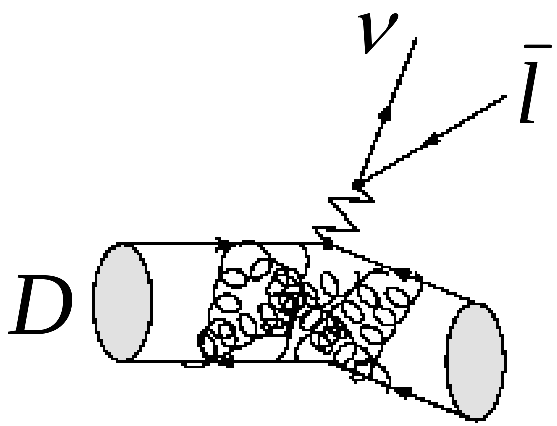
Semileptonic Decay of $D_s^+ \to \phi \bar{l} \nu$ in QCD Sum Rule

I. 介绍

重味介子的半轻衰变过程是研究弱作用和强作用的一类重要相互作用过程

1）可以用来检验和发展微扰和非微扰QCD的方法，用来测量CKM 矩阵元



2）半轻衰变过程较非轻衰变过程简单

QCD效应可以被分解成几个形状因子，这些形状因子包含的是非微扰QCD效应

- $D \to P \bar{l} \nu$ 过程（P为赝标量介子）：

$$\langle P(p_2) | J_\mu | D(p_1) \rangle$$

$$= [(p_1 + p_2)_\mu - \frac{m_D^2 - m_P^2}{q^2} q_\mu] F_1(q^2)$$

$$+ \frac{m_D^2 - m_P^2}{q^2} q_\mu F_0(q^2)$$

其中 $q = p_1 - p_2$

$D \to V \bar{l} \nu$ 过程（V为矢量介子）：

$$\langle V(\varepsilon, p_2)| j_\mu | D(p_1)\rangle$$

$$= \varepsilon_{\mu\nu\alpha\beta}\, \varepsilon^{*\nu} p_1^\alpha p_2^\beta \frac{2V(q^2)}{m_D + m_V}$$

$$- i\left(\varepsilon_\mu^* - \frac{\varepsilon^* \cdot q}{q^2} q_\mu\right)(m_D + m_V) A_1(q^2)$$

$$+ i\left[(p_1 + p_2)_\mu - \frac{m_D^2 - m_V^2}{q^2}\right] \varepsilon^* \cdot q \frac{A_2(q^2)}{m_D + m_V}$$

$$- i \frac{2 m_V \varepsilon^* \cdot q}{q^2} q_\mu A_0(q^2)$$

1）对于$D \to P \bar{l} \nu$的半轻衰变过程，只需两个参数即可描述：$F_1(q^2)$、$F_0(q^2)$

2）对于$D \to V \bar{l} \nu$的半轻衰变过程，需要4个参数来描述：$V(q^2)$、$A_0(q^2)$、$A_1(q^2)$和$A_2(q^2)$

这些参数，即形状因子，都是由非微扰的动力学所控制的，需要非微扰QCD方法来处理。

- QCD sum rule 方法可以用于计算强子跃迁的形状因子

$$D^+ \to \overline{K}^0 e^+ \nu$$

$$B \to D(D^*) l \overline{\nu}$$

$$B \to \pi \, l \overline{\nu}$$

$$D \to \pi \, e \nu \;, \rho \, e \nu$$

所作近似： 忽略轻夸克的质量，算符乘积展开中只取几个 贡献较大的算符贡献，等等。

- 我们来研究 $D_s^+ \to \phi \bar{l} \nu$ 过程

  需要考虑的一些因素：

  1）包括SU(3)破坏效应，因此 m_s不能忽略

  2）算符乘积展开到量纲为6的算符

  3）独立处理两个Borel 参数M_1^2 、M_2^2 ，降低不确定性

D.S. Du, J.W. Li, **M.Z. Yang** , Eur. Phys. J. C 37, 173 (2004)

II. 计算方法

- 基本思想：

 1）根据所考虑的物理问题构造一个关联函数，在所选的动量范围内，此关联函数即可用QCD语言来描写，也可用强子的语言来描写。

 2） 再根据夸克—强子对偶性假设，将这两种语言的描述等同起来

 3）从上述方程中求解出所要求的非微扰参数

- 关联函数

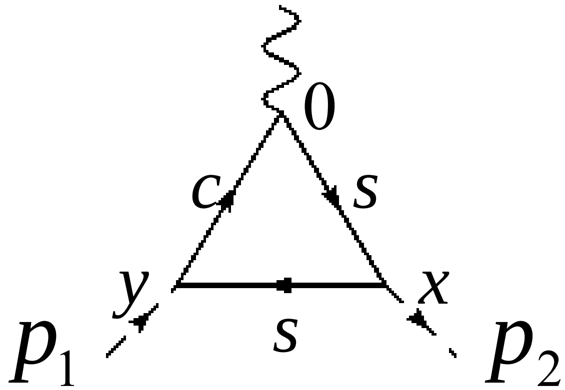
$$\Pi_{\mu\nu} = \mathrm{i}^2 \int \mathrm{d}^4x\mathrm{d}^4y \mathrm{e}^{\mathrm{i}p_2\cdot x - \mathrm{i}p_1\cdot y} \langle 0|T\{j_\nu^\phi(x) j_\mu(0) j_5^D(y)\}|0\rangle,$$

说明：此式中的"流"与所考虑的粒子有相同的量子数，它们 是：

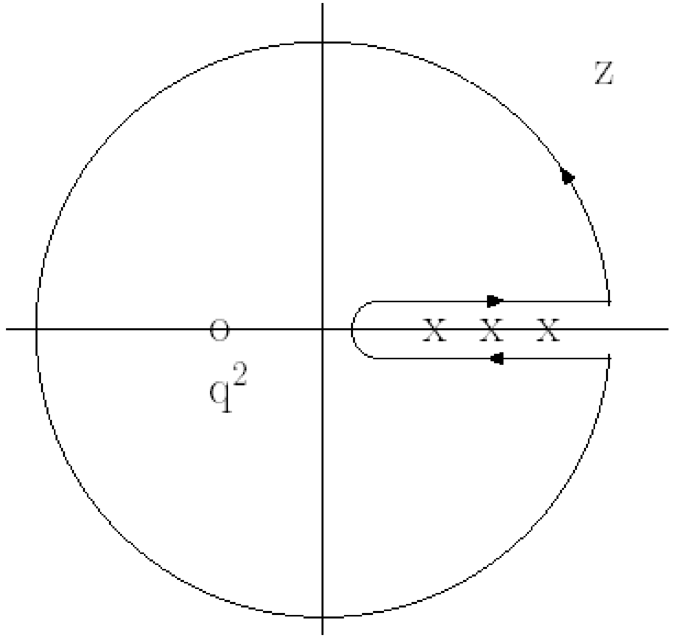
（1）与 D_s 相应的流：$j_5^D(y) = \bar{c}(y) i\gamma_5 s(y)$

（2）与 ϕ 相应的流：$j_\nu^\phi(x) = \bar{s}(x)\gamma_\nu s(x)$

（3）$c \to s$ 转移弱流：$j_\mu(0) = \bar{s}\gamma_\mu(1-\gamma_5)c$



- 我们先在强子水平上处理关联函数，将关联函数与要计算的形状因子联系起来。



色散关系：根据柯西公式

任意一个解析函数 Ш(dς)，

$$\prod\left(q^{2}\right)=\frac{1}{2\pi i}\oint_{C}dz\,\frac{\prod(z)}{z-q^{2}}$$

$$=\frac{1}{2\pi i}\oint_{|z|=R}dz\,\frac{\prod(z)}{z-q^{2}}$$

$$+\frac{1}{2\pi i}\int_{0}^{R}dz\,\frac{\prod(z+i\varepsilon)-\prod(z-i\varepsilon)}{z-q^{2}}$$

$$\prod\left(q^{2}\right)=\frac{1}{\pi}\int_{\min}^{\infty} ds\,\frac{\operatorname{Im}\prod(s)}{s-q^{2}-i\varepsilon}$$

———— 色散关系

• **定义强子谱密度函数**

$$\rho(s)=\frac{1}{\pi}\operatorname{Im}\prod(s)$$

则，色散关系可表示成为

$$\prod\left(q^{2}\right)=\int_{\min}^{\infty} ds\,\frac{\rho(s)}{s-q^{2}-i\varepsilon}$$

- 将色散关系应用到具有两个变量的情况，此时关联函数是 p_1^2，p_2^2和q^2的函数

$$\Pi_{\mu\nu} = \mathrm{i}^2 \int \mathrm{d}^4x\mathrm{d}^4y \mathrm{e}^{\mathrm{i}p_2\cdot x - \mathrm{i}p_1\cdot y}\langle 0|T\{j_\nu^\phi(x)j_\mu(0)j_5^D(y)\}|0\rangle,$$

插入完备态

$\sum_X |X\rangle\langle X|$ 和 $\sum_Y |Y\rangle\langle Y|$

关联函数可表示为

$$\Pi_{\mu\nu} = \int \mathrm{d}s_1\mathrm{d}s_2 \frac{\rho(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)},$$

其中

$$\rho(s_1, s_2, q^2) = \sum_{XY}\langle 0|j_\nu^\phi|X\rangle\langle X|j_\mu|Y\rangle\langle Y|j_5^D|0\rangle$$
$$\times\delta(s_1 - m_Y^2)\delta(s_2 - m_X^2)\theta(p_X^0)\theta(p_Y^0),$$

- 积分可得

$$\Pi_{\mu\nu} = \sum_{XY} \frac{\langle 0|j_\nu^\phi|X\rangle\langle X|j_\mu|Y\rangle\langle Y|j_5^D|0\rangle}{(m_Y^2 - p_1^2)(m_X^2 - p_2^2)} + \text{continuum states.}$$

- 进一步分离出基态粒子贡献，得

$$\Pi_{\mu\nu} = \frac{\langle 0|j_\nu^\phi|\phi\rangle\langle \phi|j_\mu|D_s\rangle\langle D_s|j_5^D|0\rangle}{(m_{D_s}^2 - p_1^2)(m_\phi^2 - p_2^2)} + \text{higher resonances and continuum states.}$$

- 利用

$$\langle 0|\bar{s}\gamma_\nu s|\phi\rangle = m_\phi f_\phi \varepsilon_\nu^{(\lambda)},$$

$$\langle 0|\bar{s}\mathrm{i}\gamma_5 c|D_s\rangle = \frac{f_{D_s} m_{D_s}^2}{m_c + m_s}.$$

- 关联函数进一步表示成

$$\Pi_{\mu\nu} = \frac{m_\phi f_\phi \varepsilon_\nu^{(\lambda)} \langle \phi(\varepsilon_\nu^{(\lambda)}, p_2) | j_\mu | D_s(p_1) \rangle f_D m_{D_s}^2}{(m_{D_s}^2 - p_1^2)(m_\phi^2 - p_2^2)(m_c + m_s)}$$

$$+\text{higher resonances and continuum states.}$$

- 下面我们将在QCD理论下用算符乘积展开的方法计算关联函数

比较两次计算的结果，即可求解出形状因子

- 算符乘积展开下，关联函数中的算符可做如下形式上的展开

$$
\begin{aligned}
&\mathrm{i}^2 \int \mathrm{d}^4 x \mathrm{d}^4 y \mathrm{e}^{\mathrm{i} p_2 \cdot x - \mathrm{i} p_1 \cdot y} T\{j_\nu^\phi(x) j_\mu(0) j_5^D(y)\} \\
&= C_{0\mu\nu} I + C_{3\mu\nu} \bar{\Psi}\Psi + C_{4\mu\nu} G^a_{\alpha\beta} G^{a\alpha\beta} \\
&+ C_{5\mu\nu} \bar{\Psi} \sigma_{\alpha\beta} T^a G^{a\alpha\beta} \Psi + C_{6\mu\nu} \bar{\Psi} \Gamma \Psi \bar{\Psi} \Gamma' \Psi + \cdots
\end{aligned}
$$

- 几点说明：
  1）I 是单位算符，$\overline{\Psi}\Psi$ 是局域费米场算符，$G^a_{\alpha\beta}$ 是胶子场强张量；
  2）$C_{i\mu\nu}$ 称为Wilson系数

- 算符乘积展开式两边夹以真空态，得关联函数在算符乘积展开下的表示

$$
\begin{aligned}
\Pi_{\mu\nu} &= \mathrm{i}^2 \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{e}^{\mathrm{i}p_2\cdot x-\mathrm{i}p_1\cdot y}\langle 0|T\{j_\nu^\phi(x)j_\mu(0)j_5^D(y)\}|0\rangle \\
&= C_{0\mu\nu}I + C_{3\mu\nu}\langle 0|\bar{\Psi}\Psi|0\rangle + C_{4\mu\nu}\langle 0|G_{\alpha\beta}^a G^{a\alpha\beta}|0\rangle \\
&+ C_{5\mu\nu}\langle 0|\bar{\Psi}\sigma_{\alpha\beta}T^aG^{a\alpha\beta}\Psi|0\rangle \\
&+ C_{6\mu\nu}\langle 0|\bar{\Psi}\Gamma\Psi\bar{\Psi}\Gamma'\Psi|0\rangle + \cdots
\end{aligned}
$$

真空凝聚

- 将上式中的Lorentz指标明确分离出来，并将系数按Lorentz指标重新整理

$$
\begin{aligned}
\Pi_{\mu\nu} = &-f_0\varepsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta \\
&- \mathrm{i}(f_1p_{1\mu}p_{1\nu} + f_2p_{2\mu}p_{2\nu} + f_3p_{1\nu}p_{2\mu} \\
&+f_4p_{1\mu}p_{2\nu} + f_5g_{\mu\nu}).
\end{aligned}
$$

每一个系数 f_i 都包括微扰贡献(三角图)和凝聚的贡献

$$f_i = f_i^{\rm pert} + f_i^{(3)} + f_i^{(4)} + f_i^{(5)} + f_i^{(6)} + \cdots ,$$

其中微扰贡献可以表示成色散积分的形式

$$f_i^{\rm pert} = \int {\rm d}s_1 {\rm d}s_2 \frac{\rho_i^{\rm pert}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)},$$

- 将强子表示的关联函数和算符乘积展开的结果等同起来，得到等式

$$\frac{m_\phi f_\phi \varepsilon_\nu^{(\lambda)} \langle \phi(\varepsilon_\nu^{(\lambda)}, p_2)|j_\mu|D_s(p_1)\rangle f_D m_{D_s}^2}{(m_{D_s}^2 - p_1^2)(m_\phi^2 - p_2^2)(m_c + m_s)}$$

$$+\text{higher resonances and continuum states}$$

$$= -f_0 \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta - \mathrm{i}(f_1 p_{1\mu} p_{1\nu} + f_2 p_{2\mu} p_{2\nu} + f_3 p_{1\nu} p_{2\mu}$$

$$+ f_4 p_{1\mu} p_{2\nu} + f_5 g_{\mu\nu})$$

Higher Resonances
+continuum
$$= \int_{s_1^h}^{\infty} ds_1 \int_{s_2^h}^{\infty} ds_2 \frac{\rho^{had.}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}$$

$$f_i^{pert.} = \int_0^{\infty} ds_1 \int_0^{\infty} ds_2 \frac{\rho^{pert.}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}$$

对偶假设

$$\int_{s_1^h}^{\infty} ds_1 \int_{s_2^h}^{\infty} ds_2 \frac{\rho^{had.}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} = \int_{s_1^0}^{\infty} ds_1 \int_{s_2^0}^{\infty} ds_2 \frac{\rho^{pert.}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}$$

- 将关联函数等式左边高激发态和连续态贡献消去

$$\frac{m_\phi f_\phi \varepsilon_\nu^{(\lambda)} \langle \phi(\varepsilon_\nu^{(\lambda)}, p_2)|j_\mu|D_s(p_1)\rangle f_D m_{D_s}^2}{(m_{D_s}^2 - p_1^2)(m_\phi^2 - p_2^2)(m_c + m_s)}$$

$$= -f_0 \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta - \mathrm{i}(f_1 p_{1\mu} p_{1\nu} + f_2 p_{2\mu} p_{2\nu} + f_3 p_{1\nu} p_{2\mu}$$

$$+ f_4 p_{1\mu} p_{2\nu} + f_5 g_{\mu\nu})$$

- 微扰贡献的积分取为

$$f_i^{pert.} = \int_0^{s_1^0} ds_1 \int_0^{s_2^0} ds_2 \frac{\rho^{pert.}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}$$

- s_1^0和 s_2^0 分别为D_s 道和 ϕ 道域值
- 但这样的计算结果会过度依赖前式的近似假设，为了减少这种依赖，给等式两边同作一个Borel变换

- 定义Borel变换为

$$\hat{B}\Big|_{p^2,M^2} f(p^2) = \lim_{\substack{n\to\infty \\ p^2\to-\infty \\ -p^2/n = M^2}} \frac{(-p^2)^n}{(n-1)!} \frac{\partial^n}{\partial(p^2)^n} f(p^2)$$

- Borel变换会对高激发态造成压低，例如

$$\hat{B}\Big|_{p^2,M^2} \frac{1}{m_R^2 - p^2} = \frac{1}{M^2} e^{-m_R^2/M^2}$$

$$\frac{m_\phi f_\phi \varepsilon_\nu^{(\lambda)} \langle \phi(\varepsilon_\nu^{(\lambda)}, p_2)|j_\mu|D_s(p_1)\rangle f_D m_{D_s}^2}{(m_{D_s}^2 - p_1^2)(m_\phi^2 - p_2^2)(m_c + m_s)}$$

$$= -f_0 \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta - \mathrm{i}(f_1 p_{1\mu} p_{1\nu} + f_2 p_{2\mu} p_{2\nu} + f_3 p_{1\nu} p_{2\mu}$$

$$+ f_4 p_{1\mu} p_{2\nu} + f_5 g_{\mu\nu})$$

- 比较等式两边的Lorentz结构，可以得到所有形状因子的表示式

$$
\begin{aligned}
V(q^2) &= \frac{(m_c+m_s)(m_{D_s}+m_\phi)}{2m_\phi f_\phi f_{D_s} m_{D_s}^2} \\
&\quad \times \mathrm{e}^{m_{D_s}^2/M_1^2} \mathrm{e}^{m_\phi^2/M_2^2} M_1^2 M_2^2 \hat{B} f_0, \\
A_1(q^2) &= -\frac{(m_c+m_s)}{m_\phi f_\phi f_{D_s} m_{D_s}^2 (m_{D_s}+m_\phi)} \\
&\quad \times \mathrm{e}^{m_{D_s}^2/M_1^2} \mathrm{e}^{m_\phi^2/M_2^2} M_1^2 M_2^2 \hat{B} f_5,
\end{aligned}
$$

$$
\begin{aligned}
A_2(q^2) &= \frac{(m_c+m_s)(m_{D_s}+m_\phi)}{m_\phi f_\phi f_{D_s} m_{D_s}^2} \\
&\quad \times \mathrm{e}^{m_{D_s}^2/M_1^2} \mathrm{e}^{m_\phi^2/M_2^2} M_1^2 M_2^2 \frac{1}{2} \hat{B}(f_1+f_3), \\
A_0(q^2) &= -\frac{(m_c+m_s)}{2m_\phi^2 f_\phi f_{D_s} m_{D_s}^2} \mathrm{e}^{m_{D_s}^2/M_1^2} \mathrm{e}^{m_\phi^2/M_2^2} M_1^2 M_2^2 \\
&\times \left[\hat{B}(f_1+f_3) \frac{m_{D_s}^2 - m_\phi^2}{2} + \hat{B}(f_1-f_3) \frac{q^2}{2} + f_5 \right],
\end{aligned}
$$

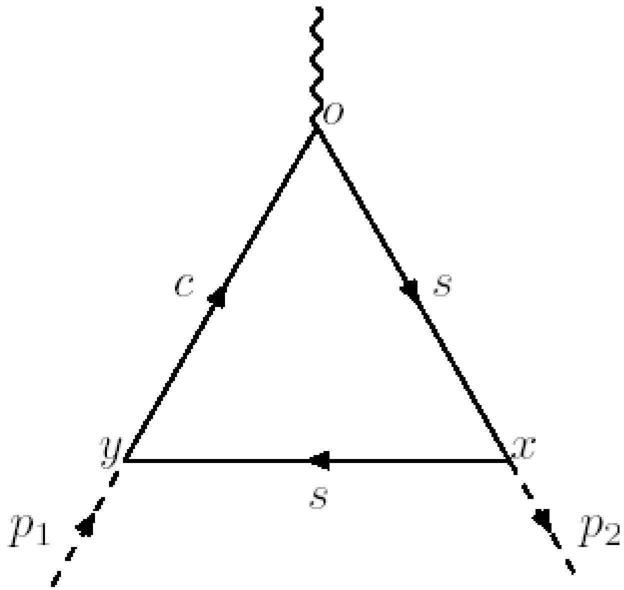
三. Wilson系数的计算

- 根据算符乘机展开

$$
\begin{aligned}
\Pi_{\mu\nu} &= \mathrm{i}^2 \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{e}^{\mathrm{i}p_2\cdot x - \mathrm{i}p_1\cdot y} \langle 0|T\{j_\nu^\phi(x) j_\mu(0) j_5^D(y)\}|0\rangle \\
&= C_{0\mu\nu} I + C_{3\mu\nu}\langle 0|\bar{\Psi}\Psi|0\rangle + C_{4\mu\nu}\langle 0|G^a_{\alpha\beta}G^{a\alpha\beta}|0\rangle \\
&+ C_{5\mu\nu}\langle 0|\bar{\Psi}\sigma_{\alpha\beta}T^a G^{a\alpha\beta}\Psi|0\rangle \\
&+ C_{6\mu\nu}\langle 0|\bar{\Psi}\Gamma\Psi\bar{\Psi}\Gamma'\Psi|0\rangle + \cdots
\end{aligned}
$$

- 考虑所有可能出现的算符形式

1) 微扰部分



$$C_0 = \mathrm{i}^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}(-1)$$
$$\times \operatorname{Tr}\left[\mathrm{i}\gamma_5 \frac{\mathrm{i}(\not{k} + m_s)}{k^2 - m_s^2 + \mathrm{i}\varepsilon}\gamma_\nu \frac{\mathrm{i}(\not{k} + \not{p}_2 + m_s)}{(k+p_2)^2 - m_s^2 + \mathrm{i}\varepsilon}\right.$$
$$\left.\times\gamma_\mu(1-\gamma_5)\ \frac{\mathrm{i}(\not{k} + \not{p}_1 + m_s)}{(k+p_1)^2 - m_c^2 + \mathrm{i}\varepsilon}\right].$$

- 2)双夸克算符的贡献 $\overline{\psi}(x)\psi(y)$ ， $\overline{\psi}(0)\psi(x)$

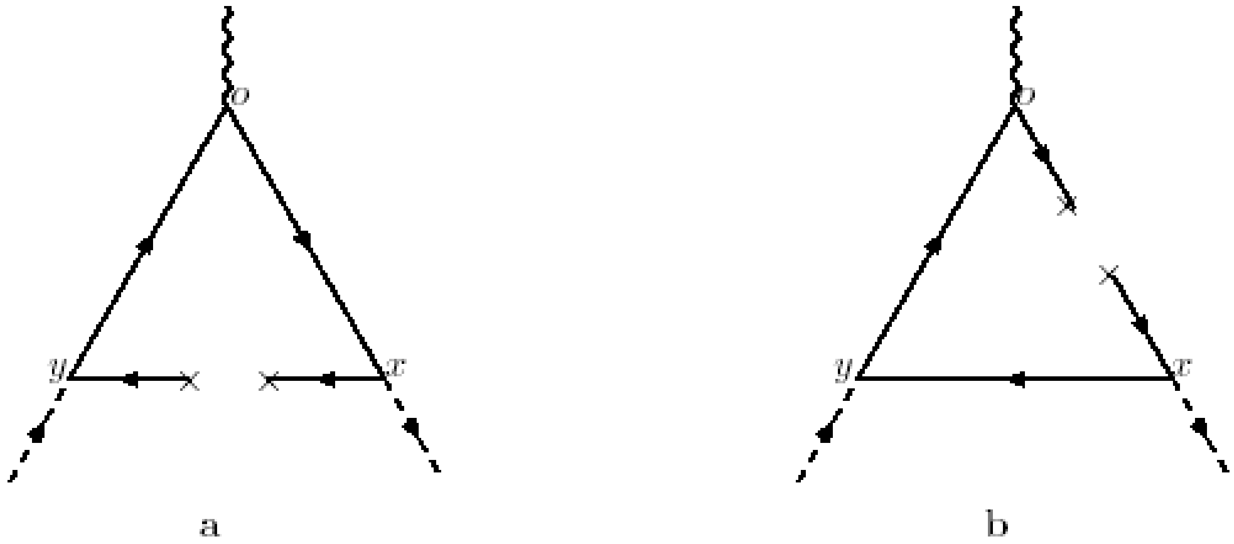


Fig. 2. Diagrams for the contributions of the non-local operators $\bar{\Psi}(x)\Psi(y)$ and $\bar{\Psi}(0)\Psi(x)$

$$\Pi^{2a}_{\mu\nu} = \mathrm{i}^2 \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{e}^{\mathrm{i}p_2\cdot x - \mathrm{i}p_1\cdot y}$$
$$\times \langle 0|\bar{\Psi}(x)\gamma_\nu \mathrm{i}S^s_{\mathrm{F}}(x)\gamma_\mu(1-\gamma_5)\mathrm{i}S^c_{\mathrm{F}}(-y)\mathrm{i}\gamma_5\Psi(y)|0\rangle$$

$$\Pi_{\mu\nu}^{2a} = \mathrm{i}^2 \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{e}^{\mathrm{i}p_2\cdot x - \mathrm{i}p_1\cdot y}$$
$$\times \langle 0|\bar{\Psi}_\alpha(x)\Psi_\beta(y)|0\rangle \left[\gamma_\nu \mathrm{i}S_\mathrm{F}^s(x)\gamma_\mu(1-\gamma_5)\right.$$
$$\left.\times \mathrm{i}S_\mathrm{F}^c(-y)\mathrm{i}\gamma_5\right]_{\alpha\beta},$$

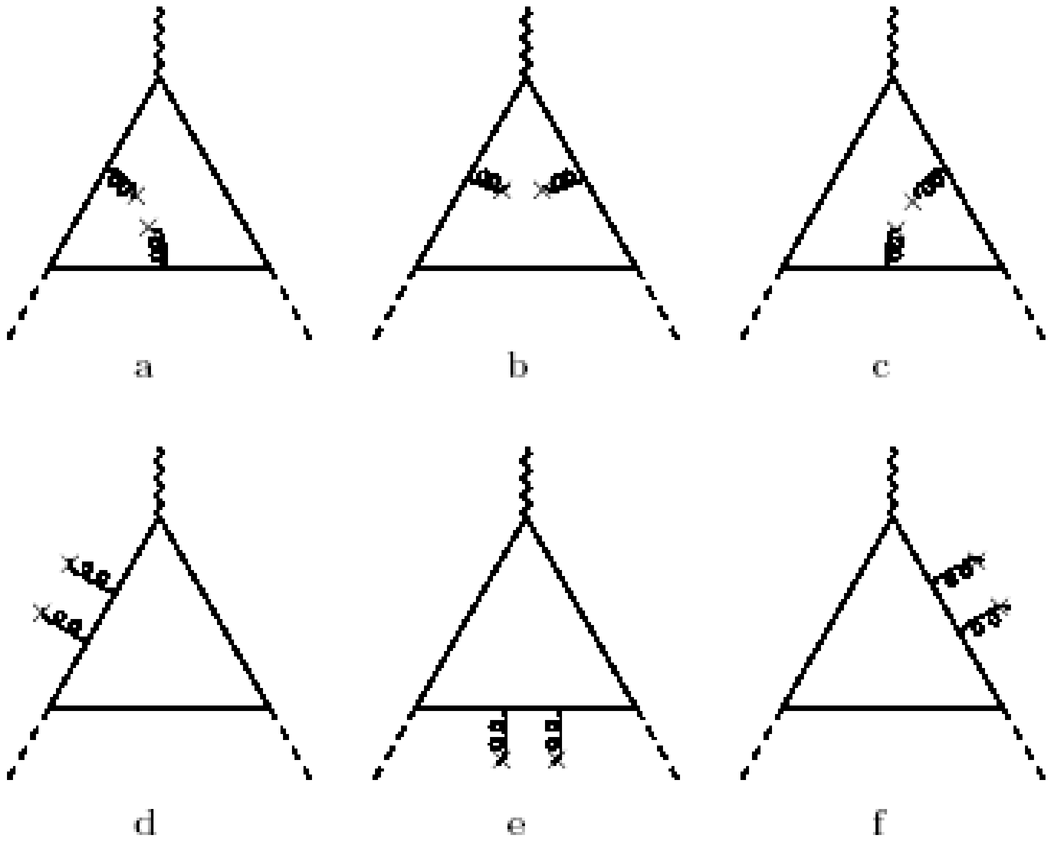
- 处理矩阵元 $\langle 0|\bar{\Psi}_\alpha(x)\Psi_\beta(y)|0\rangle$
- 场的小距离展开 (固定点规范)

$$\psi(x) = \psi(0) + x_\alpha \vec{D}^\alpha \psi(0) + \frac{1}{2} x_\alpha x_\beta \vec{D}^\alpha \vec{D}^\beta \psi(0) + \cdots$$
$$\overline{\psi}(x) = \overline{\psi}(0) + \overline{\psi}(0)\overleftarrow{D}^\alpha x_\alpha + \frac{1}{2}\overline{\psi}(0)\overleftarrow{D}^\alpha \overleftarrow{D}^\beta x_\alpha x_\beta + \cdots$$

- 其中 $D_\alpha = \partial_\alpha - igT^a A_\alpha^a$

$$
\begin{aligned}
&\langle 0|\bar{\Psi}^a_\alpha(x)\Psi^b_\beta(y)|0\rangle \\
&= \delta_{ab}\Bigg[\langle\bar{\Psi}\Psi\rangle\left(\frac{1}{12}\delta_{\beta\alpha} + \mathrm{i}\frac{m}{48}(\not{x} - \not{y})_{\beta\alpha}\right. \\
&\left. - \frac{m^2}{96}(x-y)^2\delta_{\beta\alpha} - \frac{\mathrm{i}}{3!}\frac{m^3}{96}(x-y)^2(\not{x} - \not{y})_{\beta\alpha}\right) \\
&+ g\langle\bar{\Psi}TG\sigma\Psi\rangle\left(\frac{1}{192}(x-y)^2\delta_{\beta\alpha}\right. \\
&\qquad \left. + \frac{\mathrm{i}}{3!}\frac{m}{192}(x-y)^2(\not{x} - \not{y})_{\beta\alpha}\right) \\
&- \frac{\mathrm{i}}{3!}\frac{g^2}{3^4\times 2^4}\langle\bar{\Psi}\Psi\rangle^2(x-y)^2(\not{x} - \not{y})_{\beta\alpha} + \cdots\Bigg],
\end{aligned}
$$

3)双胶子算符贡献 $G^a_{\mu\nu}G^{a\mu\nu}$



- 采用固定点规范

规范条件： $x^{\mu} A_{\mu}^{a}(x) = 0$

于是 胶子场可以表示成：

$$A_{\mu}^{a}(x) = \int_0^1 \mathrm{d}\alpha \alpha x^{\rho} G_{\rho\mu}^{a}(\alpha x)$$

证明：

1） $$\frac{d}{d\alpha}[\alpha A_{\mu}(\alpha x)] = A_{\mu}(\alpha x) + \alpha \frac{d}{d\alpha} A_{\mu}(\alpha x)$$

$$= A_{\mu}(\alpha x) + \alpha \frac{\partial A_{\mu}(y)}{\partial y_{\rho}} \cdot \frac{\partial y_{\rho}}{\partial \alpha}$$

$$= A_{\mu}(\alpha x) + y_{\rho} \frac{\partial A_{\mu}(y)}{\partial y_{\rho}}$$

$$y = \alpha x$$

2)

$$y^{\rho} G^{a}_{\rho\mu}(y) = y^{\rho}[\partial_{\rho} A^{a}_{\mu}(y) - \partial_{\mu} A^{a}_{\rho}(y) + g f^{abc} A^{b}_{\rho}(y) A^{c}_{\mu}(y)]$$

$$= y^{\rho}\partial_{\rho} A^{a}_{\mu}(y) - y^{\rho}\partial_{\mu} A^{a}_{\rho}(y)$$

$$= y^{\rho}\partial_{\rho} A^{a}_{\mu}(y) + A^{a}_{\mu}(y)$$

$y_{\rho} A^{\rho}(y) = 0$

分步积分

比较1）2）两步右边，可知

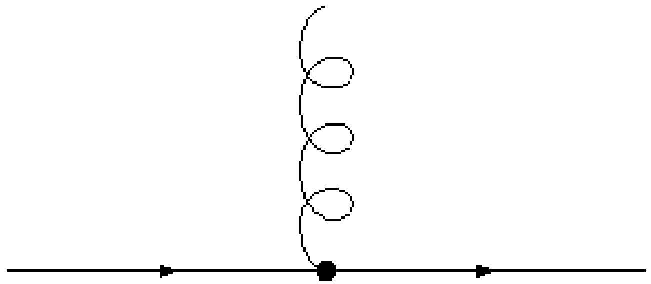
$$\frac{d}{d\alpha}[\alpha A^{a}_{\mu}(\alpha x)] = y^{\rho} G^{a}_{\rho\mu}(y)$$

$$A^{a}_{\mu}(x) = \int_{0}^{1} d\alpha \; y^{\rho} G^{a}_{\rho\mu}(y)$$

$$= \int_{0}^{1} d\alpha \; \alpha x^{\rho} G^{a}_{\rho\mu}(\alpha y)$$

$$\approx \frac{1}{2} x^{\rho} G^{a}_{\rho\mu}(0) + \cdots$$

- 有效顶点：



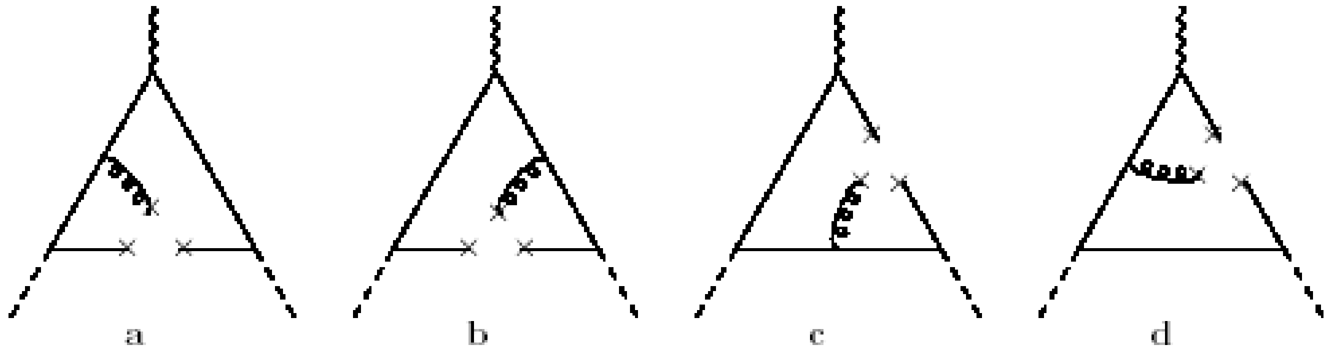
$$igT^a A_\mu^a(x)$$

$$= igT^a \frac{1}{2} x^\rho G_{\rho\mu}^a(0)$$

$$\langle 0|G_{\alpha\sigma}^a G_{\beta\rho}^b|0\rangle = \frac{1}{96}\langle GG\rangle \delta_{ab}(g_{\alpha\beta}g_{\sigma\rho} - g_{\alpha\rho}g_{\sigma\beta})$$

4)夸克胶子混合和四夸克算符的贡献:

$$\overline{\psi}(x)\psi(y)G^{a}_{\mu\nu},\ \langle\overline{\psi}\psi\rangle^{2}$$



$$\langle 0|\bar{\Psi}^{i}_{\alpha}(x)\Psi^{j}_{\beta}(y)G^{a}_{\mu\nu}|0\rangle = \frac{1}{192}\langle\bar{\Psi}\sigma TG\Psi\rangle(\sigma_{\mu\nu})_{\beta\alpha}T^{a}_{ji}$$

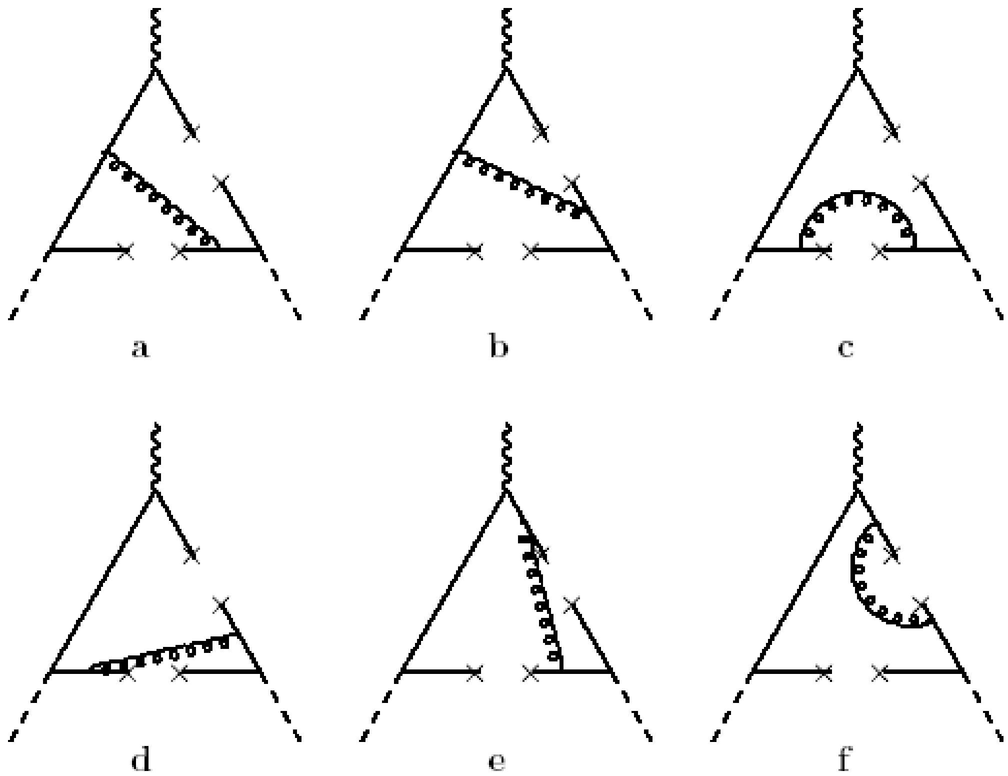
$$+\left[-\frac{g}{96\times 9}\langle\bar{\Psi}\Psi\rangle^{2}(g_{\rho\mu}\gamma_{\nu}-g_{\rho\nu}\gamma_{\mu})(x+y)^{\rho}\right.$$

$$\left.+\mathrm{i}(y-x)^{\rho}\left(\frac{g}{96\times 9}\langle\bar{\Psi}\Psi\rangle^{2}+\frac{m}{96\times 4}\langle\bar{\Psi}\sigma TG\Psi\rangle\right)\varepsilon_{\rho\mu\nu\sigma}\gamma_{5}\gamma^{\sigma}\right]_{\beta\alpha}T^{a}_{ji},$$

- 其中的两个定义:

$$\langle\bar{\Psi}\sigma TG\Psi\rangle \equiv \langle 0|\bar{\Psi}\sigma_{\mu\nu}T^{a}G^{a\mu\nu}\Psi|0\rangle$$

$$\langle\bar{\Psi}\Psi\rangle^{2} \equiv \langle 0|\bar{\Psi}\Psi|0\rangle^{2}$$



表示四夸克算符贡献的图

四. 分析和讨论

- 参数的数值选取:
- At $\mu = 1\text{GeV}$

$$\langle \bar{q}q \rangle = -(0.24 \pm 0.01\,\text{GeV})^3, \qquad \langle \bar{s}s \rangle = m_0^2 \langle \bar{q}q \rangle,$$

$$g\langle \bar{\Psi}\sigma T \Psi \rangle = m_0^2 \langle \bar{\Psi}\Psi \rangle, \qquad \alpha_s \langle \bar{\Psi}\Psi \rangle^2 = 6.0 \times 10^{-5}\,\text{GeV}^6,$$

$$m_0^2 = 0.8 \pm 0.2\,\text{GeV}^2$$

$$m_s = 140\,\text{MeV}.$$

$$m_c = 1.3\,\text{GeV}$$

$$f_{D_s} = 0.214 \pm 0.038\,\text{GeV}$$

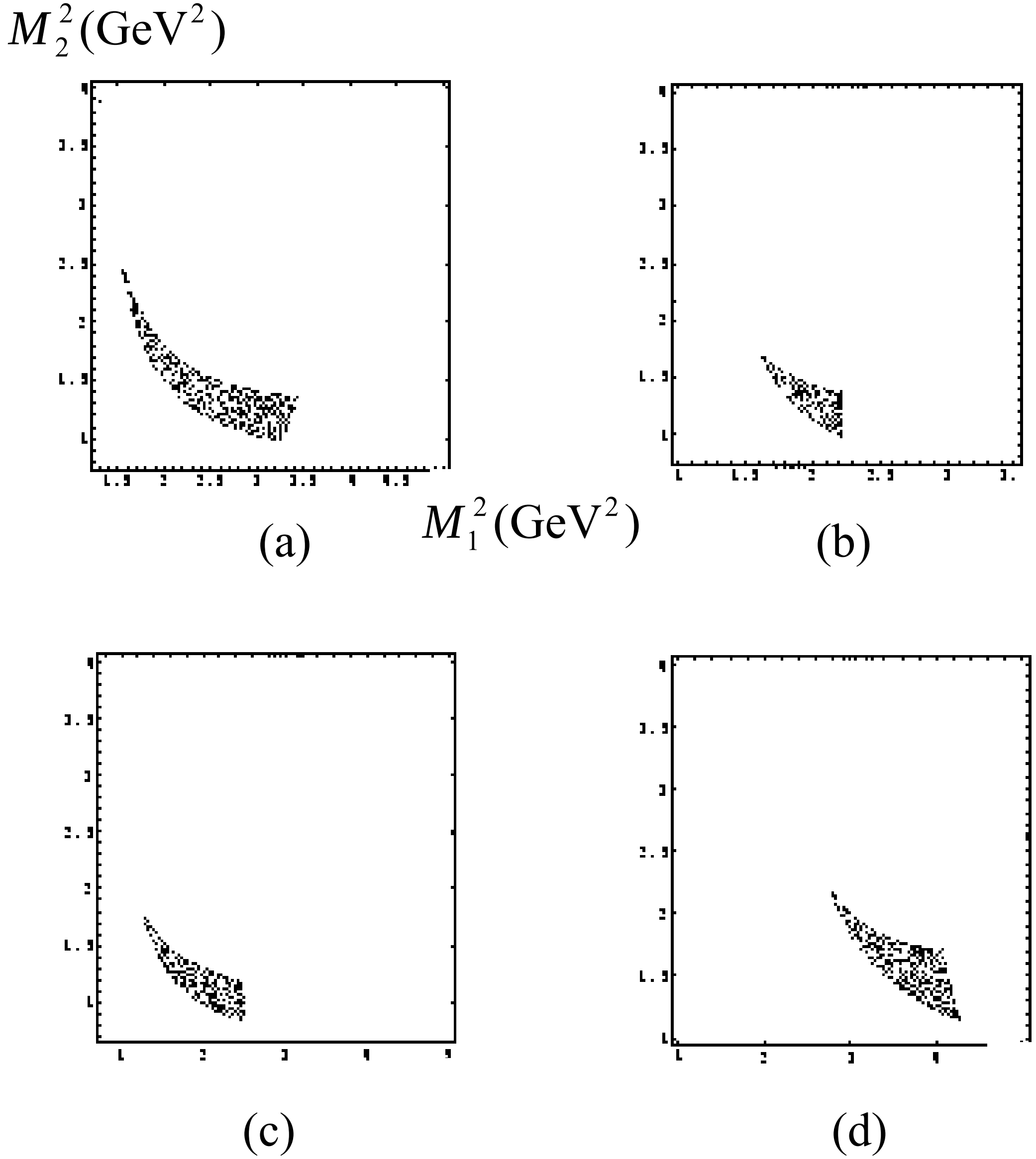
- Borel参数稳定“窗口”的选取

1） 算符乘机展开 $\sim \dfrac{1}{M_1^m M_2^n}$, 量纲越高，m、n 越大，

因此为了保障算符乘机展开有效，M1和M2不能太小

2) 高激发态的贡献 $\sim e^{-m_R^2 / M_i^2}$,

因此为了保障高激发态的影响很小， M1和M2不能太大



M_2^2(GeV^2)
M_1^2(GeV^2)
(a)
(b)
(c)
(d)

- 在域值 $s_1^0 = 5.8\text{–}6.2\,\mathrm{GeV}^2$，$s_2^0 = 1.9\text{–}2.1\,\mathrm{GeV}^2$ 处有稳定的解

$$
\begin{aligned}
V(0) &= 1.21 \pm 0.33, \\
A_0(0) &= 0.42 \pm 0.12, \\
A_1(0) &= 0.55 \pm 0.15, \\
A_2(0) &= 0.59 \pm 0.17,
\end{aligned}
$$

$$
\begin{aligned}
r_V &\equiv \frac{V(0)}{A_1(0)} = 2.20 \pm 0.85, \\
r_2 &\equiv \frac{A_2(0)}{A_1(0)} = 1.07 \pm 0.43.
\end{aligned}
$$

- 形状因子比值与实验数据的比较

Table 1 Comparison of our results of r_V and r_2 with experimental data: E791 is from [11], CLEO from [10], E687 from [9] and E653 from [8]

	r_V	r_2
E791	$2.27 \pm 0.35 \pm 0.22$	$1.57 \pm 0.25 \pm 0.19$
CLEO	$0.9 \pm 0.6 \pm 0.3$	$1.4 \pm 0.5 \pm 0.3$
E687	$1.8 \pm 0.9 \pm 0.2$	$1.1 \pm 0.8 \pm 0.1$
E653	$2.3^{+1.1}_{-0.9} \pm 0.4$	$2.1^{+0.6}_{-0.5} \pm 0.2$
average	1.92 ± 0.32	1.60 ± 0.24
our result	2.20 ± 0.85	1.07 ± 0.43

- 形状因子 $V(q^2),\ A_0(q^2),\ A_1(q^2),\ A_2(q^2),$ 依赖于动量转移的平方 q^2
- 在 $D_s^+ \to \phi \bar{\ell} \nu$ 衰变过程中：

$$0 \le q^2 \le (m_{D_s} - m_\phi)^2 \approx 0.9\mathrm{GeV}^2$$

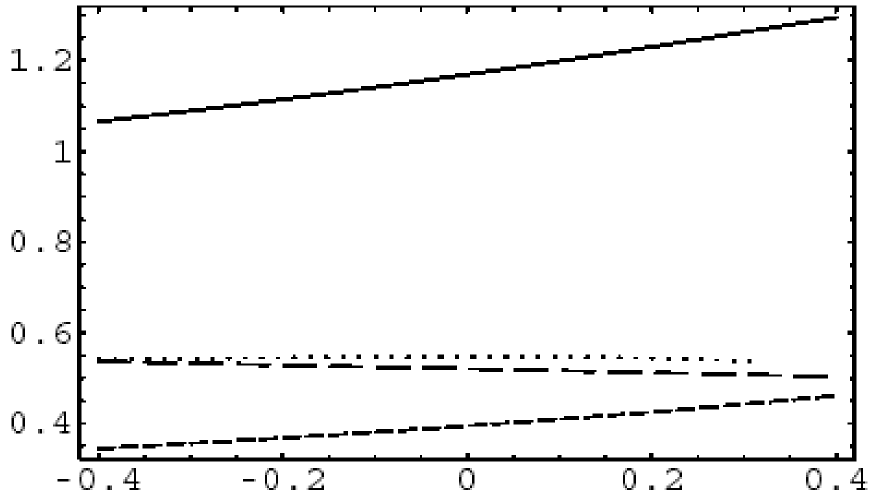


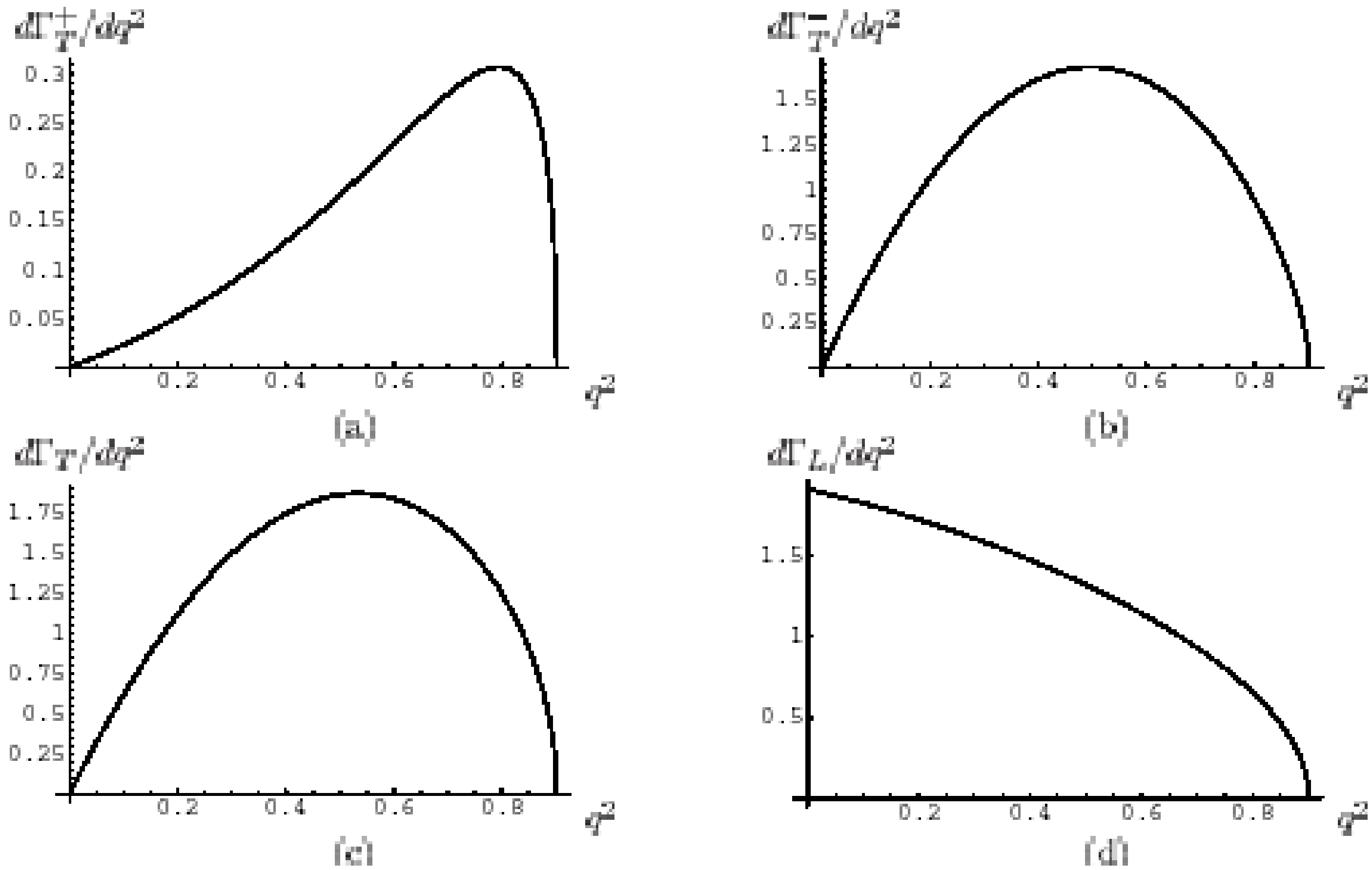
Fig. 7. The q^2 dependence of the form factors from the QCD sum rule. The solid curve is for $V(q^2)$, the short dashed curve for $A_0(q^2)$, the long dashed curve for $A_1(q^2)$, and the dotted one is for $A_2(q^2)$

- 衰变宽度 $\lambda(m_{D_s}^2, m_\phi^2, q^2) \equiv (m_{D_s}^2 + m_\phi^2 - q^2)^2 - 4m_{D_s}^2 m_\phi^2.$

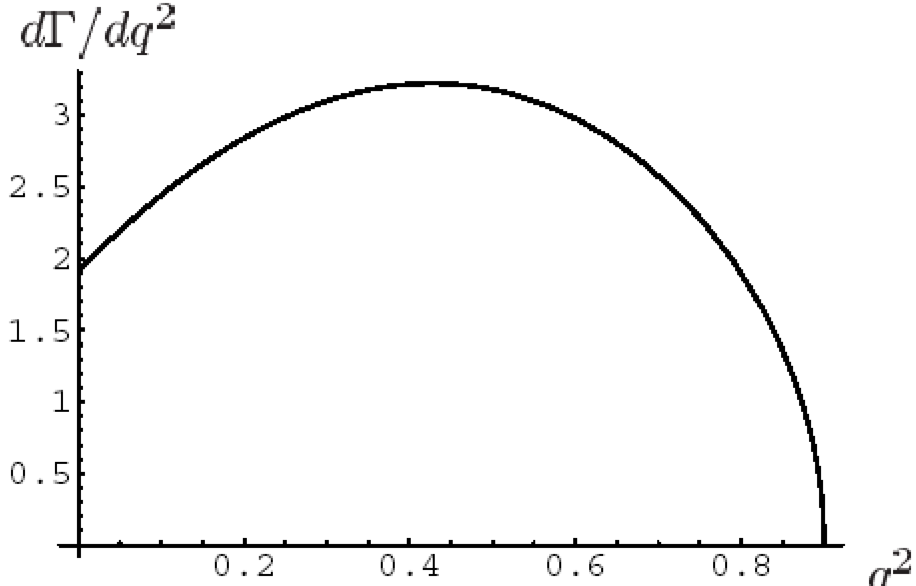
$$\frac{\mathrm{d}\Gamma_{\mathrm{L}}}{\mathrm{d}q^2} = \frac{G_{\mathrm{F}}^2 |V_{cs}|^2}{192\pi^3 m_{D_s}^3} \sqrt{\lambda(m_{D_s}^2, m_\phi^2, q^2)} \times \left| \frac{1}{2m_\phi} \left[(m_{D_s}^2 - m_\phi^2 - q^2)(m_{D_s} + m_\phi) A_1(q^2) - \frac{\lambda(m_{D_s}^2, m_\phi^2, q^2)}{m_{D_s} + m_\phi} A_2(q^2) \right] \right|^2,$$

$$\frac{\mathrm{d}\Gamma_{\mathrm{T}}^{\pm}}{\mathrm{d}q^2} = \frac{G_{\mathrm{F}}^2 |V_{cs}|^2}{192\pi^3 m_{D_s}^3} \lambda(m_{D_s}^2, m_\phi^2, q^2) \times \left| \frac{V(q^2)}{m_{D_s} + m_\phi} \mp \frac{(m_{D_s} + m_\phi) A_1(q^2)}{\sqrt{\lambda(m_{D_s}^2, m_\phi^2, q^2)}} \right|^2,$$

$$\frac{\mathrm{d}\mathit{\Gamma}_{\mathrm{T}}}{\mathrm{d}q^2} = \frac{\mathrm{d}}{\mathrm{d}q^2}(\mathit{\Gamma}_{\mathrm{T}}^{+} + \mathit{\Gamma}_{\mathrm{T}}^{-}), \qquad \frac{\mathrm{d}\mathit{\Gamma}}{\mathrm{d}q^2} = \frac{\mathrm{d}}{\mathrm{d}q^2}(\mathit{\Gamma}_{\mathrm{L}} + \mathit{\Gamma}_{\mathrm{T}}).$$



q^2 in units of $10^{-14}\,\mathrm{GeV}^{-1}$



(e)

$$\Gamma_T^+ = (1.39 \pm 0.75) \times 10^{-15}\ \mathrm{GeV},$$
$$\Gamma_T^- = (1.05 \pm 0.22) \times 10^{-14}\ \mathrm{GeV},$$
$$\Gamma_L = (1.18 \pm 0.43) \times 10^{-14}\ \mathrm{GeV},$$
$$\Gamma_T = (1.19 \pm 0.29) \times 10^{-14}\ \mathrm{GeV},$$

$$\Gamma_L/\Gamma_T = 0.99 \pm 0.43\ ,$$

	Γ_L/Γ_T
CLEO	$1.0 \pm 0.3 \pm 0.2$
E687	$1.0 \pm 0.5 \pm 0.1$
E653	$0.54 \pm 0.21 \pm 0.10$
average	0.72 ± 0.18
our result	0.99 ± 0.43

- 总分支比：

  理论 $\mathrm{Br}(D_s^+ \to \phi\bar{\ell}\nu) = (1.8 \pm 0.5)\%,$

  实验 $\mathrm{Br}(D_s^+ \to \phi\ell\nu)^{\mathrm{exp}} = (2.0 \pm 0.5)\%$