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Triple crossing, spectrahedron and positivity bounds

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CZ & **SYZ**, 2005.03047; Tolley, Wang & **SYZ**, 2011.02400; Li, Xu, Yang, **CZ** & **SYZ**, 2101.01191

Wang, Feng, **CZ** & **SYZ**, 2004.03992; Wang, **CZ** & **SYZ**, 2011.05190

CZ & **SYZ**,1808.00010; Bi, **CZ** & **SYZ**, 1902.08977; Yamashita, **CZ** & **SYZ**, 2009.04490

de Rham, Melville, Tolley & **SYZ**, 1702.06134, 1702.08577, 1706.02712, 1804.10624

in memory of my collaborator Cen Zhang

Outline

- Introduction
- Multi-field cone for s^2 coefficients
- Two-sided bounds from full crossing symmetry
- Applications
- Summary

Are all EFTs allowed?

EFTs are widely used in physics: gravity/cosmology, particle physics

$$\mathscr{L}_{\rm EFT} = \sum_i \Lambda^4 \, c_i \, \mathscr{O}_i \left(\frac{\rm boson}{\Lambda}, \frac{\rm fermion}{\Lambda^{3/2}}, \frac{\partial}{\Lambda} \right)$$

Λ: EFT cutoff $\quad c_i$: Wilson coefficients

Is every set of Wilson coefficients $\{c_i\}$ allowed? No!

UV completion satisfies: Lorentz invariance, causality/analyticity, unitarity, crossing symmetry, …

↓ *idea of bootstrapping*

Positivity bounds on Wilson coefficients

Simplest example: $P(X)$

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

2 to 2 scattering amplitude: $A(s, t=0) = \cdots + \frac{2\lambda s^2}{\Lambda^4} + \cdots$

"First" positivity bound: $\lambda > 0$

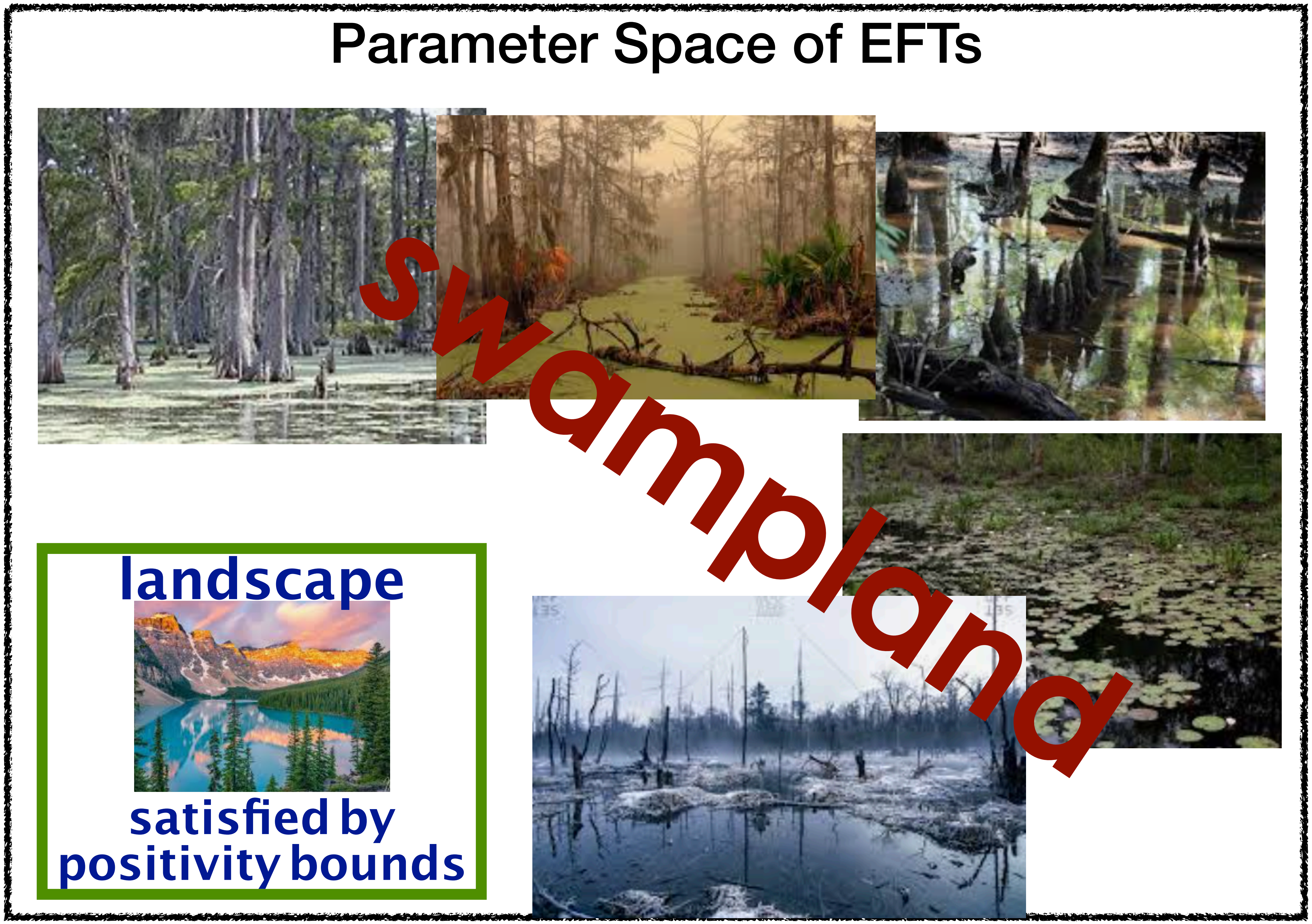
theories with $\lambda < 0$ do not have a standard UV completion

$$\mathscr{L}_{\mathrm{DBI}} \sim -\sqrt{1 + (\partial\phi)^2}$$

$$\mathscr{L}_{\overline{\mathrm{DBI}}} \sim -\sqrt{1 - (\partial\phi)^2}$$

Similar to swampland idea

But we take more conservative approach



Assumptions for positivity bounds (1)

Unitarity: conservation of probabilities $S^\dagger S = 1 \Rightarrow T - T^\dagger = iT^\dagger T$

Generalized optical theorem

$$A(I \to F) - A^*(F \to I) = i \sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_I - p_X) A(I \to X) A^*(F \to X)$$

optical theorem ($\theta = 0$): $\mathrm{Im}[A(I \to I)] \sim \sum_X \sigma(I \to X) > 0$

Partial wave expansion: $A(s,t) \sim \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos\theta) a_\ell(s)$

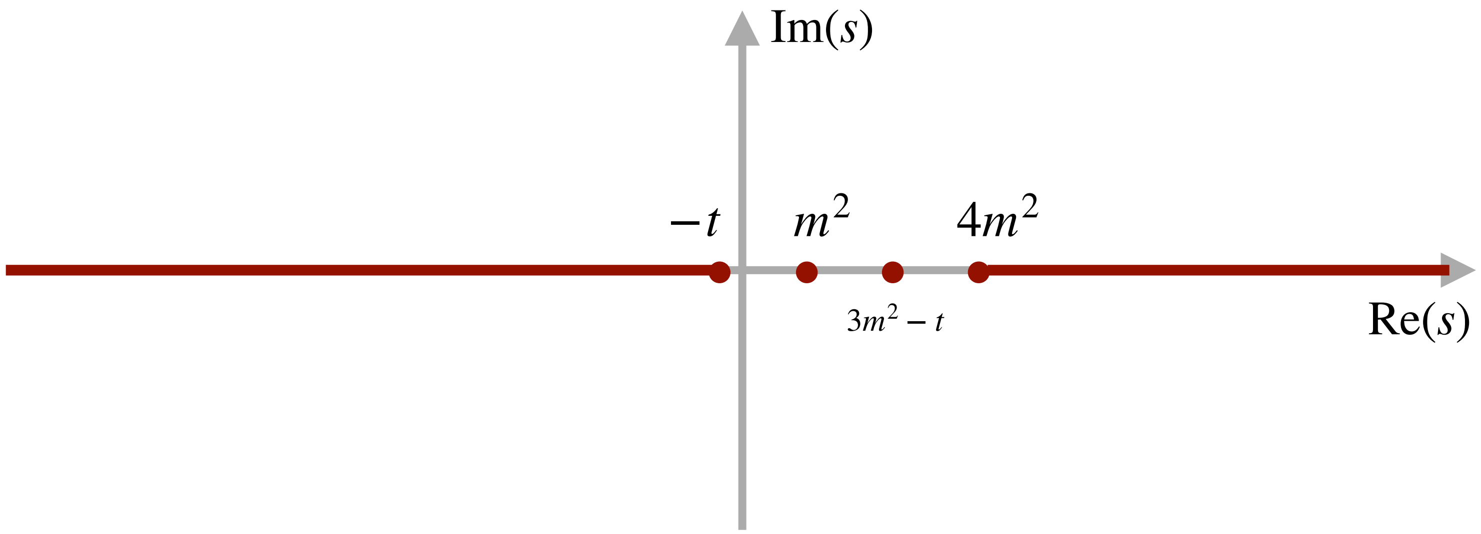
(2-2 scattering, for scalar)

Partial wave unitary bounds:

$$0 \leq |a_\ell(s)|^2 \leq \mathrm{Im}\, a_\ell(s) \leq 1$$

Assumptions for positivity bounds (2)

Causality/Analyticity: $A(s, t)$ as analytic function



s, t, u: Mandelstam variables

rigorously proven in 60'
Martin, …

Crossing symmetry: $A(s, t) = A(u, t) = A(t, s)$ (for scalar)

Locality: $A(s, t)$ is polynomially bounded at high energies

Froissart(-Martin) bound: Froissart, 1961; Martin, 1962

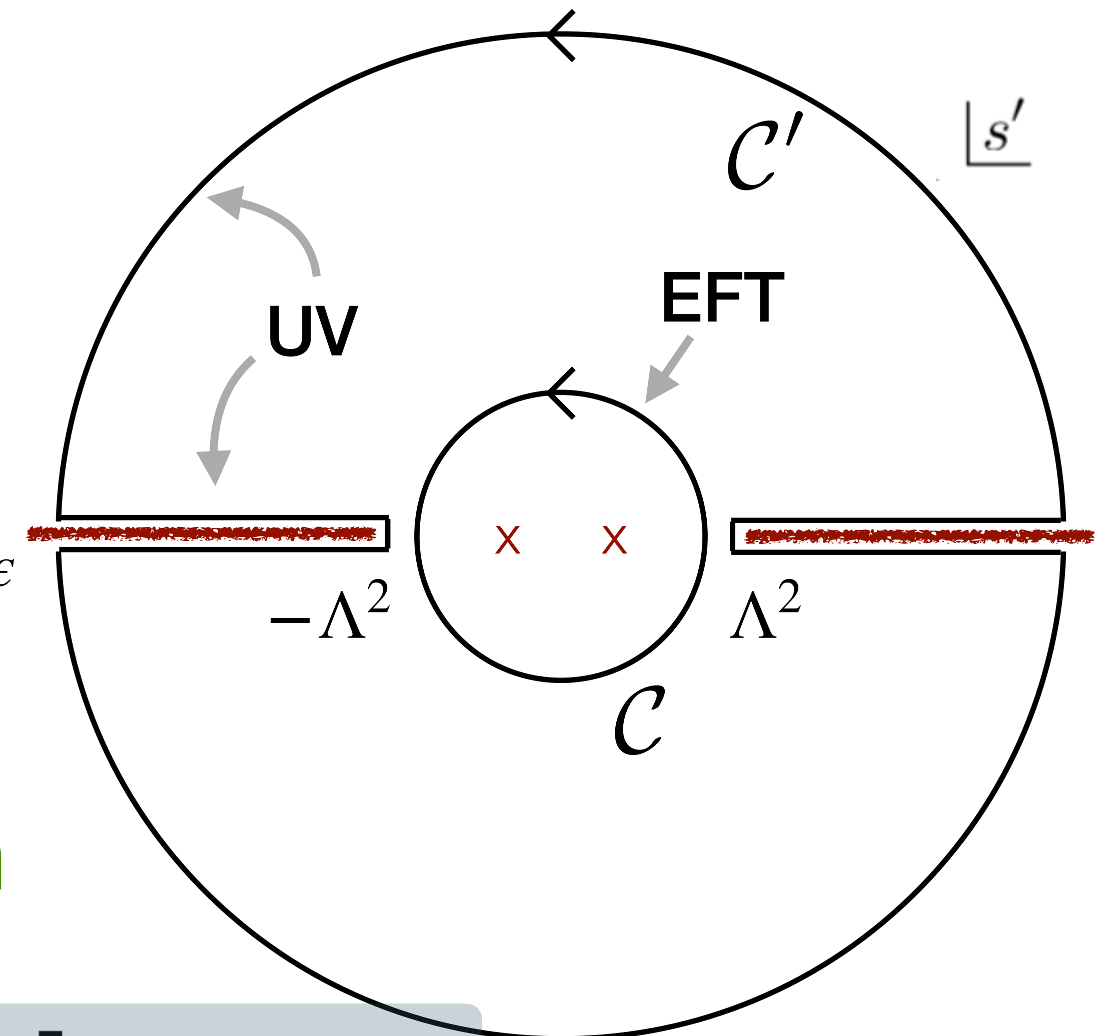
$$\lim_{s\to\infty} |A(s,t)| < Cs^{1+\epsilon(t)}, \quad t < 4m^2, \quad 0 < \epsilon(t) < 1$$

Fixed t dispersion relation

- Analyticity in complex s plane (fixed t)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s}$$

- Froissart bound $|A(s' \to \infty, t)| < s'^{2-\epsilon}$
- su crossing symmetry



su symmetric dispersion relation

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu-s} + \frac{u^2}{\mu-u}\right] \mathrm{Im}\, A(\mu,t)$$

$\mu > \Lambda^2$

EFT amplitude **IR/UV connection** **UV full amplitude**

Forward positivity bounds

- Forward limit: $\theta = 0$ (ie, $t = 0$)
- Massless limit (optional): $m \ll \Lambda$

$$A(s,0) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{s^2}{\mu + s}\right] \mathrm{Im}\, A(\mu, 0)$$

$$c_{2,0}s^2 + c_{4,0}s^4 + \cdots = \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^3}\mathrm{Im}\, A(\mu,0)\right)s^2 + \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^5}\mathrm{Im}\, A(\mu,0)\right)s^4 + \cdots$$

matching → **Sum rules:**

$$c_{2n,0} = \int \frac{2\,\mathrm{d}\mu}{\pi\mu^{1+2n}}\mathrm{Im}\, A(\mu,0)$$

Forward bounds

Optical theorem: $\mathrm{Im}[A(s,0)] = \sqrt{s(s-4m^2)}\sigma(s) > 0$

$$c_{2n,0} > 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006 + earlier works

Two different directions to go

- **one-field → multi-field for s^2 coefficients**

$$A(s,t=0)=\frac{2\lambda s^2}{\Lambda^4}+\cdots \quad \Longrightarrow \quad A_{ij\to kl}(s,t=0)=\frac{2\lambda_{ij\to kl}s^2}{\Lambda^4}+\cdots$$

  - lowest order positivity bounds — dim-8 ops
  - phenomenologically more relevant

- **higher order coefficients for s and t expansion**

$$A(s,t)\sim c_{2,0}s^2+c_{2,1}s^2t+c_{2,2}s^2t^2+\cdots$$

  - to understand naturalness in EFT
  - of theoretical importance

Multi-field cone for s^2 coefficients

based on a series of collaborations with Cen Zhang

Our fascinating universe

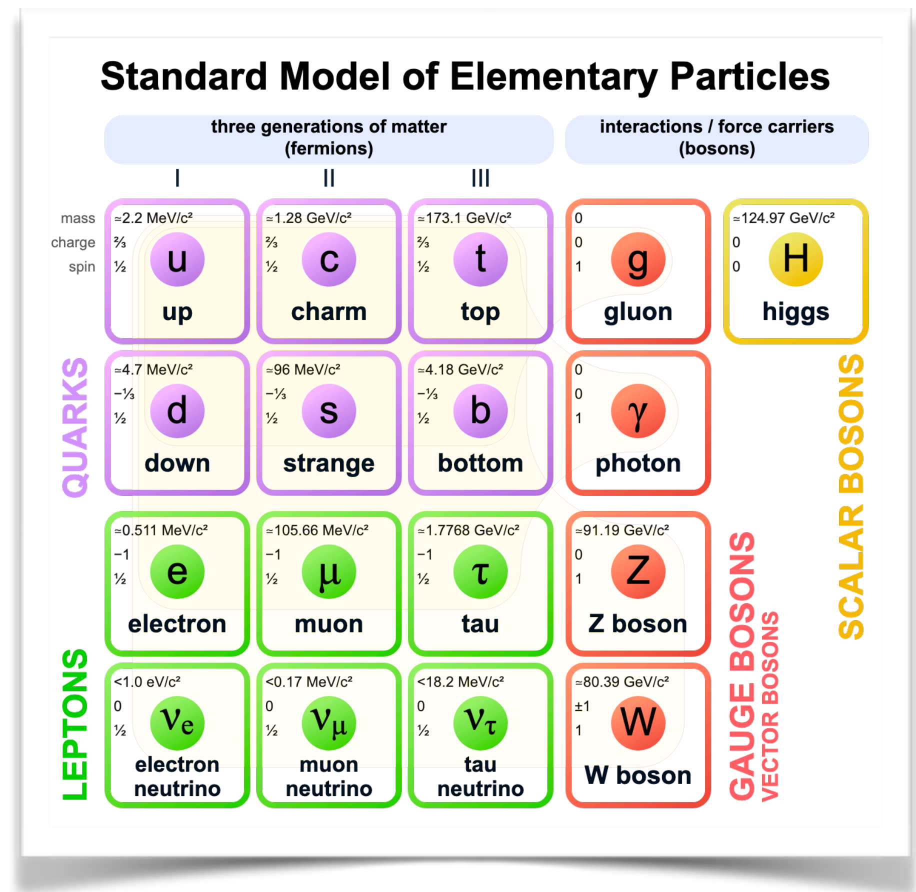
Universe is more complex than just one identical scalar!

SM Effective Field Theory (SMEFT)

$$\mathscr{L}_{\mathrm{SMEFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_j \frac{c_j^{(6)} O_j^{(6)}}{\Lambda^2} + \sum_i \frac{c_i^{(8)} O_i^{(8)}}{\Lambda^4} + \cdots$$

- SM particle contents and global symmetries
- SM gauge group structure

- Parametrize new physics
- Popular current approach



we consider up to dim-8, or s^2

still huge parameter space!

How to obtain the best forward bounds for EFTs with many DoFs?

Generalized elastic positivity bounds

Elastic scattering: $i + j \to i + j$

Consider $M(s) = A(s, t = 0)$

$$M^{ijij} = \frac{1}{2} \frac{\partial^2 M(ij \to ij)}{\partial s^2} \bigg|_{s \to 0} > 0$$

Generalized elastic scattering: $a + b \to a + b$

superposed states $|a\rangle = \sum_i u_i |i\rangle, \quad |b\rangle = \sum_j v_j |j\rangle$

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* \frac{1}{2} \frac{\partial^2}{\partial s^2} M(ij \to kl) > 0$$

Is this the most general case?

Entangled states

Is it possible such that

$$M^T = \sum_{ijkl} T_{ijkl} M^{ijkl} > 0, \text{ and } \{T_{ijkl}\} \supset \{u_i v_j u_k^* v_l^*\}?$$

Yes, T_{ijkl} is more than $u_i v_j u_k^* v_l^*$!

Example: W-boson scatterings in Standard Model EFT

$$F_{T,2} \geq 0, \quad 4F_{T,1} + F_{T,2} \geq 0$$

$$F_{T,2} + 8F_{T,10} \geq 0, \quad 8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$

$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$

$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$

scatterings of entangled states $T_{ijkl} \sim \Sigma_n \lambda_n U^n_{ij} U^n_{kl}$

Cen Zhang & **SYZ**, PRL, 2005.03047

Convex geometry: 1-slide crash course

Convex cone C:

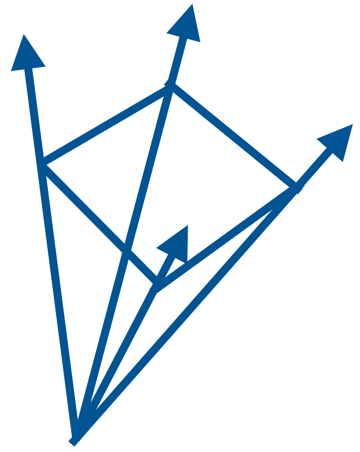
subset of a linear space that is closed under conical combinations

$$x \in C, y \in C, \alpha > 0, \beta > 0 \;\Rightarrow \alpha x + \beta y \in C$$

A conical combination of set Y, $\mathbf{cone}(Y)$, forms a convex cone.

Extremal ray (ER):

element of a convex cone that cannot be split into 2 other elements



Dual cone of C:

$$C^* = \{y \,|\, y \cdot x > 0,\, x \in C\}, \qquad (C^*)^* = C$$

Positive semi-definite matrices $\mathscr{P}_n$:

$$\mathscr{P}_n = \mathrm{cone}(\{m^I m^J \mid m^I \in \mathbb{R}\}), \qquad m^I m^J \text{ are ERs of } \mathscr{P}_n$$

Amplitude (convex) cone

s^2 sum rules

$$M^{ijkl} = \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{2i\pi\mu^3} \Big(\mathrm{Disc}(M_{ijkl}) + \mathrm{Disc}(M_{ilkj})\Big)$$

$$= \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^3} \sum_{X'} \left(m_X^{ij} m_X^{kl} + m_X^{il} m_X^{kj}\right)$$

- in absence of UV model, $m^{ij} = m_X^{ij}(\mu)$ are arbitrary numbers
- positive sum of $m^{i(j}m^{|k|l)}$

Amplitude cone (or s^2 coefficient cone)

$$\mathscr{C} \equiv \{M_{ijkl}\} = \mathrm{cone}\left(\left\{ m^{i(j}m^{|k|l)} \,\middle|\, m^{ij} \in \mathbb{R} \right\}\right)$$

$\mathscr{C}$ is cone of all amplitudes with UV completion

Dual cone $\mathscr{T}$

T_{ijkl} forms dual cone of $\mathscr{C}$

$$\mathscr{T} \equiv \left\{ T^{ijkl} \middle| T \cdot M \equiv \sum_{ijkl} T_{ijkl} M^{ijkl} > 0 \right\}$$

$\mathscr{T}$ contains the same information as $\mathscr{C}$

Crossing symmetries of amplitude

$$M^{ijkl} = M^{ilkj} = M^{kjil} = M^{jilk} \quad \Rightarrow \quad T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk}$$

$$\Rightarrow \quad T_{ijkl} \in \overrightarrow{\mathbf{S}} \equiv \left\{ T_{ijkl} \mid T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk} \right\}$$

Also, $T \cdot M > 0 \Rightarrow \Sigma_{ijkl} T_{ijkl} m^{ij} m^{kl} \Rightarrow m \,.\, T \,.\, m > 0$

positivity semi-definite (PSD) matrix

$$\Rightarrow \quad T_{ijkl} \in \mathscr{T}^{+} \equiv \left\{ T_{ijkl} \mid T_{ij,kl} \succeq 0 \right\}$$

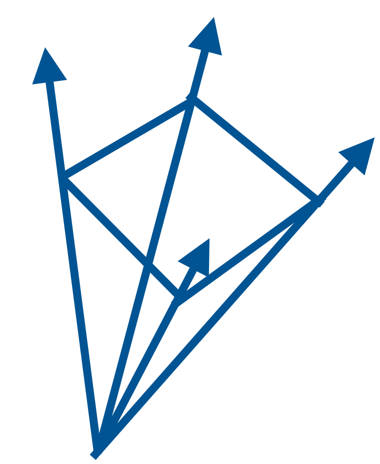
Best bounds from ERs of $\mathscr{T}$

$$T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^+ \cap \vec{\mathbf{S}} \qquad \begin{cases} \mathscr{T}^+ \equiv \left\{ T_{ijkl} \mid T_{ij,kl} \succeq 0 \right\} \\ \vec{\mathbf{S}} \equiv \left\{ T_{ijkl} \mid T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk} \right\} \end{cases}$$

T_{ijkl} forms spectrahedron $\mathscr{T}$

Li, Xu, Yang, Cen Zhang & **SYZ**, PRL, 2101.01191

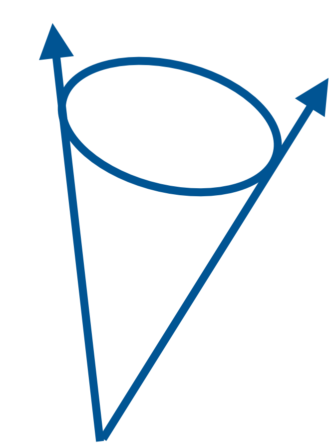
(spectrahedron) = (convex cone of PSD matrices) $\cap$ affine-linear space



To find best bounds, find all ERs of $\mathscr{T}$

all elements of $\mathscr{T}$: $T_{ijkl} = \Sigma_p \alpha_p T^{(p)}_{ijkl}, \;\; \alpha_p > 0$

p enumerates all ERs



Best positivity bounds:

$$\sum_{ijkl} T^{(p)}_{ijkl} M^{ijkl} > 0$$

Semi-definite program (SDP) for $\mathscr{C}$ cone

spectrahedron is parameter space of a semi-definite program

Use SDP to check M_{ijkl} is in $\mathscr{C}$ cone

minimize $\sum_{ijkl} T_{ijkl} M^{ijkl}$

subject to $T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^{+} \cap \overrightarrow{\mathbf{S}}$

$\min(T \cdot M) > 0$, then M_{ijkl} is within positivity bounds

Compared to elastic approach ($uvuvM > 0$)

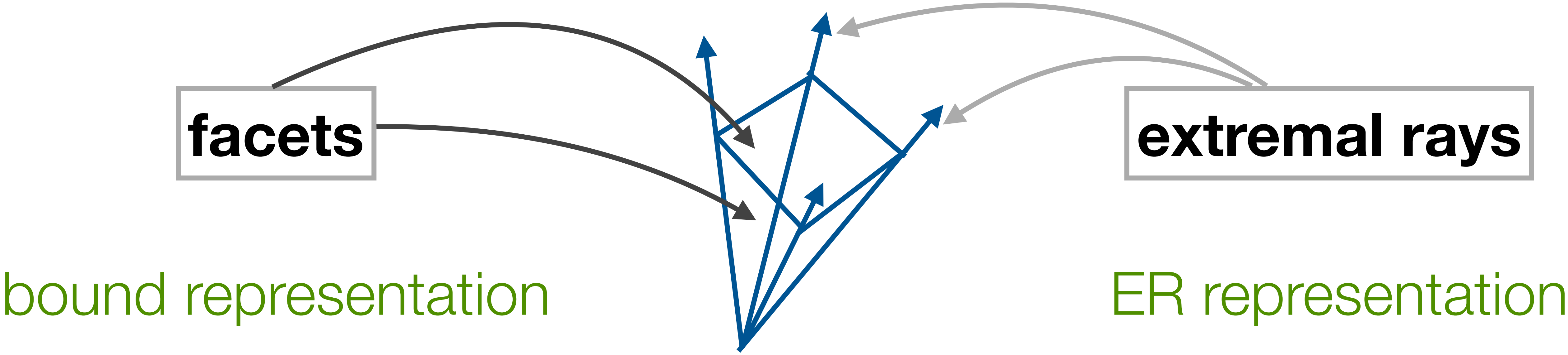
- stronger bounds
- more efficient (polynomial complexity)

Can also randomly sample and iterate to find ERs of $\mathscr{T}$

Li, Xu, Yang, Cen Zhang & **SYZ**, 2101.01191

$\mathscr{C}$ cone vs $\mathscr{T}$ cone

Two ways to describe one convex cone



facets of cone $\leftrightarrow$ ERs of dual cone

ERs of cone $\leftrightarrow$ facets of dual cone

Positivity bounds are ERs of $\mathscr{T}$ cone or facets of $\mathscr{C}$ cone

What about ERs of $\mathscr{C}$ cone?

Physical meaning of $\mathscr{C}$'s ERs

$$\mathscr{C} \equiv \{M_{ijkl}\} = \text{cone}\left(\left\{m^{i(j}m^{|k|l)}\right\}\right) \qquad m^{ij} \sim M^{ij\to X}$$

For m^{ij} to be extremal, it can not be split to two amplitudes

$$\Rightarrow \quad m^{ij}_{(\text{ER})} \sim M^{ij\to X}_{\text{irrep}}$$

$$M^{ijkl} = \int_{\Lambda^2}^{\infty} \mathrm{d}\mu \sum_{X\in \mathbf{r}}{}' \frac{|\langle X|T|\mathbf{r}\rangle(\mu)|^2}{\pi\mu^3} P_r^{i(j|k|l)}$$

CG coeff's

$$P_r^{ijkl} \equiv \sum_\alpha C_{i,j}^{r,\alpha}\left(C_{k,l}^{r,\alpha}\right)^*$$

group projector

$$\boxed{\mathscr{C} = \text{cone}\left(\left\{P_r^{i(j|k|l)}\right\}\right)}$$

Cen Zhang & **SYZ**, 2005.03047

Example: W boson

Symmetries: tensor project of the following

- gauge: $\mathbf{3}$ of $SU(2)_L$ with $N=3$
- forward rotation: $\mathbf{2}$ of $SO(2)$ with $N=2$

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}$$

$$\mathbf{2}\otimes\mathbf{2}=\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{2}$$

$$P^1_{\alpha\beta\gamma\sigma}=\frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\sigma},\quad P^2_{\alpha\beta\gamma\sigma}=\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\sigma}-\delta_{\alpha\sigma}\delta_{\beta\gamma})$$
$$P^3_{\alpha\beta\gamma\sigma}=\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\sigma}+\delta_{\alpha\sigma}\delta_{\beta\gamma})-\frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\sigma}$$

ERs:

$$E_{1,1},E_{1,3},E_{2,2},E_{3,1},E_{3,3},$$
$$E_{1,2},E_{2,1},E_{2,3},E_{3,2}$$

$E_{3,3}$ is not ER due to $P_r^{i(j|k|l)}$

Positivity bounds:

$$F_{T,2}\geq 0,\quad 4F_{T,1}+F_{T,2}\geq 0$$
$$F_{T,2}+8F_{T,10}\geq 0,\quad 8F_{T,0}+4F_{T,1}+3F_{T,2}\geq 0$$
$$12F_{T,0}+4F_{T,1}+5F_{T,2}+4F_{T,10}\geq 0$$
$$4F_{T,0}+4F_{T,1}+3F_{T,2}+12F_{T,10}\geq 0$$

The inverse problem

For weakly coupled UV completion

Cen Zhang & **SYZ**, 2005.03047

ER ⟷ UV particle

Example: Higgs $\mathscr{C}$ cone in SMEFT

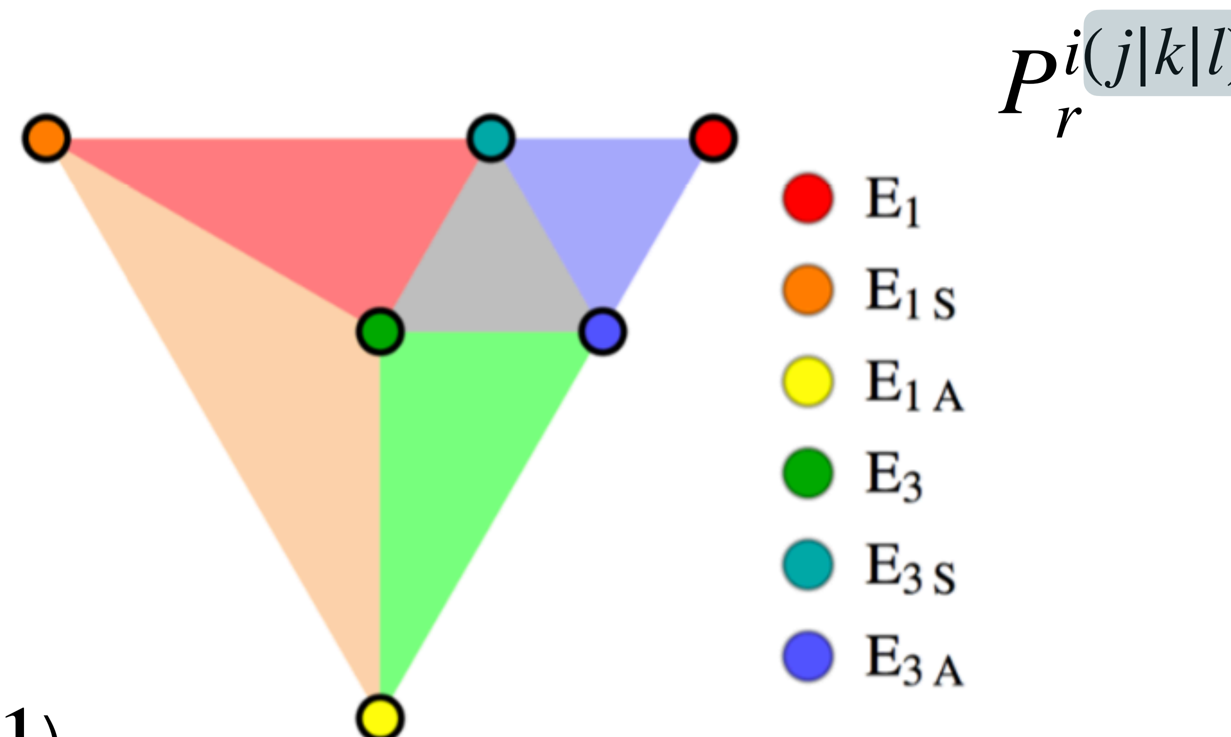
Wilson coeff's fall in blue region

↓

E_1 must exit

↓

new UV state ($SU(2)_L$ singlet, $Y = 1$)

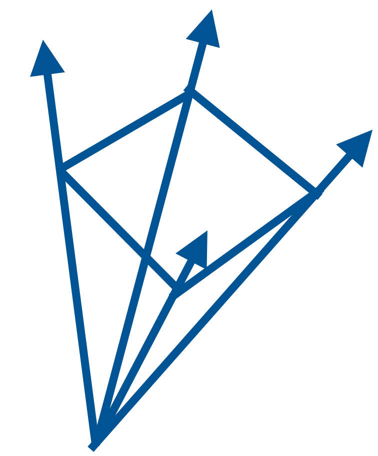


ERs of $\mathscr{C}$ (or positivity bounds) are important to inverse-engineer UV physics!

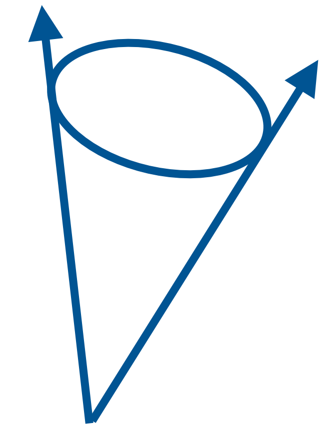
From $\mathscr{C}$ cone to positivity bounds

ERs of $\mathscr{C}$ are easier to identify if enough symmetries!

- sufficient symmetries ⇒ finite # of ERs
  - easy to obtain ERs of $\mathscr{T}$, ie, positivity bounds
  - by vertex enumeration
  - efficient codes available



- insufficient symmetries ⇒ infinite # of ERs
  - continuous parameters appear in the ERs of $\mathscr{C}$
  - difficult to obtain ERs of $\mathscr{T}$
  - more efficient to obtain positivity bounds via SDP



Two-sided bounds from full crossing symmetry

What about other coefficients?

(switch back to single scalar)

forward bounds

non-forward bounds

$$A(s,t) \sim c_{2,0}s^2 + c_{2,1}s^2t + c_{2,2}s^2t^2 + \cdots$$

S-matrix principles of UV theory

new gradient:
full crossing symmetry

New bounds:

All $c_{i,j}$ with $i > 2,\ j \geq 0$ have two-sided bounds

(in units of $c_{2,0}$)!

Tolley, Wang & **SYZ**, 2011.02400

Caron-Huot & Duong, 2011.02957

su symmetric non-forward bounds

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu-s}+\frac{u^2}{\mu-u}\right]\mathrm{Im}\,A(\mu,t)$$

Partial wave unitarity + Positivity of Legendre polynomial

$$\frac{\partial^n}{\partial t^n}\mathrm{Im}[A(s,t)] > 0, \quad s \geq 4m^2,\ 0 \leq t < 4m^2$$

Recurrent Y bounds:

de Rham, Melville, Tolley & **SYZ**,1702.06134,1706.02712

$$Y^{(2N,M)} = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} + \frac{1}{\mathcal{M}^2}\sum_{k\ \mathrm{even}}^{(M-1)/2}(2(N+k)+1)\beta_k Y^{(2(N+k),M-2k-1)} > 0$$

$c_{i\geq 2,j}$ are typically bounded, but with open sides

Arkani-Hamed, Huang & Huang (*EFTHedron*) + other authors

su symmetric sum rules

$$\sum_{i,j} c_{i,j} s^i t^j = A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t)$$

$$A(s,t) \sim \Sigma_\ell P_\ell(1 + 2t/s) a_\ell(s)$$

expand dispersion relation and matching $s^i t^j$ on both sides

Sum rules: $$c_{i,j} \sim \int_{\Lambda^2}^{\infty} d\mu \frac{D_{i,j}(\eta)}{\mu^{i+j}}$$ $$\eta = \ell(\ell + 1)$$

$D_{i,j}$ is polynomial of η that is bounded below

su **symmetric bounds** $$c_{i,j} \sim \int_{\Lambda^2}^{\infty} d\mu \frac{D_{i,j}(\eta)}{\mu^{i+j}} > D_{i,j}^{\min} \int_{\Lambda^2}^{\infty} d\mu \frac{1}{\mu^{i+j}} = D_{i,j}^{\min} c_{2,0}$$

Tolley, Wang & **SYZ**, 2011.02400

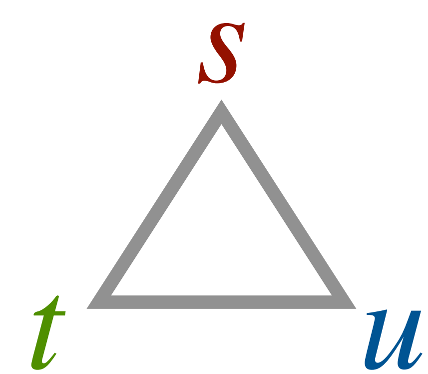
$$D_{i,j}^{\min} = \min_\eta [D_{i,j}(\eta)]$$

these su bounds are like Y bounds

reason for su symmetric bounds often being open-sided

New ingredient: sum rules from st symmetry

But we actually also have



$$A(u,t) = A(s,t) = A(t,s)$$

Impose st symmetry on su dispersion relation

$$\int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{t^2}{\mu - t} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, s)$$

partial wave expansion + matching on both sides for $s^i t^j$

Null constraints

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

$$\int_{\Lambda^2}^{\infty} d\mu \frac{\Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}} = 0$$

$\Gamma^{(n)}_{i,j}$ are polynomials of $\eta = \ell(\ell+1)$

bounded for physical η

Two-sided bounds

Add null constraints to sum rules:

$$\int_{\Lambda^2}^{\infty} d\mu \frac{\Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}} = 0$$

$$c_{i,j} \sim \int_{\Lambda^2}^{\infty} d\mu \frac{D_{i,j}(\eta) + \sum_n \alpha_n \Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}}$$

can choose α_n to make $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ bounded from blow and above

before: $D_{i,j}$ only has **min**

now: $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ can have **min and max**

α_n can be positive or negative

→ **$c_{i,j}$ is bounded from both sides!**

Tolley, Wang & **SYZ**, 2011.02400

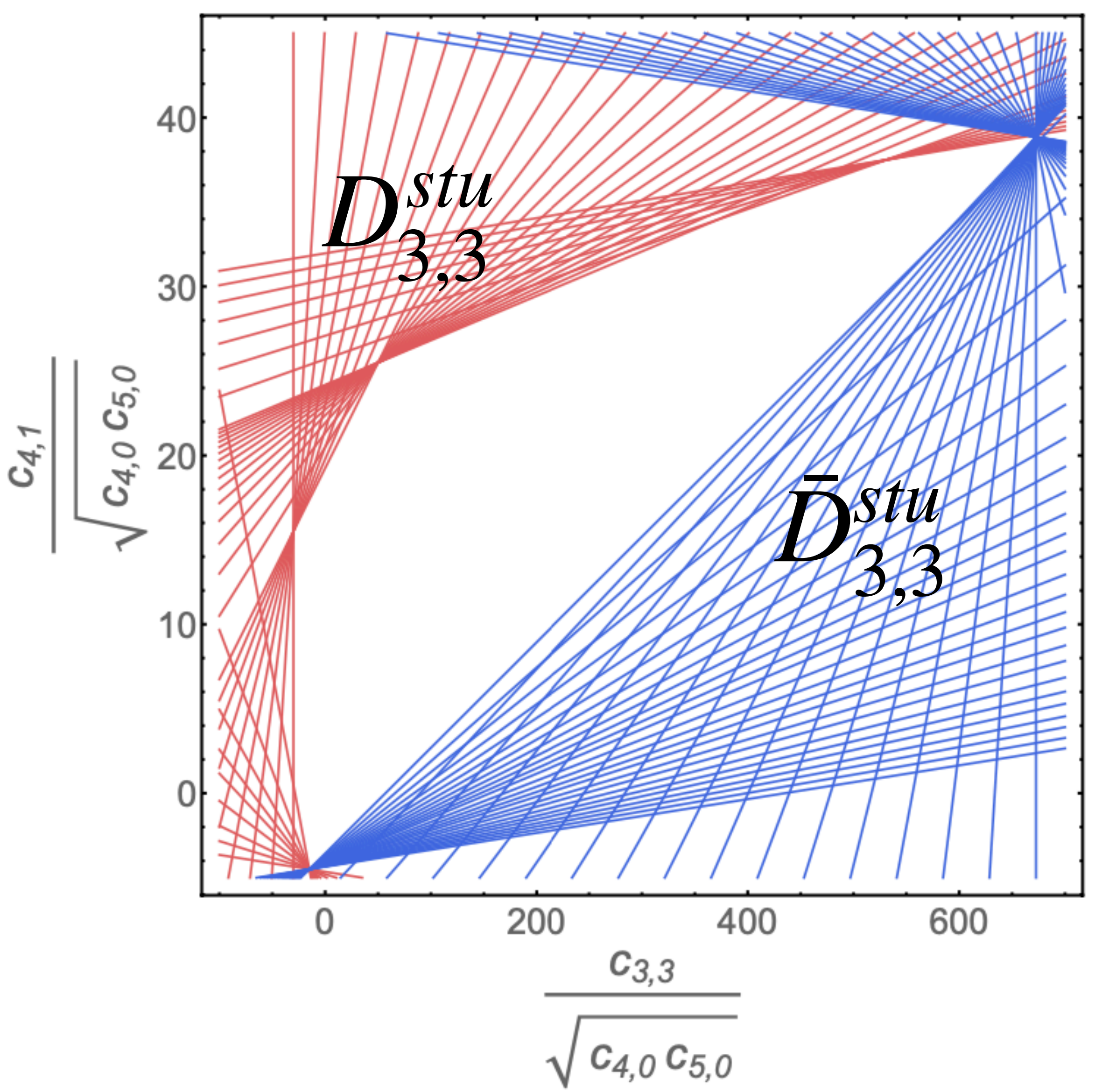
Two-sided bounds

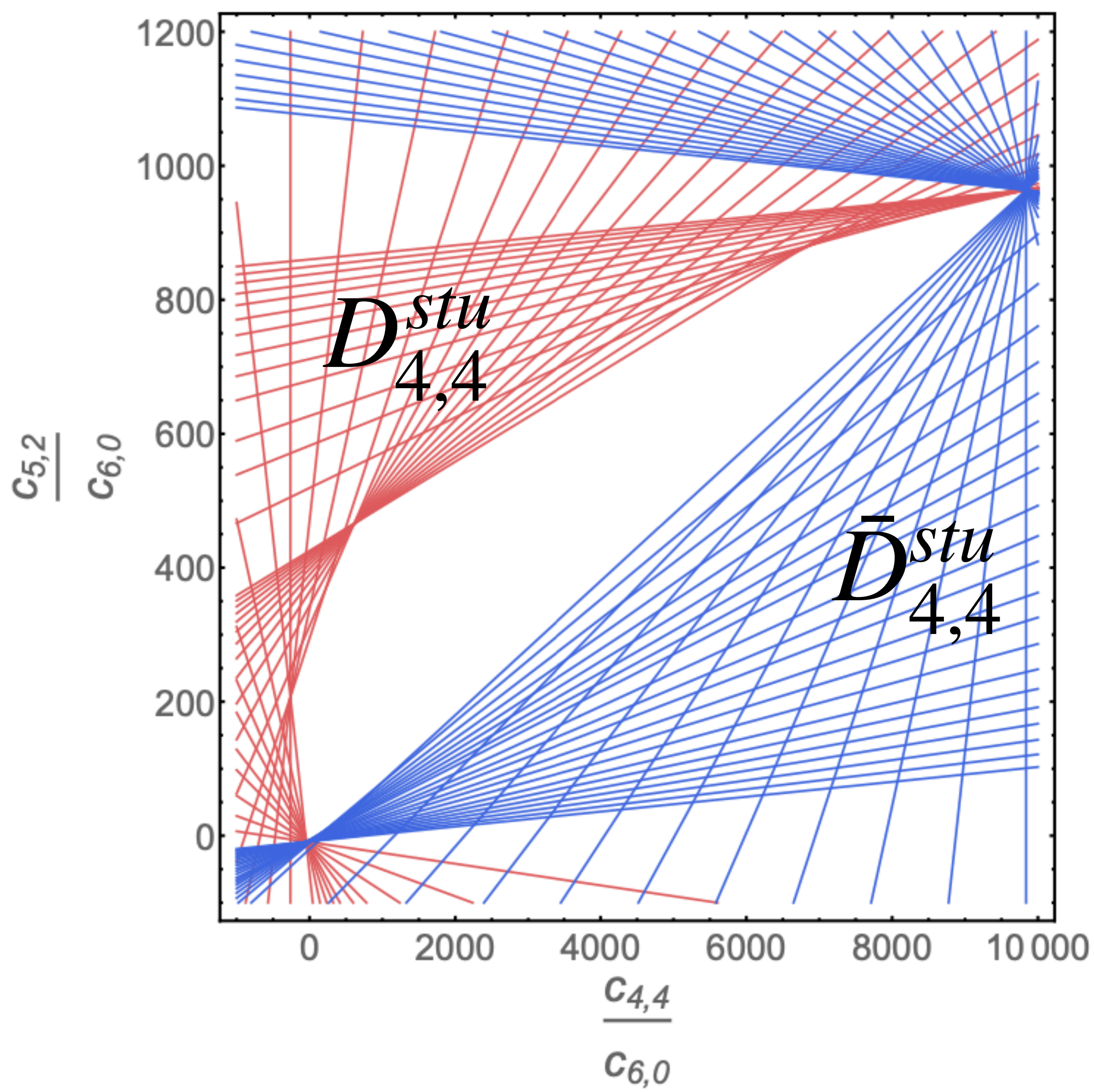
Tolley, Wang & **SYZ**, 2011.02400

(m,n)	**Lower bounds**	**Upper bounds**
$(1,1)$	$c_{1,1} > -\frac{3}{2}\sqrt{c_{1,0}c_{2,0}}$	$c_{1,1} < 8\sqrt{c_{1,0}c_{2,0}}$
$(2,1)$	$c_{2,1} > -\frac{5}{2}\sqrt{c_{2,0}c_{3,0}}$	$c_{2,1} < \frac{465}{38}\sqrt{c_{2,0}c_{3,0}}$
$(2,2)$	$c_{2,2} > -\frac{9}{2}c_{3,0}$	$c_{2,2} < \frac{2961}{58}c_{3,0}$
$(3,1)$	$c_{3,1} > -\frac{7}{2}\sqrt{c_{3,0}c_{4,0}}$	$c_{3,1} < \frac{1097}{58}\sqrt{c_{3,0}c_{4,0}}$
$(3,2)$	$c_{3,2} > -7c_{4,0}$	$c_{3,2} < \frac{10027}{59}c_{4,0}$
$(3,3)$	$c_{3,3} + \frac{3}{4}c_{4,1} > -\frac{147}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 8c_{4,1} > -154\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - \frac{481}{12}c_{4,1} > -\frac{7777}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 104c_{4,1} > -3369\sqrt{c_{4,0}c_{5,0}}$	$c_{3,3} - \frac{650}{41}c_{4,1} < -\frac{2310}{41}\sqrt{c_{4,0}c_{5,0}}$
$(4,2)$	$c_{4,2} > -\frac{17}{2}c_{5,0}$	$c_{4,2} < \frac{3923}{12}c_{5,0}$
$(4,3)$	$c_{4,3} + \frac{3}{4}c_{5,1} > -\frac{253}{8}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{180}{41}c_{5,1} > -\frac{8705}{82}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{325}{12}c_{5,1} > -\frac{16825}{24}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{169}{2}c_{5,1} > -\frac{11187}{4}\sqrt{c_{5,0}c_{6,0}}$ $c_{4,3} - \frac{743}{4}c_{5,1} > -\frac{63279}{8}\sqrt{c_{5,0}c_{6,0}}$	$c_{4,3} - \frac{73153}{1748}c_{5,1} < -\frac{708543}{3496}\sqrt{c_{5,0}c_{6,0}}$
$(4,4)$	$c_{4,4} + \frac{25}{24}c_{5,2} > -\frac{147}{8}c_{6,0}$, $c_{4,4} - \frac{125}{37}c_{5,2} > -\frac{71175}{74}c_{6,0}$, $c_{4,4} - \frac{785}{52}c_{5,2} > -\frac{83490}{13}c_{6,0}$, $c_{4,4} - \frac{2485}{69}c_{5,2} > -\frac{1144125}{46}c_{6,0}$	$c_{4,4} - 15c_{5,2} < -\frac{195}{2}c_{6,0}$, $c_{4,4} + \frac{368085}{36544}c_{5,2} < -\frac{2365845}{18272}c_{6,0}$

Enclosed regions from two-sided bounds

Tolley, Wang & **SYZ**, 2011.02400





Further developments

Linear programming

Caron-Huot & Duong, 2011.02957

bounds can be improved by mixing different orders of $\Gamma^{(n)}_{i,j}(\eta)$

use SDPB numerically

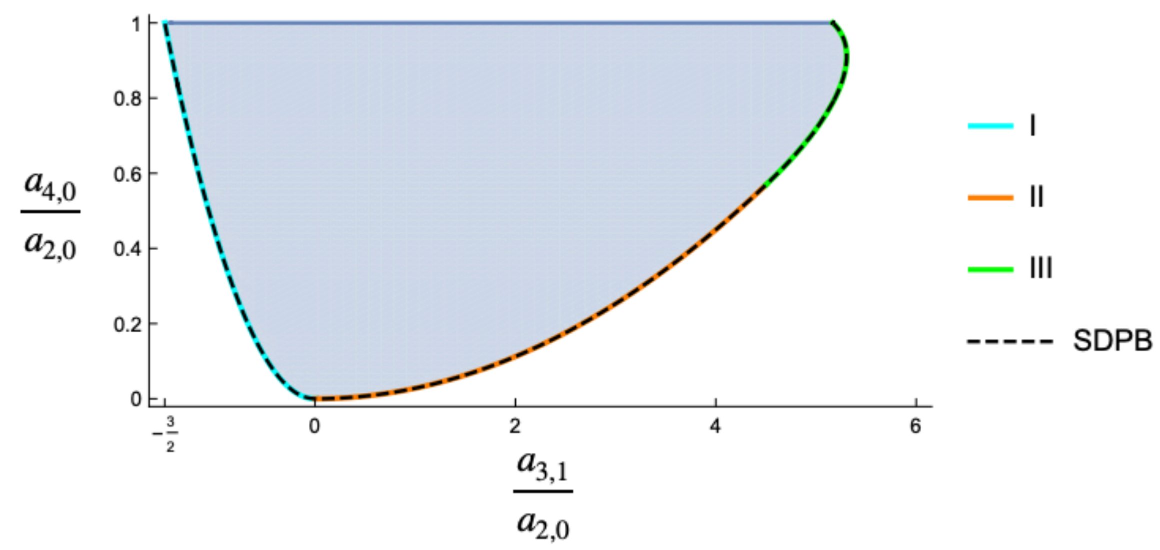
Fully crossing symmetric dispersion relation

Sinha & Zahed, 2012.04877

Analytical approach

reduce to bi-variate moment problem
(GL rotations + triple-crossing slices)

Chiang, Huang, Li, Rodina & Weng, 2105.02862



Bounds from fixed impact parameter

can deal with spin-2 t-pole

Caron-Huot, Mazac, Rastelli, Simmons-Duffin,2102.08951

Applications

Application (1): Ruling out Galileon

$$\pi \to \pi + c + b_\mu x^\mu, \quad c, b_\mu = const$$

- linked to dRGT massive gravity
- applications in cosmology

original Galileon marginally ruled out by Adams et al, 2006

Weakly broken Galileon theories

$$\mathcal{L} \sim \mathcal{L}_{\rm galileon} - \frac{m^2}{2}\pi^2$$

stu symmetric bounds

$\Lambda \sim m$ **not a valid EFT**

may also add $\alpha(\partial\phi)^4, \ |\alpha| \ll 1$
leads to same conclusion

Tolley, Wang & **SYZ**, 2011.02400

Constraining SMEFT

Standard model EFT

$$\mathscr{L}_{\mathrm{SMEFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_j \frac{f_j^{(6)} O_j^{(6)}}{\Lambda^2} + \sum_i \frac{f_i^{(8)} O_i^{(8)}}{\Lambda^4} + \cdots$$

huge parameter space!

Application (2): Transversal VBS

Vector Boson Scattering

Yamashita, Cen Zhang & **SYZ**, 2009.04490

$$V_1 + V_2 \to V_3 + V_4 \qquad V_i \in \{W_x^i, W_y^i, B_x, B_y\}$$

$$\begin{aligned}
O_{T,0} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}] & O_{T,1} &= \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}] \\
O_{T,2} &= \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}] & O_{T,10} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}] \\
O_{T,5} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta} & O_{T,6} &= \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\hat{B}_{\mu\beta}\hat{B}^{\alpha\nu} \\
O_{T,7} &= \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha} & O_{T,11} &= \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta} \\
O_{T,8} &= \hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta} & O_{T,9} &= \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta}\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}
\end{aligned}$$

already 10D parameter space

They lead to anomalous Quartic Gauge Couplings (aQGCs)

ERs of amplitude cone $\mathscr{C}$

Mixing W and B leads to projectors

$$P^{(1)}(r)_{\alpha\beta\gamma\sigma} = \frac{1}{3} d_{\alpha\beta}(r) d_{\gamma\sigma}(r)$$

$$d_{\alpha\beta}(r) = \begin{cases} 1 & \alpha=\beta=1,2,3 \\ r & \alpha=\beta=4 \\ 0 & \text{otherwise} \end{cases}$$

$$f^1_{\alpha\beta}(r_1,r_2) = \begin{pmatrix} 0 & 0 & 0 & r_1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ r_2 & 0 & 0 & 0 \end{pmatrix}$$

$$P_S^{(2)}(r_1,r_2)_{\alpha\beta\gamma\sigma} = \frac{1}{2}\sum_{i=1}^{3} f^i_{\{\alpha,\beta\}}(r_1,r_2) f^i_{\{\gamma,\sigma\}}(r_1,r_2)$$

$$f^2_{\alpha\beta}(r_1,r_2) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & r_1 \\ 1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \end{pmatrix}, \quad f^3_{\alpha\beta}(r_1,r_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 \\ 0 & 0 & r_2 & 0 \end{pmatrix}$$

$$P_A^{(2)}(r_1,r_2)_{\alpha\beta\gamma\sigma} = \frac{1}{2}\sum_{i=1}^{3} f^i_{[\alpha,\beta]}(r_1,r_2) f^i_{[\gamma,\sigma]}(r_1,r_2)$$

$$P^{(3)}_{\alpha\beta\gamma\sigma} = \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\gamma}) - \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\sigma}$$

ERs:

$$\vec{e}_1 = (1,0,0,0,0,0,0,0,0,0,0,0,0)$$
$$\vec{e}_2 = (0,1,0,0,0,0,0,0,0,0,0,0,0)$$
$$\vec{e}_3 = (0,0,1,0,0,0,0,0,0,0,0,0,0)$$
$$\vec{e}_4 = (0,0,0,1,0,0,0,0,0,0,0,0,0)$$
$$\vec{e}_5 = \left(-\frac{1}{6},\frac{1}{6},0,0,-\frac{5}{3},0,0,\frac{5}{3},0,0,\frac{5}{6},0,0\right)$$
$$\vec{e}_6 = \left(0,0,-1,1,0,-\frac{3}{4},0,0,\frac{3}{4},0,0,0,1\right)$$

$$\vec{e}_7(r) = (0,0,0,0,1,r,r^2,0,0,0,0,0,0)$$
$$\vec{e}_8(r) = (0,0,0,0,0,0,0,1,r,r^2,0,0,0)$$
$$\vec{e}_9(r) = (0,0,0,0,0,0,0,0,0,0,1,r,r^2)$$
$$\vec{e}_{10}(r) = \left(-\frac{1}{3},\frac{1}{3},-\frac{4r}{3},\frac{4r}{3},-\frac{1}{3},0,-r^2,\frac{1}{3},0,r^2,-\frac{1}{3},0,-\frac{4r}{3}\right)$$
$$\vec{e}_{11}(r) = \left(\frac{1}{2},\frac{1}{2},\frac{r^2}{2},\frac{r^2}{2},-1,-\frac{3r^2}{8},0,-1,-\frac{3r^2}{8},0,-\frac{1}{2},r,-\frac{r^2}{2}\right)$$
$$\vec{e}_{12}(r) = \left(1,0,r^2,0,-2,-\frac{3r^2}{4},0,0,0,0,1,-2r,r^2\right)$$

Convex cone bounds on transversal aQGCs

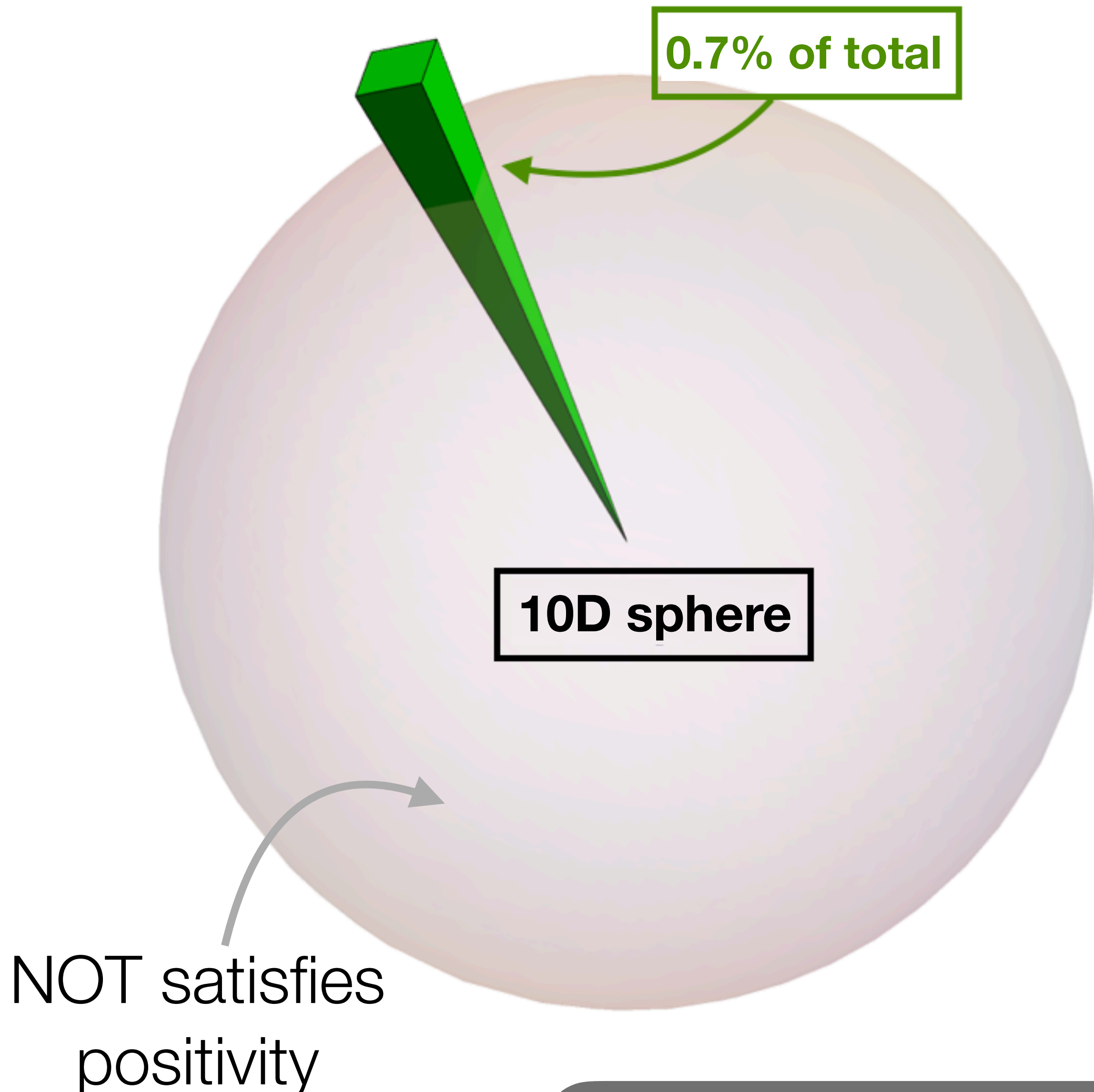
Conservative analytic positivity bounds:

$$F_{T,2} \geq 0$$
$$4F_{T,1} + F_{T,2} \geq 0$$
$$F_{T,2} + 8F_{T,10} \geq 0$$
$$8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$
$$4F_{T,6} + F_{T,7} \geq 0$$
$$F_{T,7} \geq 0$$
$$2F_{T,8} + F_{T,9} \geq 0$$
$$F_{T,9} \geq 0$$

$$F_{T,9}\left(F_{T,2} + 4F_{T,10}\right) \geq F_{T,11}^2$$
$$16\left(2\left(F_{T,0} + F_{T,1}\right) + F_{T,2}\right)\left(2F_{T,8} + F_{T,9}\right) \geq \left(4F_{T,5} + F_{T,7}\right)^2$$
$$32\left(2F_{T,8} + F_{T,9}\right)\left(3F_{T,0} + F_{T,1} + 2F_{T,2} + 4F_{T,10}\right) \geq 3\left(4F_{T,5} + F_{T,7}\right)^2$$
$$2\sqrt{2}\sqrt{F_{T,9}\left(F_{T,2} + 8F_{T,10}\right)} \geq \max\left(4F_{T,6} + F_{T,7} - 4F_{T,11}, F_{T,7} + 4F_{T,11}\right)$$
$$4\sqrt{\left(8F_{T,0} + 4F_{T,1} + 3F_{T,2}\right)\left(2F_{T,8} + F_{T,9}\right)} \geq \max\left(-8F_{T,5} - F_{T,7}, 8F_{T,5} + 4F_{T,6} + 3F_{T,7}\right)$$
$$4\sqrt{F_{T,9}\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left(4F_{T,6} + F_{T,7} - 4F_{T,11}, F_{T,7} + 4F_{T,11}\right)$$
$$4\sqrt{6}\sqrt{\left(2F_{T,8} + F_{T,9}\right)\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left[-3\left(8F_{T,5} + F_{T,7}\right), 3\left(8F_{T,5} + 4F_{T,6} + 3F_{T,7}\right)\right]$$
$$\sqrt{6}\sqrt{\left(4F_{T,8} + 3F_{T,9}\right)\left(6F_{T,0} + 2F_{T,1} + 3F_{T,2} + 6F_{T,10}\right)} \geq \max\left[-3\left(2F_{T,5} + F_{T,11}\right), 3\left(2F_{T,5} + F_{T,7} + F_{T,11}\right)\right]$$
$$2\sqrt{\left(12F_{T,8} + 7F_{T,9}\right)\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left(-12F_{T,5} - F_{T,7} - 2F_{T,11}, -12F_{T,5} + 4F_{T,6} - F_{T,7} - 2F_{T,11}, -12F_{T,5} - F_{T,7} + 2F_{T,11}, 12F_{T,5} + 4F_{T,6} + 5F_{T,7} + 2F_{T,11}\right)$$

Yamashita, Cen Zhang & **SYZ**, 2009.04490

How effective are the bounds?



Conservative analytic positivity bounds:

0.687 %

Full numeric positivity bounds:

0.681 %

only **~0.7%** of transversal aQGC parameter space is physical!

Application (3): 4-gluon SMEFT operators

$Q_{G^4}^{(1)}$	$(G_{\mu\nu}^A G^{A\mu\nu})(G_{\rho\sigma}^B G^{B\rho\sigma})$
$Q_{G^4}^{(2)}$	$(G_{\mu\nu}^A \widetilde{G}^{A\mu\nu})(G_{\rho\sigma}^B \widetilde{G}^{B\rho\sigma})$
$Q_{G^4}^{(3)}$	$(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^A G^{B\rho\sigma})$
$Q_{G^4}^{(4)}$	$(G_{\mu\nu}^A \widetilde{G}^{B\mu\nu})(G_{\rho\sigma}^A \widetilde{G}^{B\rho\sigma})$

$Q_{G^4}^{(7)}$	$d^{ABE}d^{CDE}(G_{\mu\nu}^A G^{B\mu\nu})(G_{\rho\sigma}^C G^{D\rho\sigma})$
$Q_{G^4}^{(8)}$	$d^{ABE}d^{CDE}(G_{\mu\nu}^A \widetilde{G}^{B\mu\nu})(G_{\rho\sigma}^C \widetilde{G}^{D\rho\sigma})$
Q_G	$f^{ABC}G_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$

(dim-6)^2 contribute negatively to bounds

$$\vec{c} \equiv \left[C_{G^4}^{(1)} \quad C_{G^4}^{(2)} \quad C_{G^4}^{(3)} \quad C_{G^4}^{(4)} \quad C_{G^4}^{(7)} \quad C_{G^4}^{(8)} \quad c_G^2 \right]$$

Positivity bounds region: 1.6628% ± 0.0007%

obtained bounds both in $\mathscr{C}$ and $\mathscr{T}$ cones

Li, Xu, Yang, Cen Zhang & **SYZ**, PRL, 2101.01191

Rough estimate: $\sim 1/2^N$, N **= parameter space's dimensions**

More applications:

- Cosmology

  Tolley, Wang & **SYZ**, 2011.02400
  de Rham, Melville, Tolley & **SYZ**, 1702.08577, 1804.10624
  Wang, Zhang & **SYZ**, 2011.05190

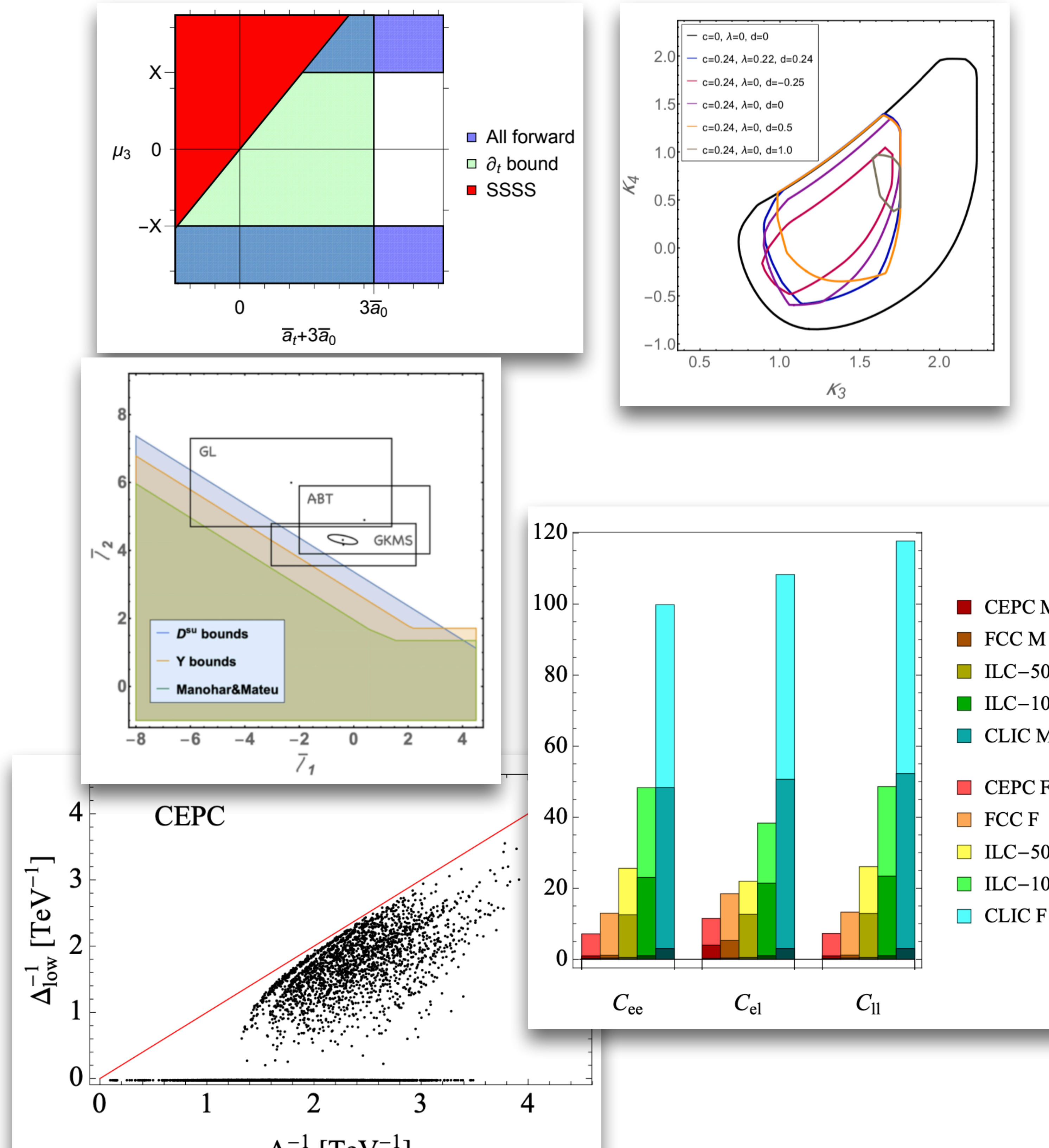
- Chiral PT

  Wang, Feng, Zhang & **SYZ**, 2004.03992
  Tolley, Wang & **SYZ**, 2011.02400

- SMEFT

  Li, Xu, Yang, Zhang & **SYZ**, 2101.01191
  Yamashita, Zhang & **SYZ**, 2009.04490
  Fuks, Liu, Zhang & **SYZ** 2009.02212
  Bi, Zhang & **SYZ**, 1902.08977
  Zhang & **SYZ**, 2005.03047, 1808.00010

+ many works by other authors



Summary

- Positivity bounds from fundamental principles of QFT
- s^2 positivity bounds for **multi-fields** form a convex **cone**.
- **Extreme rays** of the s^2 cone correspond to **UV states**.
- Wilson coeff's are bounded from both sides: $c_i \sim O(1)$

  used to be a folklore, but now is a theorem!
- Scalar theories with **soft amplitudes can not** have standard **UV completion**.
- SMEFT's parameter space is highly constrained.

Backup slides

An infinite number of positivity bounds

Recurrence relation:

de Rham, Melville, Tolley & **SYZ**, arXiv:1702.06134

$$Y^{(2N,M)} = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} + \frac{1}{\mathcal{M}^2} \sum_{k \text{ even}}^{(M-1)/2} (2(N+k)+1)\beta_k Y^{(2(N+k),M-2k-1)} > 0$$

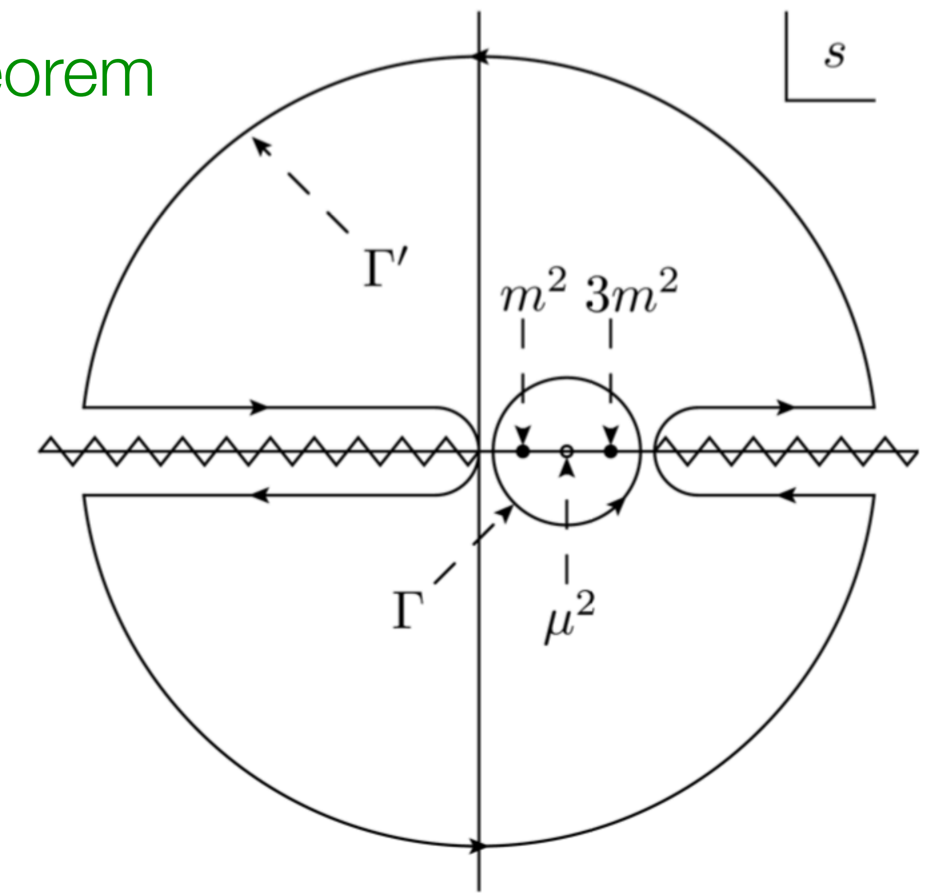
$$B^{(2N,M)}(t) = \frac{1}{M!}\partial_v^{2N}\partial_t^M \tilde{B}(v,t)\Big|_{v=0}$$

$$\text{sech}(x/2) = \sum_{k=0}^{\infty} c_k x^{2k} \quad \text{and} \quad \tan(x/2) = \sum_{k=0}^{\infty} \beta_k x^{2k+1} \qquad \mathcal{M}^2 = (t+4m^2)/2$$

Dispersion relation

$$f := \frac{1}{2\pi i}\oint_{\Gamma} \mathrm{d}s \frac{A(s,0)}{(s-\mu^2)^3}$$

Cauchy's theorem



Frossart bound: as $s \to \infty, |A(s,0)| < C s \ln^2 s$

$$f = \frac{1}{2\pi i}\left(\int_{-\infty}^{0} + \int_{4m^2}^{+\infty}\right) \mathrm{d}s \frac{\mathrm{Disc}\, A(s,0)}{(s-\mu^2)^3}$$

Crossing and $\mathrm{Disc}\, A(s,0) = 2i\, \mathrm{Im} A(s,0)$

$$f = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s \left[\frac{\mathrm{Im} A(s,0)}{(s-\mu^2)^3} + \frac{\mathrm{Im} A^{(u)}(s,0)}{(s+\mu^2-4m^2)^3}\right]$$

Positivity bound

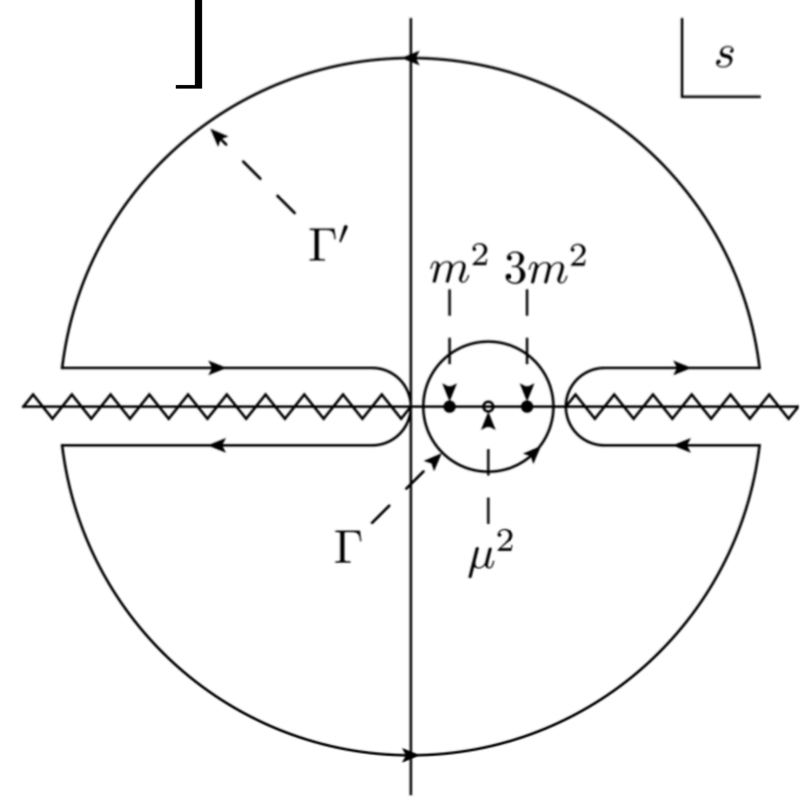
Optical theorem: $\mathrm{Im}[A(s,0)] = \sqrt{s(s-4m^2)}\sigma(s) > 0$

$$f = \frac{1}{\pi}\int_{4m^2}^{\infty} ds \left[\frac{\sqrt{s(s-4m^2)}}{(s-\mu^2)^3}\sigma(s) + \frac{\sqrt{s(s-4m^2)}}{(s+\mu^2-4m^2)^3}\sigma^{(u)}(s) \right]$$

For $s > 4m^2$, $0 < \mu^2 < 4m^2$

$$f > 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006



$$f = \sum_{\Gamma} \mathrm{Res}\left[\frac{A(s,0)}{(s-\mu^2)^3}\right]$$

Calculable within low energy EFT!