Luminosity calculation

These expressions are derived from the first principle

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General expression for luminosity

- The number of events per unit time and per unit volume observed in an arbitrary reference frame

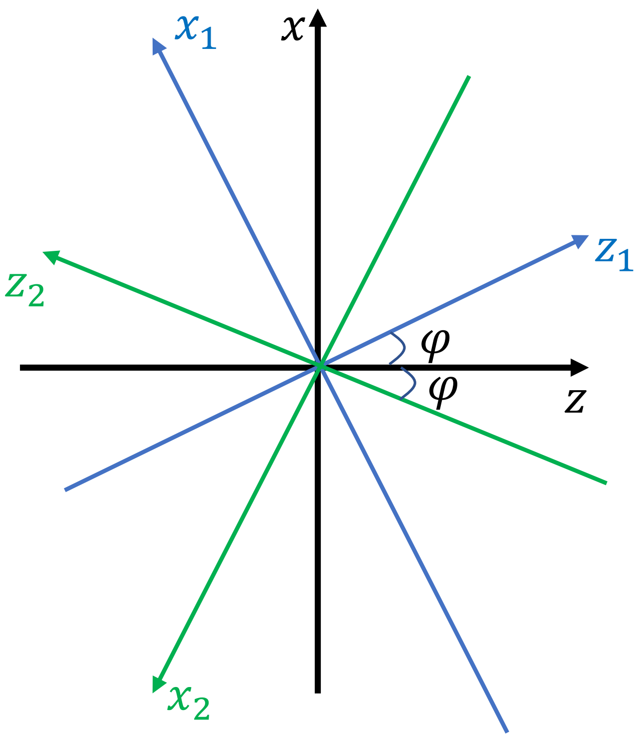

Fig. 1 Geometry of the crossing region

$$\frac{dN}{dtdV} = \sigma n_1 n_2 \cdot \sqrt{(\vec{v_1} - \vec{v_2})^2 - \frac{(\vec{v_1} \times \vec{v_2})^2}{c^2}}$$

σ	Total cross section
$n_{1,}n_2$	Densities of particle 1 and 2 For coasting beam, $n_i = \lambda_i f_i(x_i, y_i, z_i)$ $\lambda_i - line\ density\ of\ the\ beam;$ $\int f_i(x_i, y_i, z_i) dx_i dy_i = 1$ For bunched beam, $n_i = N_i f_i(x_i, y_i, z_i)$ $N_i - number\ of\ particles\ in\ a\ bunch$ $\int f_i(x_i, y_i, z_i, t) dx_i dy_i dz_i = 1$
$\vec{v_1}, \vec{v_2}$	Velocity vectors of particles 1 and 2
c	Densities of light
dV	Volume element of the interaction region

Ref: T. Suzuki, General formulae of luminosity for various of collider beam machines, Report No. KEK-76-83, 1976.

Definition of luminosity

$$\frac{dN}{dtdV} = \sigma n_1 n_2 \cdot \sqrt{(\vec{v_1} - \vec{v_2})^2 - \frac{(\vec{v_1} \times \vec{v_2})^2}{c^2}}$$

$$|\vec{v_1}| = |\vec{v_2}| = \mathrm{c}$$

$$(\vec{v_1} - \vec{v_2})^2 = |\vec{v_1}|^2 + |\vec{v_2}|^2 - 2\vec{v_1} \cdot \vec{v_2}$$

$$= c^2 + c^2 - 2c^2 \cos(\pi - 2\Phi)$$

$$= 2c^2 + 2c^2 \cos 2\Phi$$

$$= 4c^2 \cos^2 \Phi$$

$$\frac{(\vec{v_1} \times \vec{v_2})^2}{c^2} = \frac{[|\vec{v_1}||\vec{v_2}|\sin(\pi - 2\Phi)]^2}{c^2} = c^2 \sin^2(2\Phi)$$

$$= c^2(1 - cos^2(2\Phi))$$

$$= -4c^2 \, cos^4 \Phi + 4c^2 \, cos^2 \Phi$$

$$\frac{dN}{dtdV} = \sigma n_1 n_2 \cdot 2c \cdot cos^2(\Phi)$$

σ	Total cross section
n_1, n_2	Densities of particle 1 and 2
	For coasting beam, $n_i = \lambda_i f_i(x_i, y_i, z_i)$ $\lambda_i - line\ density\ of\ the\ beam;$ $\int f_i(x_i, y_i, z_i) dx_i dy_i = 1$
	For bunched beam, $n_i = N_i f_i(x_i, y_i, z_i)$ $N_i - number\ of\ particles\ in\ a\ bunch$ $\int f_i(x_i, y_i, z_i, t) dx_i dy_i dz_i = 1$
c	Densities of light
2Φ	Crossing angle

- The luminosity is defined as:

$$\frac{dN}{dt} = \sigma \mathrm{L}$$

$$\mathrm{L} = \int n_1 n_2 \cdot 2c \cdot cos^2(\Phi) dV$$

Definition of coordinate system

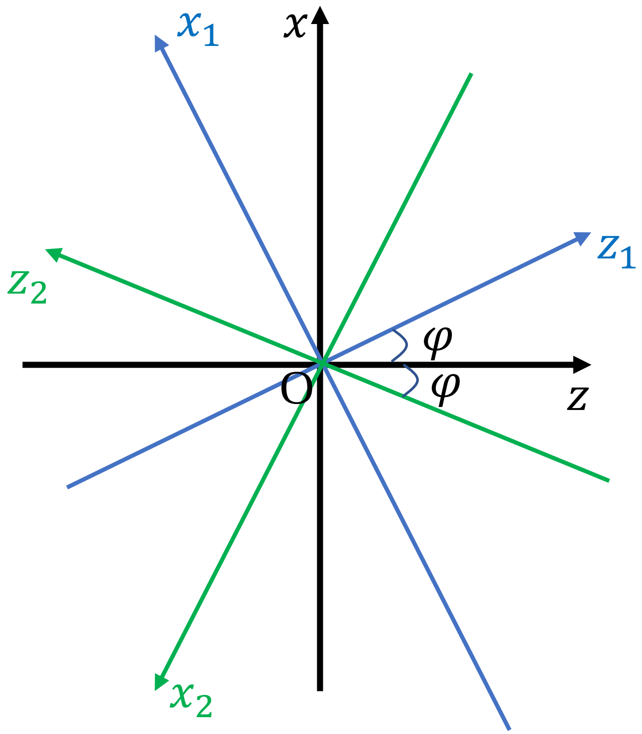

Fig. 1 Geometry of the crossing region

(x_i, y_i, z_i), i = 1,2分别代表着两束流。O为interaction point.

$$\begin{cases} x_1 = x cos\Phi - z sin\Phi \\ y_1 = y \\ z_1 = x sin\Phi + z cos\Phi \end{cases}$$

$$\begin{cases} x_2 = -x cos\Phi - z sin\Phi \\ y_2 = y \\ z_2 = x sin\Phi - z cos\Phi \end{cases}$$

Definition of luminosity

- The luminosity is expressed in the following way according to the types of collisions:

(1) Coasting beam + coasting beam

$$L = \int n_1 n_2 \cdot 2c \cdot \cos(\Phi)\, dV = \lambda_1 \lambda_2 2c \cdot cos^2\,(\Phi) \int f_1(x_1, y_1, z_1) f_2(x_2, y_2, z_2) dxdydz$$

(2) Coasting beam + bunched beam

$$L = \int n_1 n_2 \cdot 2c \cdot \cos(\Phi)\, dV = \int N_1 f_1(x_1, y_1, z_1, t) \lambda_2 f_2(x_2, y_2, z_2) \cdot 2c \cdot cos^2(\Phi) dxdydzdt$$

(3) bunched beam + bunched beam

$$L = \int n_1 n_2 \cdot 2c \cdot \cos(\Phi)\, dV = \int N_1 f_1(x_1, y_1, z_1, t) N_2 f_2(x_2, y_2, z_2, t) \cdot 2c \cdot cos^2(\Phi) dxdydzdt$$

The distribution function of beam

For bunched beam:

$$f_i(x_i, y_i, z_i, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sigma_{xi}\sigma_{yi}\sigma_{zi}} \exp\left[-\frac{1}{2}\left(\frac{x_i^2}{\sigma_{xi}^2} + \frac{y_i^2}{\sigma_{yi}^2} + \frac{(z_i - ct)^2}{\sigma_{zi}^2}\right)\right]$$

For coasting beam:

$$f_i(x_i, y_i, z_i) = \frac{1}{2\pi} \frac{1}{\sigma_{xi}\sigma_{yi}} \exp\left[-\frac{1}{2}\left(\frac{x_i^2}{\sigma_{xi}^2} + \frac{y_i^2}{\sigma_{yi}^2}\right)\right]$$

Calculation of luminosity

$$L = \int n_1 n_2 \cdot 2c \cdot \cos(\Phi)\, dV = \int N_1 f_1(x_1, y_1, z_1, t) \lambda_2 f_2(x_2, y_2, z_2) \cdot 2c \cdot cos^2(\Phi) dxdydzdt$$

- Integrate the parameter t

$$\int_{-\infty}^{+\infty} -\frac{1}{2}\frac{(z_i - ct)^2}{\sigma_{zi}^2} dt = \frac{\sqrt{2\pi}}{\sqrt{\frac{c^2}{\sigma_{zi}^2}}}$$

$$L = 2c \cdot cos^2(\Phi) N_1 \lambda_2 \frac{\sqrt{2\pi}}{\sqrt{\frac{c^2}{\sigma_{zi}^2}}} \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_{xi}\sigma_{yi}\sigma_{zi} 2\pi \sigma_{xi}\sigma_{yi}} \int \exp\left[-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x1}^2} + \frac{y_1^2}{\sigma_{y1}^2} + \frac{x_2^2}{\sigma_{x2}^2} + \frac{y_2^2}{\sigma_{y2}^2}\right)\right] dxdydz \,(11)$$

- Integrate the parameter x,y,z

$$L = \frac{cos\Phi}{sin\Phi} \frac{1}{\sqrt{2\pi}} N_1 \lambda_2 \frac{1}{\sqrt{\sigma_{y1}^2 + \sigma_{y2}^2}}$$

Calculation of luminosity

In[7]:=
```
Clear["Global`*"]
清除
x1 = x*Cos[ϕ] - z*Sin[ϕ];
       余弦          正弦
y1 = y;
z1 = x*Sin[ϕ] + z*Cos[ϕ];
       正弦          余弦
x2 = -x*Cos[ϕ] - z*Sin[ϕ];
        余弦          正弦
y2 = y;
z2 = x*Sin[ϕ] - z*Cos[ϕ];
       正弦          余弦
```

In[∘]:=

$$equ6 = N1*N2*2*c*\mathrm{Cos}[\phi]*\mathrm{Cos}[\phi]*\int_{-Infinity}^{Infinity}\int_{-Infinity}^{Infinity}\int_{-Infinity}^{Infinity}\int_{-Infinity}^{Infinity}\frac{1}{(2\pi)^{\frac{3}{2}}\,\sigma x1*\sigma y1*\sigma z1}\mathrm{Exp}\left[-\frac{1}{2}*\left(\frac{(x1)^2}{\sigma x1^2}+\frac{(y1)^2}{\sigma y1^2}+\frac{(z1-ct)^2}{\sigma z1^2}\right)\right]*\frac{1}{2\pi*\sigma x2*\sigma y2}\mathrm{Exp}\left[-\frac{1}{2}*\left(\frac{(x2)^2}{\sigma x2^2}+\frac{(y2)^2}{\sigma y2^2}\right)\right]dx\,dy\,dz\,dt$$

余弦 余弦 指数形式 指数形式

In[5]:=

$$intege\_t = \int_{-Infinity}^{Infinity}\mathrm{Exp}\left[-\frac{(z-c*t)^2}{2*\sigma z1^2}\right]dt$$

指数形式

Out[5]=

$$\mathrm{ConditionalExpression}\left[\frac{\sqrt{2\pi}}{\sqrt{\frac{c^2}{\sigma z1^2}}}, \mathrm{Re}\left[\frac{c^2}{\sigma z1^2}\right]\geq 0\right]$$

In[14]:=

$$intege\_xyz = N1*N2*2*c*\mathrm{Cos}[\phi]*\mathrm{Cos}[\phi]*\frac{\sqrt{2\pi}}{\sqrt{\frac{c^2}{\sigma z1^2}}}*\frac{1}{(2\pi)^{\frac{3}{2}}\,\sigma x1*\sigma y1*\sigma z1*2\pi*\sigma x2*\sigma y2}\int_{-Infinity}^{Infinity}\int_{-Infinity}^{Infinity}\int_{-Infinity}^{Infinity}\mathrm{Exp}\left[-\frac{1}{2}*\left(\frac{(x1)^2}{\sigma x1^2}+\frac{(y1)^2}{\sigma y1^2}\right)\right]*\mathrm{Exp}\left[-\frac{1}{2}*\left(\frac{(x2)^2}{\sigma x2^2}+\frac{(y2)^2}{\sigma y2^2}\right)\right]dx\,dy\,dz$$

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••• Integrate: 警告：一个或者更多的假设的计算结果为 False.

Out[14]=

$$\mathrm{ConditionalExpression}\left[\frac{c\,N1\,N2\,\mathrm{Cos}[\phi]^2}{\sqrt{2\pi}\,\sigma x1\,\sigma x2\,\sigma y1\sqrt{\frac{1}{\sigma y1^2}+\frac{1}{\sigma y2^2}}\,\sigma y2\sqrt{\frac{c^2}{\sigma z1^2}}\,\sigma z1\sqrt{\left(\frac{1}{\sigma x1^2}+\frac{1}{\sigma x2^2}\right)\mathrm{Cos}[\phi]^2}\sqrt{\frac{\mathrm{Sin}[\phi]^2}{\sigma x1^2+\sigma x2^2}}}, \mathrm{Re}\left[\frac{\mathrm{Sin}[\phi]^2}{\sigma x1^2+\sigma x2^2}\right]>0\right]$$

$$luminosity = \frac{N1\,N2\,\mathrm{Cos}[\phi]}{\sqrt{2\pi}\sqrt{\sigma y1^2+\sigma y2^2}\;\mathrm{Sin}[\phi]}$$

Calculation of luminosity

- For electron bunch and CW laser,

碰撞一次的亮度是:

$$L = \frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}N_1\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

N_1: *number particles in a bunch*
$\lambda_2 = \frac{P_L\lambda}{hc^2}$

- For electron bunch and CW laser,

1s亮度是:

$$L = BF\frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}N_1\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

$$= \frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}\frac{I_e}{e}\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

BF — Number of collisions per time (B—bunch number; F---revolution frequency)

Calculation of luminosity

Electron parameter(Z-pole)	
Beam energy	E = 45.5 GeV
Bunch current	461mA
electrons number/bunch	8×10^{10}
Bunch number	12000
Bunch length σ_z	8.5mm (28ps)
Laser-electron IP β function	$\beta_x = 16.6895[m]$
	$\beta_y = 39.539[m]$
Laser-electron IP Beam size	$\sigma_x = 0.0543$ [mm]
	$\sigma_y = 0.0079$ [mm]

laser parameter	
Operated on continuous wave mode	
Average power	5 [W]
wavelength	1064[nm]
Waist size	$\sigma_0 = 300$ [μm]
Rayleigh length	$z_R = \frac{\pi\sigma_0^2}{\lambda} = 26.5$ [cm]

- Per collision:

$$Ł = \frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}N_1\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

- Per second:

$$Ł = BF\frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}N_1\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}} = \frac{cos\Phi}{sin\Phi}\frac{1}{\sqrt{2\pi}}\frac{I_e}{e}\lambda_2\frac{1}{\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

$N = \sigma Ł$

$\sigma = 393$mb

- Per collision: $N \approx 0.32$
- Per second: $N \approx 1.17 \times 10^7$

Calculation of luminosity

- The luminosity for pulsed laser

Parameters	meaning	value
	Nd:YAG laser operation mode: pulsed	
λ	Wavelength	1064nm
Pulsed repetition frequency	10 laser pulses are emitted in one second.	1Hz
P_L	Peak power = Laser energy / pulsed width	0.1GW
E_{laser}	Laser energy	2.8mJ
Pulsed width	duration of one laser pulse per shot or the duration of one laser pulse	28ps
σ_γ	Rms beam size	$\sigma_\gamma = 100\mu m$

For a pulsed laser, the γe luminosity is given by:

$$\mathcal{L} = N_e N_\gamma f \frac{\cos(\alpha/2)}{2\pi} \frac{1}{\sqrt{(\sigma_{e,y}^2 + \sigma_{\gamma,y}^2)}\sqrt{(\sigma_{\gamma,x}^2 + \sigma_{e,x}^2)cos^2\left(\frac{\alpha}{2}\right) + (\sigma_{\gamma,z}^2 + \sigma_{e,z}^2)sin^2\left(\frac{\alpha}{2}\right)}}$$

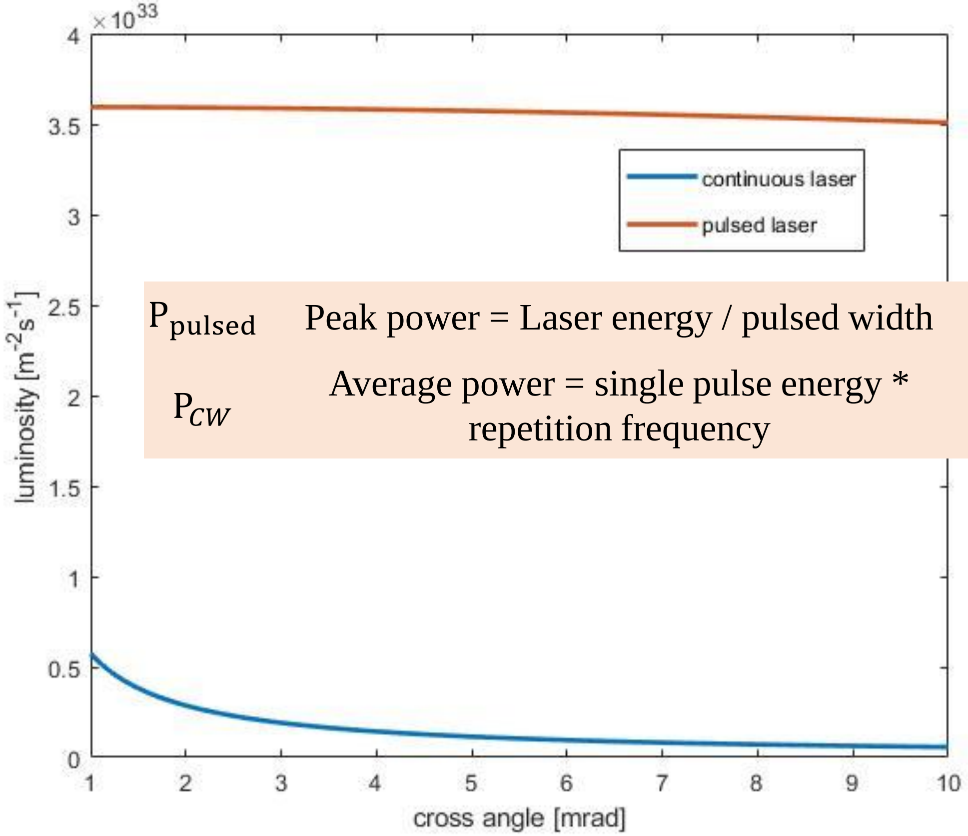
N_e — number of electrons per bunch， N_γ — number of photons per laser pulse
f — number of bunch crossing per second, α — cross angle of laser and electron (2.35mrad)
σ_e and σ_γ is the horizontal size of electron and laser

Compare Luminosity

➢ Compare the continuous wave(CW) laser and pulsed laser

$$\mathcal{L}_{pulse} = N_e N_\gamma f \frac{1+\cos\alpha}{2\pi} \frac{1}{\sqrt{\left(\sigma_{e,y}^2+\sigma_{\gamma,y}^2\right)}\sqrt{\left(\sigma_{\gamma,x}^2+\sigma_{e,x}^2\right)(1+\cos\alpha)^2+\left(\sigma_{\gamma,z}^2+\sigma_{e,z}^2\right)sin^2(\alpha)}}$$

$$\mathcal{L}_{CW} = \frac{(1+cos\alpha)}{\sqrt{2\pi}} \frac{I_e}{e} \frac{P_L\lambda}{hc^2} \frac{1}{\sqrt{\sigma_{e,y}^2+\sigma_{\gamma,y}^2}} \frac{1}{sin\alpha}$$



P_{pulsed}	Peak power = Laser energy / pulsed width	0.1GW
P_{CW}	Average power = single pulse energy * repetition frequency	2.8mW

Compare Luminosity with the CW laser and pulsed laser:

- Peak Power: For CW laser, the average power is relative low. A pulsed laser has high peak power that require more protection (for mirror system or coating process).
- For the beam disturbance: scattered events per collision for CW laser is less than for pulsed laser, which corresponding to the relative large beam disturbance.
- For timing system: The requirement of timing of the laser pulse and electron bunch is high for pulsed laser, but for CW laser don't need to consider.

Back-up