRR: sub-diagram (b)

- take $y = \eta/s$
- define dimensional integrals

$$\mathcal{D}_1^{\mathrm{t}} = (Q-k_2)^2 - m_t^2,$$
$$\mathcal{D}_1^{+} = (k_1+l_1^{+})^2 - m_t^2,$$
$$\mathcal{D}_2^{+} = (k_2-l_1^{+})^2 - m_t^2,$$

$$\hat{F}(\{\nu_1^{\mathrm{t}}, \nu_1^{+}, \nu_2^{+}\}; x, y) = s^{3-\frac{3}{2}D+\nu_1^{\mathrm{t}}+\nu_1^{+}+\nu_2^{+}} \int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}\,\nu_1^{\mathrm{t}}}} \int \frac{\mathrm{d}^D l_1^{+}}{(2\pi)^D} \frac{1}{(\mathcal{D}_1^{+}+\mathrm{i}\eta)^{\nu_1^{+}}(\mathcal{D}_2^{+}+\mathrm{i}\eta)^{\nu_2^{+}}}$$

- after reduction, 7 MIs for finite η (or y)

$$\Big\{\hat{F}(\{0,0,1\};x,y), \hat{F}(\{-1,0,1\};x,y), \hat{F}(\{0,1,1\};x,y), \hat{F}(\{-1,1,1\};x,y),$$
$$\hat{F}(\{0,1,2\};x,y), \hat{F}(\{1,1,1\};x,y), \hat{F}(\{1,1,2\};x,y)\Big\},$$

- 6 MIs for $\eta \to 0^+$

$$\Big\{\hat{F}(\{0,0,1\};x,0^+), \hat{F}(\{-1,0,1\};x,0^+), \hat{F}(\{0,1,1\};x,0^+),$$
$$\hat{F}(\{0,1,2\};x,0^+), \hat{F}(\{1,1,1\};x,0^+), \hat{F}(\{1,1,2\};x,0^+)\Big\}.$$

Auxiliary mass flow

VRR: sub-diagram (b)

- choose $x = 1/2$ (regular point) and set up DEs

$$
\frac{\partial}{\partial y}\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};\frac{1}{2},y) =
$$

$$
\begin{bmatrix}
\frac{-8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\mathrm{i} & 0 & 0 \\
\frac{8(-1+\epsilon)}{\mathrm{i}+8y} & 0 & \mathrm{i}(-1+2\epsilon) & 0 & \frac{1}{2(-\mathrm{i}+8y)} & 0 & 0 \\
\frac{-16\left(\substack{5\mathrm{i}+20y-11\mathrm{i}\epsilon- \\ 52y\epsilon+6\mathrm{i}\epsilon^2+32y\epsilon^2}\right)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)^2} & \frac{192\mathrm{i}(-1+\epsilon)^2}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)^2} & \frac{2(1-2\epsilon)\left(\substack{5-20\mathrm{i}y- \\ 6\epsilon+40\mathrm{i}y\epsilon}\right)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{8(1-2\epsilon)(-3+4\epsilon)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{\left(\substack{1+16\mathrm{i}y+128y^2- \\ 2\epsilon-16\mathrm{i}y\epsilon-384y^2\epsilon}\right)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\mathrm{i} \\
\frac{-64(5-11\epsilon+6\epsilon^2)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{768(-1+\epsilon)^2}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{8(1-2\epsilon)\left(\substack{5-4\mathrm{i}y- \\ 6\epsilon+8\mathrm{i}y\epsilon}\right)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{32(1-2\epsilon)(-3+4\epsilon)}{y(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{-4(-\mathrm{i}+4y)(-1+2\epsilon)}{y(-\mathrm{i}+8y)} & \frac{16\mathrm{i}(1-2\epsilon)(3\epsilon-1)}{(-\mathrm{i}+8y)(\mathrm{i}+8y)} & \frac{4(\mathrm{i}+8y-4\mathrm{i}\epsilon-64y\epsilon)}{(-\mathrm{i}+8y)(\mathrm{i}+8y)}
\end{bmatrix}
$$

$$
\hat{F}(\{\nu_1^{\mathrm{t}},\nu_1^{+},\nu_2^{+}\};\frac{1}{2},y)
$$

- take boundaries at $\eta \to \infty$

Auxiliary mass flow

Expansion for tree propagators at $\eta \to \infty$

- scalar products among external momenta and cut momenta are finite

$$\frac{1}{\mathcal{D}_\alpha^{\mathrm{t}} + \eta} \xlongequal{\eta\to\infty} \frac{1}{\eta} \sum_{j=0}^{+\infty} \left(\frac{-\mathcal{D}_\alpha^{\mathrm{t}}}{\eta} \right)^j$$

$$\frac{1}{\mathcal{D}_\alpha^{\mathrm{t}}} \xlongequal{\eta\to\infty} \frac{1}{\mathcal{D}_\alpha^{\mathrm{t}}},$$

- If η is introduced, tree propagators are removed
- else, tree propagators remain

Auxiliary mass flow

Expansion for loop propagators at $\eta \to \infty$

- loop momenta can be any large value
- at $\eta \to \infty$, linear combinations of loop momenta can be either at the order of $|\eta|^{1/2}$ or much smaller than it
- decompose $\mathcal{D}_\alpha^+$ into two parts

$$\mathcal{D}_\alpha^+ = \widetilde{\mathcal{D}}_\alpha^+ + K_\alpha$$

  - $\widetilde{\mathcal{D}}_\alpha^+$: only including the part at order $|\eta|$
  - K_α: other parts

Auxiliary mass flow

Expansion for loop propagators at $\eta \to \infty$

$$\frac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}\eta} \xlongequal{\eta\to\infty} \frac{1}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}\eta} \sum_{j=0}^{+\infty} \left(\frac{-K_\alpha}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}\eta} \right)^j,$$

$$\frac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}0^+} \xlongequal{\eta\to\infty} \begin{cases} \dfrac{1}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}0^+} \displaystyle\sum_{j=0}^{+\infty} \left(\frac{-K_\alpha}{\widetilde{\mathcal{D}}_\alpha^+ + \mathrm{i}0^+} \right)^j & \text{if } \widetilde{\mathcal{D}}_\alpha^+ \neq 0, \\ \dfrac{1}{\mathcal{D}_\alpha^+ + \mathrm{i}0^+} & \text{if } \widetilde{\mathcal{D}}_\alpha^+ = 0, \end{cases}$$

- expansion for $\mathcal{D}_\alpha^-$ are similar
- propagators (η exits and $\widetilde{\mathcal{D}}_\alpha^+ = 0$) removed
- decouple some loop momenta at order $|\eta|^{1/2}$

$\longrightarrow$ single-scale vacuum integrals factored

eg: $l_\alpha^+ \cdot k_i$, $l_\alpha^+ \cdot q_i$

Auxiliary mass flow

Expansion at $\eta \to \infty$

- $F(\vec{v};\vec{s},\eta)$ is simplified to a linear combination of integrals with fewer inverse propagators

$$F(\vec{v};\vec{s},\eta) \longrightarrow \sum c \times F^{\mathrm{cut}} \times F^{\mathrm{bub}}$$

- c are rational functions of $\vec{s}$ and η
- F^{bub}: single-scale vacuum bubble integrals
  - studied up to five-loop order

Luthe, Maier, Marquard et al., *JHEP* **03**, (2017) 020

- F^{cut} : basal phase-space integrations with the integrands being polynomials of scalar products between cut momenta.
  - also studied for $m_i = 0$ or m (no more than 2)

Bernreuther, Bogner, Dekkers, *JHEP* **06**, (2011) 032
Liu, Ma, Tao, Zhang, *Chin. Phys. C* **45**, (2021) 013115

Auxiliary mass flow

VRR: sub-diagram (b)

$$F^{\mathrm{bub}}_{1,1}(D) \equiv \int \frac{\mathrm{d}^D l_1^+}{(2\pi)^D} \frac{1}{{l_1^+}^2 + \mathrm{i}}$$

$$\hat{F}(\{0,0,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{4-\frac{3}{2}D}\eta^{\frac{D}{2}-1} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{-1,0,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{3-\frac{3}{2}D}\eta^{\frac{D}{2}-1} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3 \mathcal{D}_1^{\mathrm{t}}\right)_{x=1/2}$$

$$\hat{F}(\{0,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{5-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{\mathrm{i}(D-2)}{2} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{-1,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{4-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{\mathrm{i}(D-2)}{2} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3 \mathcal{D}_1^{\mathrm{t}}\right)_{x=1/2}$$

$$\hat{F}(\{0,1,2\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{6-\frac{3}{2}D}\eta^{\frac{D}{2}-3} \frac{(4-D)(D-2)}{8} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3\right)_{x=1/2}$$

$$\hat{F}(\{1,1,1\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{6-\frac{3}{2}D}\eta^{\frac{D}{2}-2} \frac{\mathrm{i}(D-2)}{2} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}}}\right)_{x=1/2}$$

$$\hat{F}(\{1,1,2\};\frac{1}{2},y) \xlongequal{\eta\sim\infty} s^{7-\frac{3}{2}D}\eta^{\frac{D}{2}-3} \frac{(4-D)(D-2)}{8} F^{\mathrm{bub}}_{1,1}(D)\left(\int \mathrm{dPS}_3 \frac{1}{\mathcal{D}_1^{\mathrm{t}}}\right)_{x=1/2}$$

Auxiliary mass flow

basal phase-space integrations

- $\int \mathrm{dPS}_3(\mathcal{D}_1^t)^i$ can be reduced to two MIs of RR process
- $F_{r,N,n}^{\mathrm{cut}}$ denote the *n*-th MI for *N*-particle-cut integrals with $m_1 = \cdots = m_r = m$ and $m_{r+1} = \cdots = m_N = 0$
- for $N = 3$, two MIs: $F_{2,3,1}^{\mathrm{cut}}$ and $F_{2,3,2}^{\mathrm{cut}}$
- definition

$$F_{2,N,n}^{\mathrm{cut}} \equiv \int \mathrm{dPS}_N \left((k_1+k_2)^2\right)^{n-1}$$

Auxiliary mass flow

basal phase-space integrations

- MI result of RR

$$F^{\text{cut}}_{2,N,n} \equiv \int \mathrm{dPS}_N \left((k_1+k_2)^2\right)^{n-1} = \frac{2^{5+2N(\epsilon-2)-2\epsilon}\pi^{3+N(\epsilon-2)-\epsilon}\Gamma(1+n-2\epsilon)\Gamma(1-\epsilon)^{N-1}\Gamma(n-\epsilon)}{\Gamma(2-2\epsilon)\Gamma(n-1+N-N\epsilon)\Gamma(n-2+N+\epsilon-N\epsilon)} s^{n-3+N+\epsilon-N\epsilon}$$

$$\times {}_3F_2\left(\epsilon-\frac{1}{2}, 2-n-N+N\epsilon, 3-n-N-\epsilon+N\epsilon; 1-n+\epsilon, 2\epsilon-n; \frac{4m^2}{s}\right)$$

$$+\frac{2^{4+2N(\epsilon-2)}\pi^{\frac{7}{2}+N(\epsilon-2)-\epsilon}\Gamma(1-\epsilon)^{N-1}\Gamma(\epsilon-n)}{\Gamma\left(\frac{3}{2}-n\right)\Gamma((N-1)(1-\epsilon))\Gamma((N-2)(1-\epsilon))} s^{n-3+N+\epsilon-N\epsilon}\left(\frac{4m^2}{s}\right)^{n-\epsilon}$$

$$\times {}_3F_2\left(n-\frac{1}{2}, 3-N-2\epsilon+N\epsilon, 2-N-\epsilon+N\epsilon; 1+n-\epsilon, \epsilon; \frac{4m^2}{s}\right)$$

$$+\frac{2^{4+2N(\epsilon-2)}\pi^{\frac{7}{2}+N(\epsilon-2)-\epsilon}\Gamma(1-\epsilon)^{N-2}\Gamma(\epsilon-1)\Gamma(2\epsilon-1-n)}{\Gamma\left(\frac{1}{2}-n+\epsilon\right)\Gamma((N-2)(1-\epsilon))\Gamma((N-3)(1-\epsilon))} s^{n-3+N+\epsilon-N\epsilon}\left(\frac{4m^2}{s}\right)^{1+n-2\epsilon}$$

$$\times {}_3F_2\left(\frac{1}{2}+n-\epsilon, 4-N-3\epsilon+N\epsilon, 3-N-2\epsilon+N\epsilon; 2+n-2\epsilon, 2-\epsilon; \frac{4m^2}{s}\right).$$

Comment

basal phase-space integrations

- without η, MIs of RRR are not the basal phase-space integrations
- add η and make expansion at $\eta \to \infty$ (see Page 11)
- all $\mathcal{D}_\alpha^t$ come to the numerators
- then MIs are all basal phase-space integrations

Auxiliary mass flow

Flow of $\eta\ (y)$

- Set up DEs w.r.t. y (as shown in page 10)
- boundary condition: fixed x$=1/2$ and $y \to \infty$
- solve DEs with the flow of y from ∞ to 0^+
- eg: $\hat{F}(\{1,1,2\};1/2,0)$

$$
\begin{aligned}
\hat{F}\left(\{1,1,2\};\frac{1}{2},0\right) =&(7.7879044672106926250285009377 4\times 10^{-6}+2.9131946977223739413535630834 8\times 10^{-6}\mathrm{i})\\
&+(0.000130430373015787655604488198861+0.0000684041694582011842919200921 23\mathrm{i})\epsilon\\
&+(0.00107743481382819190978718636243 2+0.000750926876250745472210277589430\mathrm{i})\epsilon^2\\
&+(0.00584278150920839062615612508136+0.00527570101382158661589031061691\mathrm{i})\epsilon^3\\
&+(0.0233461280012444372334494219123+0.0270859736951617524563966282868\mathrm{i})\epsilon^4\\
&+(0.0730918539437148667076104654800+0.1095165249743204252589933869672\mathrm{i})\epsilon^5\\
&+(0.1859753738831259864886138815 20+0.3663937700427084433315648015 09\mathrm{i})\epsilon^6\\
&+(0.3930939861885190765124246945 64+1.0521721707656381162578254106 32\mathrm{i})\epsilon^7\\
&+(0.6977527729960686104870625004 7+2.6728254612200838302261510428 9\mathrm{i})\epsilon^8+\cdots.
\end{aligned}
$$

Summary

- Reverse unitarity relation transforms the delta function to inverse propagators on cut
- With IBP reduction, complex integrals can be reduced to linear combination of MIs
- set up DEs w.r.t. kinematical invariants
- use AMF to calculate the boundary conditions
  - add auxiliary mass on inverse propagators
  - set up DEs w.r.t. η
  - at $\eta \to \infty$, integrals are reduced to a linear combination of basal phase-space integrals multiplied by single-scale vacuum bubble integrals
  - flow $\eta \to \infty$ to $\eta \to 0$ with DEs
- The method is systematic and efficient
- Its high-precision nature makes it possible to obtain analytical results with a proper ansatz

Thank you!