Thermodynamics in Curved Spacetime; Self-gravitating Systems

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Introduction:

- In general, thermodynamics is treated in flat spacetime since the curvature effects are mostly not relevant even for astrophysical objects.

- But of course there are systems that the curvature could be important, namely compact objects; black holes, neutron stars, possible exotic objects (2-2-holes), etc.

- Black hole thermodynamics has been heavily studied. But, due the event horizon, it is different than a self-gravitating system.

- We set out to investigate the curvature effects in thermodynamics and statistical mechanics.

- As a case study, we consider classical ideal gas. massless and massive, in canonical and microcanonical ensembles.

“First” things first:

- Zeroth law in flat spacetime: a system in thermal equilibrium $\longleftrightarrow$ constant T
- In curved spacetime; this is obviously still true locally, for small volume element. But once the curvature effects are important, this no longer holds: thermal equilibrium $\not\longleftrightarrow$ constant T
- The system configures itself under gravity. Establishes distribution. Position dependent state parameters $p(r)$, $\rho(r)$, $T(r)$..
- Gravity yields scaling relations. The temperature scaling is “universal” for any gravitational system; known as the (generalized) Tolman’s law.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Tolman’s law:

$$T(r)\sqrt{g_{00}(r)} = constant \equiv T_\infty$$

→ Commonly known version, originally derived for blackbody radiation; photon gas with μ=0. (The generalized version; next slide)

[1] R. C. Tolman, Phys. Rev. 35, 904 (1930).
[2] R. C. Tolman and P. Ehrenfest, Phys. Rev. 36, 1791 (1930).

Generalized Tolman's law

The general metric for a static, spherically symmetric spacetime is given as

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,.$$

Energy-momentum tensor

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta - p g^{\alpha\beta}$$

We start with the energy conservation law $u_\mu T^{\mu\nu}_{;\nu} = 0$, which yields

$$p' + (p+\rho)\frac{B'}{2B} = 0\,,$$

where $(')$ denotes derivative with respect to r and the semi-colon is the covariant derivative.

- Use the the thermodynamical relation $\bar{s} = (\rho + p)/(nT) - \mu/T$

Generalized Tolman's law:

Generalized Tolman's law:

$$\frac{B'}{2B} + \frac{T'}{T} = -\frac{nT}{\rho + p}\left(\frac{\mu}{T}\right)' = -\frac{\mu}{\bar{s}T}\left(\frac{B'}{2B} + \frac{\mu'}{\mu}\right)$$

Recently noticed by Lima et. al,
Phys.Rev.D 100 (2019) 10, 104042, arXiv:1911.09060

The common version
for μ=0 or μ/T=constant

$$T(r)\sqrt{B(r)} = T_\infty$$

(The latter is a good
approximation in our case)

Or, additionally we have

$$\mu(r)\sqrt{B(r)} = \mu_\infty$$

Scaling relation for particle energy:

$$E_i(r)\sqrt{B(r)} = E_{i,\infty} \quad \text{for particle } i$$

- Expected since we have $T(r)\sqrt{B(r)} = T_\infty$ and $E \propto T$ for thermal gas.
- Also through the definition of conserved energy $E_{i,\infty} = \xi^\mu p_{i,\mu}$,

where $\xi^\mu = (1, \mathbf{0})$ is the Killing vector of the static spacetime and p^μ is the four-momentum

Hence, $E_{i,\infty} = g_{00} p_i^0 = \sqrt{g_{00}} E_i(r)$

Recall the commonly used definition of internal energy of the gas:

$$U = \int \rho\sqrt{B}\, dV$$

Local-global correspondence:

In the canonical ensemble, the partition function is given as

$$Z(\beta) = \frac{1}{(2\pi)^3} \int dV \int e^{-\beta\sqrt{\mathbf{p}^2+m^2}} d^3\mathbf{p} \,,$$

where $dV = \sqrt{A}\, d^3r$ in the curved spacetime. Seeing the local-global correspondence is more explicit in the zero-mass case. For $m = 0$, we have

$$Z(\beta) = \frac{1}{2\pi^2} \int \sqrt{A}\, d^3r \int e^{-\beta E} E^2 dE \,. \qquad \text{locally treated}$$

Use the scaling relations:

$$Z = \frac{1}{2\pi^2} \int \sqrt{\frac{A}{B^3}}\, d^3r \int e^{-\beta_\infty E_\infty} E_\infty^2 dE_\infty \,, \qquad \text{its global dual}$$

Provided that the thermodynamical volume element (for $m = 0$) is defined as

$$\boxed{dV_0^{th} = \sqrt{\frac{A(r)}{B(r)^3}}\, d^3r}$$

the problem reduces the flat spacetime thermodynamics with this volume, which encodes all the curvature effects.

Local dynamics; general

$$dU = -pdV + TdS + \mu dN$$

$$d\rho = \frac{\rho + p}{n} dn + nTd\bar{s}$$

where n is the number density and $\bar{s}$ is the entropy per particle

$$\left(\frac{\partial \bar{s}}{\partial \rho}\right)_n = \frac{1}{nT} \quad \text{and} \quad \left(\frac{\partial \bar{s}}{\partial n}\right)_\rho = -\frac{(\rho + p)}{n^2 T}$$

$$\bar{s} = (\rho + p)/nT - \mu/T.$$

Different thermodynamical potential based on the characteristics of the system;
Helmholtz free energy, Gibbs free energy, etc.

$$f = \rho - n\bar{s}T \qquad g = \rho + p - n\bar{s}T$$

Massless ideal gas — local picture (1)

Choose the ensemble; begin with local parameters, determine the partition function, find the equation(s) of state.

Canonical ensemble:

$$Z(\beta) = \frac{\Delta V}{(2\pi)^3} \int e^{-\beta\sqrt{\mathbf{p}^2+m^2}} d^3p \qquad \Delta V = \Delta N / n$$

$$Z(\beta) = \frac{8\pi}{(2\pi)^3} \frac{\Delta N}{n(r)} \frac{1}{\beta^3(r)} \,,$$

Partition for ΔN particle:

$$Z_{\Delta N}(\beta) = \frac{Z^{\Delta N}(\beta)}{\Delta N!}$$

Massless ideal gas — local picture

Entropy per particle:

$$\bar{s} = \frac{1}{\Delta N}\left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln\left[Z_{\Delta N}\right] = \left(\ln\left[\frac{T^3}{n\pi^2}\right] + 4\right)$$

Pressure:

$$p = -n^2 T\left(\frac{\partial\bar{s}}{\partial n}\right)_T = \underline{nT}$$

Total energy within the small volume ΔV

$$\epsilon \;=\; -\left(\frac{\partial \ln Z_{\Delta N}}{\partial\beta}\right)_n \quad\Rightarrow\quad \rho = \underline{3nT}$$

Energy density

$$\boxed{\rho = 3p.}$$

✓

$$\rho = C\,T^4 \qquad C = \frac{3N}{V_{th}T_\infty^3}$$

A cross check: Noting that $n = e^{\mu/T}T^3/\pi^2$, the parameters above satisfy the Gibbs relation

$$\boxed{\bar{s} = (\rho + p)/nT - \mu/T}$$

✓

$$N^\mu = \int \frac{d^3p}{p^0} f^J p^\mu$$ particle four-vector

$$f^J = \frac{1}{(2\pi)^3} e^{(\mu - p^\alpha U_\alpha)/T}$$ Maxwell-Jüttner distr.

$$n = N^\mu U_\mu$$

Massless ideal gas;

From the local parameters to the global ones

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

- Let's start with total particle number:

Recall that $\frac{\mu}{T} = \frac{\mu_\infty}{T_\infty} \equiv \alpha \to constant$

and $T(r)\sqrt{B(r)} = T_\infty$ Tolman's law

$$N = \int n\, dV = \frac{e^\alpha}{\pi^2}\int T^3(r)\sqrt{A(r)}\, d^3r$$

$$= \underbrace{\frac{e^\alpha\, T_\infty^3}{\pi^2}}_{n_\infty} \underbrace{\int \sqrt{\frac{A(r)}{B^3(r)}}\, d^3r}_{V_{th}}$$

V_{th} Recall our thermodynamic volume

n_∞ ↕ via Tolman's law

$$n = e^{\mu/T} T^3/\pi^2 \longrightarrow \boxed{n(r)B^{3/2}(r) = n_\infty}$$

Scaling rule for the number density

$$\boxed{N = n_\infty V_{th}}$$

Massless ideal gas;
From the local parameters to the global ones, cont.

Energy:

$$U = \int \left(\sqrt{B(r)}\, \rho(r) \right) \sqrt{A(r)}\, d^3 r = 3NT_\infty$$

$$\rho = C\,T^4 \qquad C = \frac{3N}{V_{th} T_\infty^3}$$

$$U = \rho_\infty V_{th} \qquad\qquad \rho(r) B^2(r) = \rho_\infty$$

Entropy:

$$S = \int s\, dV = N \left(4 + \ln \left[\frac{V_{th}\, T_\infty^3}{\underbrace{\pi^2 N}_{e^{-\mu_\infty / T_\infty}}} \right] \right)$$

$s = n\bar{s}$ (entropy density)

Pressure:

$$p_\infty = \frac{NT_\infty}{V_{th}} = n_\infty T_\infty = \frac{\rho_\infty}{3}$$

$$p(r) B^2(r) = p_\infty$$

- So, the system globally mimics a homogeneous thermodynamic system.

The Gibbs's relation

$$U = T_\infty S - \underbrace{T_\infty N}_{p_\infty V_{th}} + \mu_\infty N$$

Massless ideal gas: global picture

- Recall the partition function in the global case

$$Z = \frac{1}{2\pi^2}\int \sqrt{\frac{A}{B^3}}\, d^3r \int e^{-\beta_\infty E_\infty} E_\infty^2 dE_\infty \qquad Z(\beta_\infty) = \frac{V_{th}}{\pi^2\beta_\infty^3} \qquad Z_N(\beta_\infty) = \frac{Z^N}{N!} \approx \left(\frac{Ze}{N}\right)^N$$

Then, using The Helmholtz free energy, $F = -\beta_\infty^{-1} \ln Z_N$, we obtain

$$\begin{aligned} S &= \beta_\infty^2 \left(\frac{\partial F}{\partial \beta_\infty}\right)_{V_{th},N} = N\left(\ln\left[\frac{T_\infty^3 V_{th}}{\pi^2 N}\right] + 4\right) && ✓ \\ U &= \left(\frac{\partial\, \beta_\infty F}{\partial \beta_\infty}\right)_{V_{th},N} = 3NT_\infty && ✓ \\ p_\infty &= -\left(\frac{\partial F}{\partial V_{th}}\right)_{\beta_\infty,N} = \frac{NT_\infty}{V_{th}}, && ✓ \end{aligned}$$

The first law:

$$dU = T_\infty dS - p_\infty dV_{th} + \mu_\infty dN \qquad ✓$$

Microcanonical ensemble:

- **So far we have focused on canonical ensemble. How about microcanonical ensemble?**

The number of microstates for N indistinguishable particles can be estimated as, as

$$\Omega = \frac{V_{\rm PS}}{h^{3N}}\frac{1}{N!}$$

where h^3 denotes our coarse-graining as the volume of a single cell, $1/N!$ is the estimation factor for not overcounting indistinguishable particles, and $V_{\rm PS} = \int dV^{(x)} dV^{(p)}$ is the total phase space volume

- **Local picture**

$$\Delta\Omega = \left(\frac{e^4}{27\pi^2}\frac{\rho^3}{n^4}\right)^{\Delta N}$$

$$s(r)\Delta V = \ln \Delta\Omega(r)$$

$$\left(\frac{\partial \bar{s}}{\partial \rho}\right)_n = \frac{1}{nT} \quad \text{and} \quad \left(\frac{\partial \bar{s}}{\partial n}\right)_\rho = -\frac{(\rho+p)}{n^2 T}$$

$$\rho + p = 4nT \quad \text{and} \quad \rho = 3nT$$

✓

- **Global picture**

$$S = \ln\Omega = \left(4 + \ln\left[\frac{V_{th}T_\infty^3}{\pi^2 N}\right]\right)$$

$$\frac{1}{T_\infty} = \left(\frac{\partial S}{\partial U}\right)_{V_{th},N} = \frac{3N}{U}$$

$$\frac{p_\infty}{T_\infty} = \left(\frac{\partial S}{\partial V_{th}}\right)_{U,N} = \frac{N}{V_{th}}$$

✓

Agrees with the canonical ensemble. Expected but not trivial in the curved spacetime.

Massive ideal gas:

- So far we have considered the simple case of massless ideal gas. Things get a little complicated in the massive case but still remarkably self-consistent.

- **Local picture**

$$Z(\beta) = \frac{\Delta V}{(2\pi)^3}\int e^{-\beta\sqrt{\mathbf{p}^2+m^2}}d^3p = \frac{\Delta V}{(2\pi)^3}\left(4\pi m^3\frac{K_2(b)}{b}\right) \qquad b{=}m/T$$

$$\rho = nT\left(3+b\frac{K_1(b)}{K_2(b)}\right)$$

$$p = nT = \frac{e^{\mu/T}m^2}{2\pi^2}T^2K_2(m/T)$$

$$\bar{s} = 4+b\frac{K_1(b)}{K_2(b)}+\ln\left[\underbrace{\frac{T^3}{\pi^2 n}\frac{b^2K_2(b)}{2}}_{e^{-\mu/T}}\right]$$

$$\bar{s} = (\rho+p)/nT - \mu/T.$$

✓

- **Global picture**

$$Z(\beta) = \frac{1}{(2\pi)^3}\int\sqrt{A(r)}\,d^3r\int e^{-\beta\sqrt{\mathbf{p}^2+m^2}}d^3p$$

$$U = \left(\frac{\partial\,\beta_\infty F}{\partial\beta_\infty}\right)$$

$$S = \beta_\infty^2\left(\frac{\partial F}{\partial\beta_\infty}\right)$$

agrees with the local case

Difficulty in finding the pressure via

$$p_\infty = -\left(\frac{\partial F}{\partial V_{th}}\right)_{\beta_\infty,N} \quad ?$$

Not clear how to identify the thermodynamic volume

Thermodynamic volume for the massive case: not straightforward

- **Easy in the massless case**

$$Z = \frac{1}{2\pi^2}\int\sqrt{\frac{A}{B^3}}\,d^3r\int e^{-\beta_\infty E_\infty}E_\infty^2 dE_\infty$$

$$\downarrow$$

Curvature and the energy parts can be seperated

- **Does not seem possible for the massive case**

$$Z(\beta) = \frac{1}{(2\pi)^3}\int dV\int e^{-\beta\sqrt{\mathbf{p}^2+m^2}}d^3\mathbf{p} \quad\longrightarrow\quad Z(\beta) = \frac{1}{2\pi^2\beta_\infty^3}\int d^3r\,\sqrt{\frac{A}{B^3}}\,b^2K_2(b)$$

Curvature and energy are tangled.

- **Yet we can still isolate dV_{th} from the first law of thermodynamics.**

$$dV_{th} = \frac{1}{b_\infty^2 K_2(b_\infty)}d\left(\int d^3r\sqrt{\frac{A}{B^3}}\,b^2K_2(b)\right)_{T_\infty,N}$$

$$\begin{aligned} p_\infty &= -\left(\frac{\partial F}{\partial V_{th}}\right)_{\beta_\infty,N} \\ &= \frac{e^\alpha m^2}{2\pi^2}T_\infty^2 K_2(m/T_\infty) \end{aligned} \quad ✓$$

Outlook and Conclusion:

- We have initiated a study of the curvature effects in thermodynamics and statistical mechanics.
- We have so far considered the massless and massive classical ideal gas both in canonical and microcanonical ensembles.
- There exists a thermodynamic dual picture globally in the curved spacetime.
- The global characteristics of the system mimics the flat spacetime thermodynamics with the identification of thermodynamic volume, which is different than the geometric volume.
- The curvature effects in the global thermodynamics is encoded in this thermodynamical volume.
- Next, 2-2-holes, neutron stars, quantum gas, grand canonical ensemble.

Additional Slides

Modified Bessel functions:

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$$K_n(z) = \frac{2^n n!}{(2n)!} \frac{1}{z^n} \int_z^\infty d\tau (\tau^2 - z^2)^{n-1/2} \exp(-\tau)$$

which, via partial integration, could also be given as

$$K_n(z) = \frac{2^{n-1}(n-1)!}{(2n-2)!} \frac{1}{z^n} \int_z^\infty d\tau (\tau^2 - z^2)^{n-3/2} \tau \exp(-\tau)$$