Spin Hydrodynamics: Analytic Solutions, Causality, and Stability

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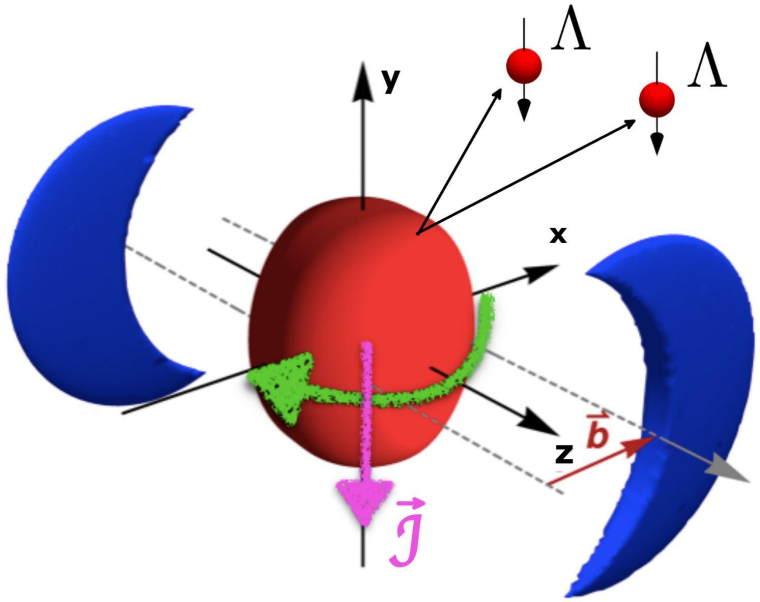
Outline

- **Introduction**
- **Analytic solutions of first order spin hydrodynamics**
  - Solutions in Bjorken expansion
  - Solutions in Gubser expansion
- **Causality and stability analysis**
  - First order spin hydrodynamics
  - Minimal causal (second order) spin hydrodynamics
- **Summary**

Introduction

Rotation and Polarization



picture from W. Florkowski, A. Kumar, and R. Ryblewski, Prog. Part. Nucl. Phys. 108, 103709 (2019)

- **Large angular momentum in noncentral heavy ion collisions**

$$J \sim \frac{A\sqrt{s}}{2} b \sim 10^5 \hbar$$

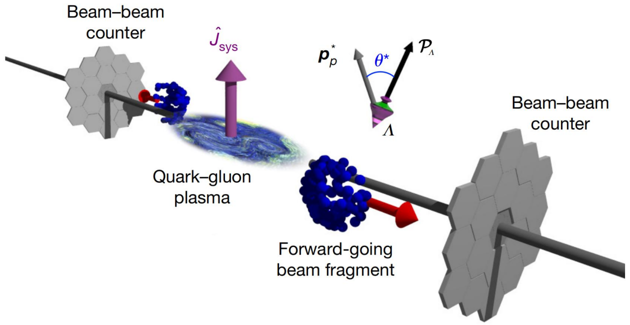
Vorticity of quark gluon plasma

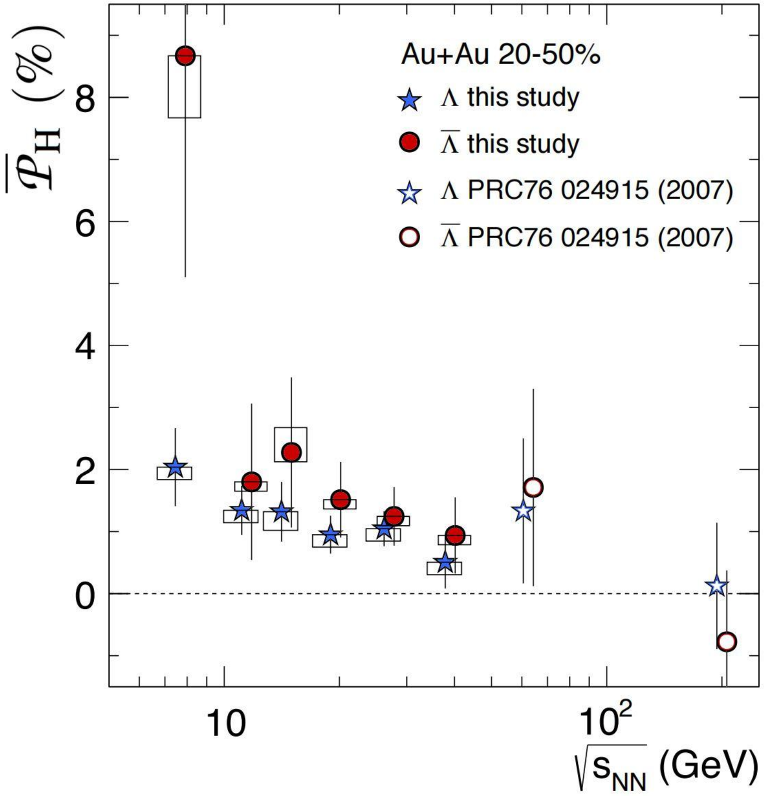
↓ **(Total angular momentum conservation)**

Polarization of particles along vorticity

Liang and Wang, PRL 94,102301(2005); PLB 629, 20(2005)
Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

Most Vortical Fluid and Spin Polarization





Adamczyk et al. (STAR), Nature 548, 62 (2017)

- **Weak decay:** $\Lambda \to p + \pi^-$
- **Proton distribution:**

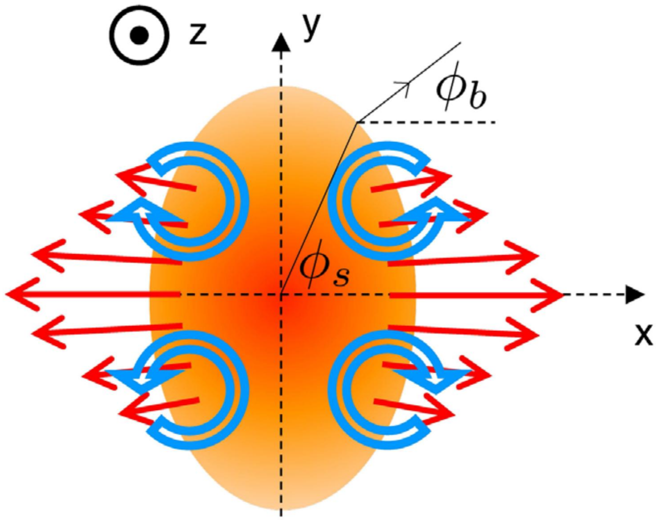
$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}(1 + \alpha_{\rm H}|\vec{\mathcal{P}}_{\rm H}|\cos\theta^*)$$

- **Large vorticity:**

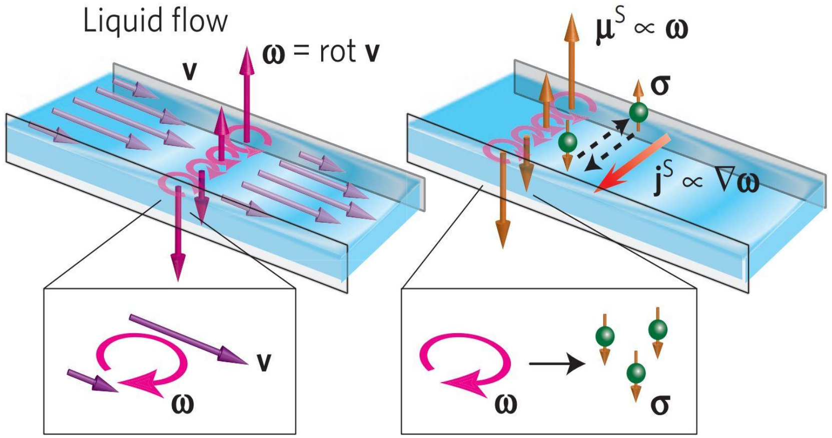
$$\omega \approx (9 \pm 1) \times 10^{21} {\rm s}^{-1}$$

Most vortical fluid ever observed!

Spin-Orbit Coupling in Hydrodynamics



Takahashi R, et al. Nature Physics, 2016, 12(1): 52-56.



How to describe the spin-orbit coupling in hydrodynamic level ?

Canonical Spin Hydrodynamics

Modified thermodynamic relations:

$$\boxed{\begin{aligned} e+p &= Ts+\mu n+\underline{\omega_{\mu\nu}S^{\mu\nu}} \\ de &= Tds+\mu dn+\underline{\omega_{\mu\nu}dS^{\mu\nu}} \end{aligned}}$$

energy density: e pressure: p

temperature: T entropy density: s

spin density: $S^{\mu\nu}$ spin chemical potential: $\omega_{\mu\nu}$

Florkowski, Ryblewsk, Kumar, Prog.Part.Nucl.Phys. (2019)
Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021); JHEP (2022)
Fukushima, Pu, Lecture Note (2020); PLB (2021);
Li, Stephanov, Yee, PRL (2021); **DLW**, Fang, Xie, Pu, PRD (2021); PRD (2022)
Weickgenannt, Wagner, Speranza, Rischke, PRD (2022);
Cao, Hattori, Hongo, Huang, Taya, PTEP (2022)
Hu, PRD (2021); PRC (2023); She, Huang, Hou, Liao, Sci. Bull (2023)
Pu, Huang, Acta Phys.Sin. (2023)

Conservation equations:

$$\boxed{\partial_\mu \Theta^{\mu\nu}=0, \quad \partial_\mu j^\mu=0, \quad \partial_\lambda J^{\lambda\mu\nu}=0}$$

$\Downarrow$ energy-momentum $\Downarrow$ particle number $\Downarrow$ total angular momentum

$$J^{\lambda\mu\nu}=x^\mu\Theta^{\lambda\nu}-x^\nu\Theta^{\lambda\mu}+\Sigma^{\lambda\mu\nu}$$

$$\Sigma^{\lambda\mu\nu}=u^\lambda S^{\mu\nu}+\Sigma_\perp^{\lambda\mu\nu}$$

$$\Downarrow$$

$$\partial_\lambda\Sigma^{\lambda\mu\nu}=-2\Theta^{[\mu\nu]}$$

Convertibility between spin and orbital angular momentum

Constitutive Relations from Entropy Principle

First order constitutive relations:

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa[\Delta^{\mu\nu}\partial_\nu T - T(u\cdot\partial)u^\mu],$$
$$\pi^{\mu\nu} = \eta_s \partial^{<\mu}u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu},$$
$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$
$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - \frac{2}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right),$$

entropy principle $\leftarrow$

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

Entropy current:

$$\mathcal{S}^\mu = su^\mu + \frac{1}{T}h^\mu + \frac{1}{T}q^\mu - \frac{\mu}{T}\nu^\mu$$

$$\Omega^{\mu\nu} \equiv -\Delta^{\mu\rho}\Delta^{\nu\sigma}\partial_{[\rho}(u_{\sigma]}/T) \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

$$\Theta^{\mu\nu} = (e+p)u^\mu u^\nu - pg^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \Theta^{[\mu\nu]},$$
$$J^{\lambda\mu\nu} = x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu},$$
$$j^\mu = nu^\mu + \nu^\mu,$$

$$\Theta^{[\mu\nu]} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}, \quad \Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_\perp^{\lambda\mu\nu}.$$
$$\sim \mathcal{O}(\partial)$$

Halori, Hongo, Huang, Matsuo, Taya, PLB(2019); JHEP(2022);
Fukushima, Pu, PLB(2021); Li, Stephanov, Yee, PRL(2021);

Analytic Solutions of Spin hydrodynamics

- **Spin hydrodynamics with Bjorken expansion (longitudinal expansion)**

  DLW, Fang, Pu, PRD 104, 11404 (2021)
- **Spin hydrodynamics with Gubser expansion (longitudinal & transverse expansion)**

  DLW, Xie, Fang, Pu, PRD 105, 114050 (2022)

Spin Hydrodynamics with Bjorken Expansion

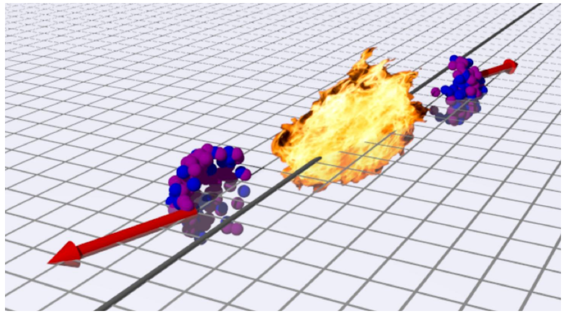
$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

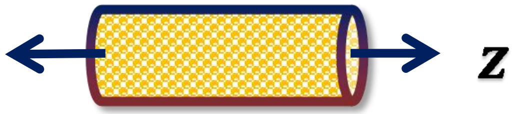
$$e = 3p$$
$$S^{\mu\nu} = aT^2\omega^{\mu\nu}$$
$$n \sim 0$$

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

J. D. Bjorken, Phys. Rev. D 27, 140(1983)

picture from F. Becattini, M. A. Lisa, ARNPS 70 (2020)





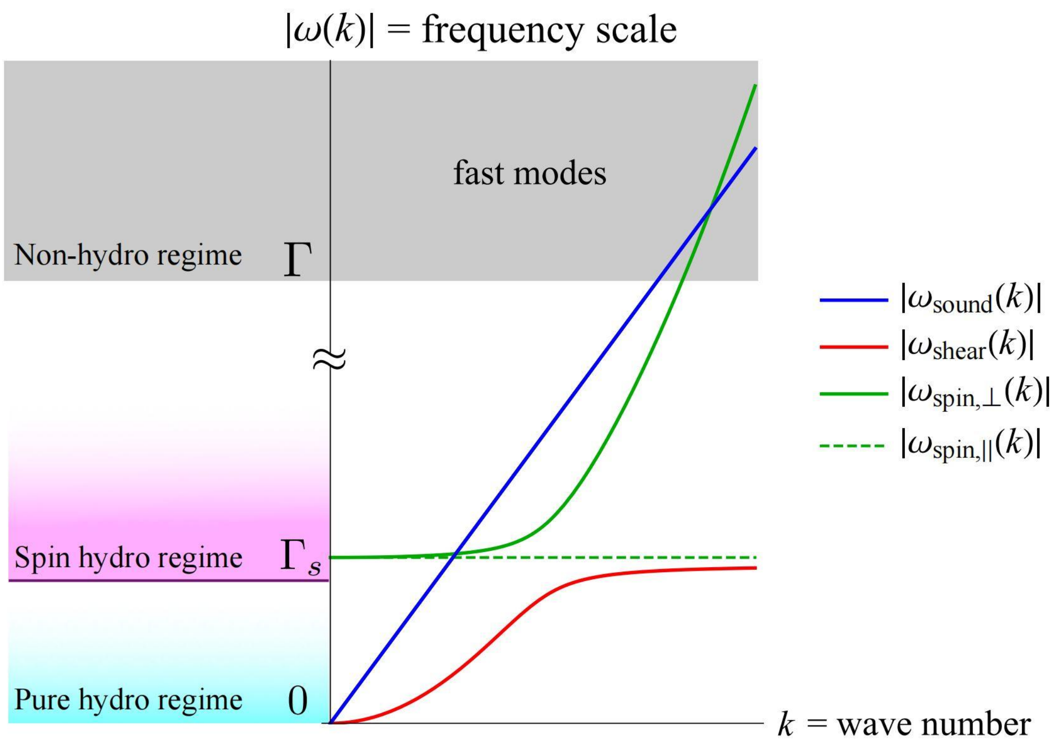
✓ **The hydrodynamic equations in a Bjorken flow are self-consistent if and only if**

$$\boxed{S^{\mu\nu} = 0 \text{ for } (\mu\nu) \neq (xy)}$$

✓ **Bjorken flow in Spin hydrodynamics**

DLW, Fang, Pu, PRD 104, 11404 (2021)

Analytic Solutions in Bjorken Expansion



picture from Hongo, Huang, Kaminski, Stephanov, Yee, JHEP(2021)

non-hydro-variables ?

$$S^{xy} \propto \tau^{-1} \exp\left[-\tilde{\alpha}(\tau^2/\tau_0^2 - 1)\right]$$

$$\omega^{xy} \propto \tau^{-1/3} \exp\left[-\tilde{\alpha}(\tau^2/\tau_0^2 - 1)\right]$$

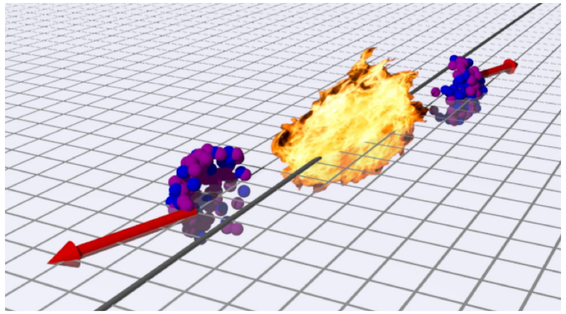
$$\begin{cases} \tilde{\alpha} = \dfrac{2\gamma\tau_0}{aT_0^3} \ll 1 & \Rightarrow \textbf{hydro.} \\[2ex] \tilde{\alpha} = \dfrac{2\gamma\tau_0}{aT_0^3} \gtrsim 1 & \Rightarrow \textbf{non-hydro.} \end{cases}$$

DLW, Fang, Pu, PRD 104, 11404 (2021)

Spin Hydrodynamics with Gubser Expansion

- **Equations are changed under Weyl rescaling**

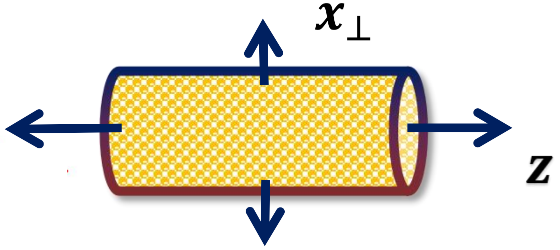
picture from F. Becattini, M. A. Lisa, ARNPS 70 (2020)



$$\mathbb{R}^{3,1} \quad \xleftrightarrow{\text{Weyl rescaling}} \quad dS_3 \times \mathbb{R}$$

$$\hat{A}^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n}(x) = \tau^{\Delta_A} A^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n}(x)$$

$$\Delta_A = [A] + m - n$$



$$\begin{aligned} \nabla_\mu \Theta^{\mu\nu} &= 0 \\ \nabla_\lambda (u^\lambda S^{\mu\nu}) &= -2\Theta^{[\mu\nu]} \end{aligned} \quad \xleftrightarrow{\text{Weyl rescaling}} \quad \begin{aligned} \hat{\nabla}_\mu \hat{\Theta}^{\mu\nu} &= 2\tau^{-1}\hat{\Theta}^{[\mu\nu]}\hat{\nabla}_\mu \tau \\ \hat{\nabla}_\lambda(\hat{u}^\lambda \hat{S}^{\mu\nu}) &= -2\hat{\Theta}^{[\mu\nu]} \\ &\quad + f^{\beta\mu\nu}\tau^{-1}\hat{\nabla}_\beta \tau \end{aligned}$$

$$f^{\beta\mu\nu} \equiv \hat{u}_\alpha \hat{S}^{\alpha\nu}\hat{g}^{\mu\beta} + \hat{u}_\alpha \hat{S}^{\mu\alpha}\hat{g}^{\nu\beta} + \hat{u}^\mu \hat{S}^{\nu\beta} - \hat{u}^\nu \hat{S}^{\mu\beta}$$

$$u_\mu = \left(-\frac{1}{\cosh\rho}\frac{L^2+\tau^2+x_\perp^2}{2L\tau}, \frac{1}{\cosh\rho}\frac{x_\perp}{L}, 0, 0\right)$$

$$\sinh\rho = -\frac{L^2-\tau^2+x_\perp^2}{2L\tau}$$

Gubser symmetry is broken !

DLW, Xie, Fang, Pu, PRD 105, 114050 (2022)

Gubser, PRD 82, 085027 (2010)
Gubser, Yarom, NPB 846, 469 (2011)

Analytic Solutions in Gubser Expansion

No exponential decay factor!

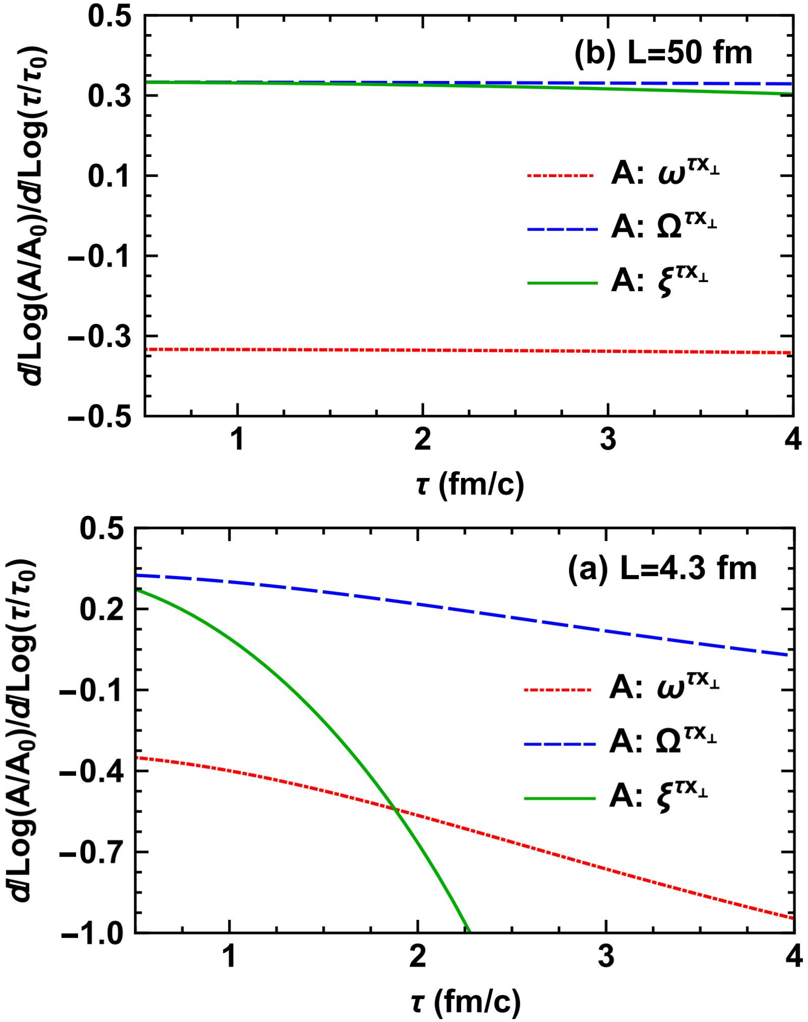
$$e \propto \tau^{-4/3} \quad T \propto \tau^{-1/3}$$

$$S^{\tau x_\perp} \propto L^{-2}\tau^{-1} \quad \omega^{\tau x_\perp} \propto L^{-2}\tau^{-1/3}$$

⬇

Spin density and spin chemical potential are hydrodynamic variables !

DLW, Xie, Fang, Pu, PRD 105, 114050 (2022)



thermal vorticity

thermal shear tensor

$$\Omega^{\mu\nu} \equiv \frac{1}{2}\nabla^\nu(u^\mu/T) - \frac{1}{2}\nabla^\mu(u^\nu/T)$$

$$\xi^{\mu\nu} \equiv \frac{1}{2}\nabla^\mu(u^\nu/T) + \frac{1}{2}\nabla^\nu(u^\mu/T)$$

Causality and Stability Analysis

- **First order spin hydrodynamics**
- **Minimal causal (second order) spin hydrodynamics**

  Xie, **DLW**, Yang, Pu, arXiv: 2306.13880

Acausal mode in First Order Spin Hydrodynamics

- **Independent small perturbations on top of equilibrium satisfy**

$$\partial_\mu \delta\Theta^{\mu\nu} = 0, \quad \partial_\lambda \delta J^{\lambda\mu\nu} = 0, \quad \partial_\mu \delta j^\mu = 0.$$

- **Nontrivial solutions exist iff the corresponding determinant vanishes**

$$\begin{vmatrix} -i\omega + (\gamma_\perp + \gamma')k^2 & iD_s k \\ -2i\gamma' k & -i\omega + 2D_s \end{vmatrix}^2 = 0$$

(B.2) of Phys.Lett.B 795 (2019) by Hattori, Hongo, Huang, Matsuo, Taya

- **One mode behaves as**

$$\boxed{\omega \propto k^2 \quad \text{when } k \to \infty}$$

acausal mode !

Why Is It an Acausal Mode ?

- **Suppose that the perturbations have compact support at t = 0.**

- **If propagation speed is finite, the perturbations have compact support at any time.**

$$\delta\varphi(t;\vec{k}) \equiv \int_{\Omega(t)} d\vec{x} e^{i\vec{k}\cdot\vec{x}} \delta\varphi(t,\vec{x}) \quad e^{i\omega t} \sim \left\|\delta\varphi(t;\vec{k})\right\| \leq C(t)\cdot e^{\mathrm{Im}(\vec{k}\cdot\vec{a})} \qquad a \in \Omega(t)$$

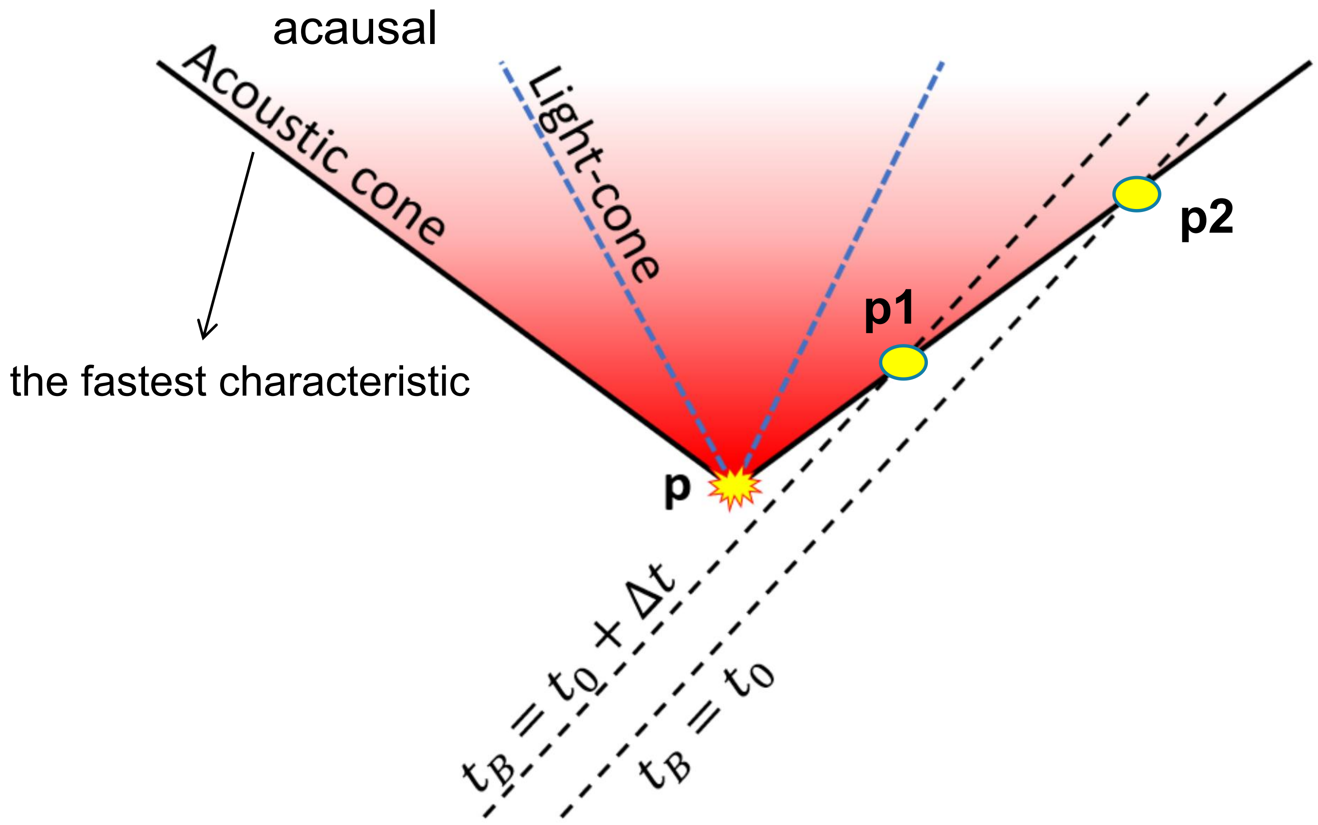
perturbations: $\delta\varphi(t,\vec{x})$ compact support: $\Omega(t)$

Finite propagation speed $\Rightarrow$ $\lim_{k\to\infty}\left|\frac{\omega}{k}\right|$ **is bounded**

E. Krotscheck, W. Kundt, Communications in Mathematical Physics 60, 171 (1978)
P. D. Lax, Hyperbolic Partial Differential Equations (2006)

Therefore, $\omega \propto k^2$ is an acausal mode,
even if it is a pure imaginary (damping) mode.

Acausality Implies Instability



picture from Gavassino, PRX (2022)

- **Damped perturbation**

$$\delta\varphi(p) > \delta\varphi(p_1) > \delta\varphi(p_2)$$

- **In Alice's frame** $t_A(p) < t_A(p_1) < t_A(p_2)$
- **In Bob's frame** $t_B(p) > t_B(p_1) > t_B(p_2)$
- **Instability: the perturbation is growing in Bob's frame**

First order spin hydrodynamics is acausal and unstable !

Minimal Causal Spin Hydrodynamics

- **Introduce nonzero relaxation times:**

$$
\begin{aligned}
\tau_q \Delta^{\mu\nu} \frac{d}{d\tau} q_\nu + q^\mu &= \lambda (T^{-1} \Delta^{\mu\alpha} \partial_\alpha T + Du^\mu - 4\omega^{\mu\nu} u_\nu), \\
\tau_\phi \Delta^{\mu\alpha} \Delta^{\nu\beta} \frac{d}{d\tau} \phi_{\alpha\beta} + \phi^{\mu\nu} &= 2\gamma_s \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_{[\alpha} u_{\beta]} + 2\omega_{\alpha\beta}), \\
\tau_\pi \Delta^{\alpha<\mu} \Delta^{\nu>\beta} \frac{d}{d\tau} \pi_{\alpha\beta} + \pi^{\mu\nu} &= 2\eta \partial^{<\mu} u^{\nu>}, \\
\tau_\Pi \frac{d}{d\tau} \Pi + \Pi &= -\zeta \partial_\mu u^\mu,
\end{aligned}
$$

Liu, Huang, Nucl. Sci. Tech. (2020)
Biswas, Daher, Das, Florkowski, Ryblewski, arXiv:2304.01009
Xie, **DLW**, Yang, Pu, arXiv:2306.13880

- **The thermodynamic flux cannot instantaneously vanish/appear when thermodynamic force is suddenly switched off/on.**

Israel, Stewart, Annals Phys. (1979); Muronga, PRC (2004); Koide, Denicol, Mota, Kodama, PRC (2007)

Complete second order spin hydrodynamics: Asaad Daher's talk on July 17th

Causality Analysis

By linear mode analysis, we study the causality of the minimal causal spin hydrodynamics.

- **The modes $\omega \propto k^2$ with infinite propagation speed disappear.**
- **As $k \longrightarrow \infty$, all modes behave as (finite propagation speed)**

$$\omega = Ck + \mathcal{O}(k^0) \ \text{ or } \ \omega \ll k$$

C: the functions of transport coefficients and relaxation times

- **Under certain conditions, the minimal causal spin hydrodynamics is causal.**

$$\lim_{k\to\infty} \left| \mathrm{Re}\, \frac{\omega}{k} \right| \leq 1$$

Xie, **DLW**, Yang, Pu, arXiv: 2306.13880

Stability and Equations of State

- **Stability conditions:**

Hattori, Hongo, Huang, Matsuo, Taya, PLB(2019); Sarwar, Hasanujjaman, Bhatt, Mishra, Alam, PRD (2023); Daher, Das, Ryblewski, PRD (2023); Xie, **DLW**, Yang, Pu, arXiv:2306.13880

$$\delta\varphi \sim e^{i\omega t - i\vec{k}\cdot\vec{x}} \qquad \mathrm{Im}\ \omega(k) > 0 \Longrightarrow$$

Constraints for transport coefficients and relaxation times

$$D_s > 0,\ D_b < 0$$

Spin susceptibility

$$D_b = 4\gamma_s \frac{\delta\omega^{0i}}{\delta S^{0i}}$$

$$D_s = 4\gamma_s \frac{\delta\omega^{ij}}{\delta S^{ij}}$$

- **Equations of state in linear region (isotropic background):**

Xie, **DLW**, Yang, Pu, arXiv:2306.13880

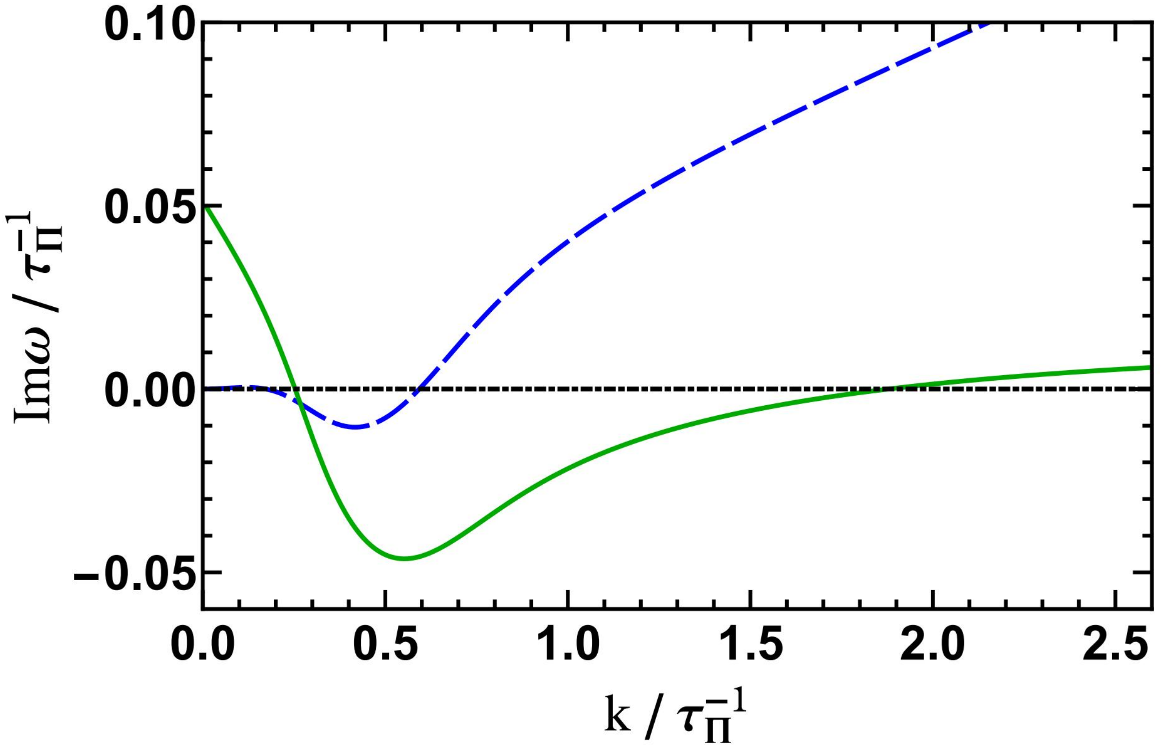
$$\delta\omega^{\mu\nu} = \chi_1 \delta S^{\mu\nu} + \chi_2 \Delta^{\mu\alpha}\Delta^{\nu\beta}\delta S_{\alpha\beta}$$

$$\Downarrow$$

$$D_s > 0, D_b < 0 \quad \text{if} \quad \chi_2 > -\chi_1 > 0$$

Puzzle in Stability Conditions

Stability conditions obtained in small and large k limits are necessary but not sufficient !



$$\delta\varphi \sim e^{i\omega t - i\vec{k}\cdot\vec{x}} \qquad \mathrm{Im}\ \omega(k) > 0$$

- **stable for $k \to 0$ and $k \to \infty$**
- **unstable for some finite k**

Summary

Summary

- The analytic solutions to canonical spin hydrodynamics imply that the intrinsic spin of fluid cell can be hydrodynamic variable under certain conditions.
- The canonical first order spin hydrodynamics is acausal and unstable.
- The minimal causal (second order) spin hydrodynamics can be causal.
- The stability for the minimal causal spin hydrodynamics depends on the EoS.
- Causality and Stability in rotating background?

Thank you !

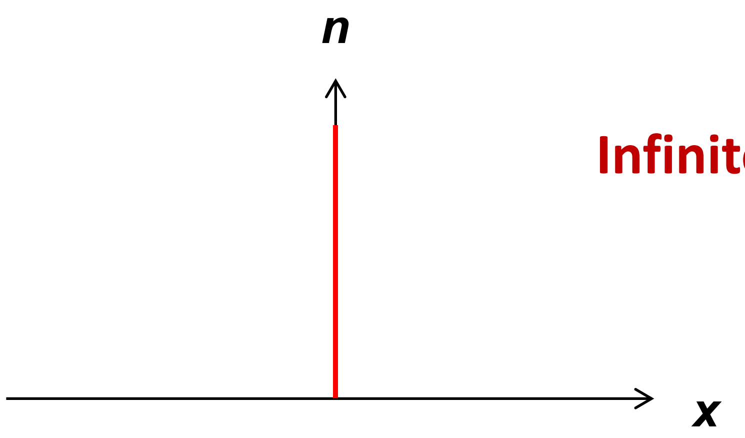
Back up

Infinite Propagation Speed

Diffusion equation: $\partial_t n - D_n \partial_x^2 n = 0$ **Dispersion relation:** $\omega \propto k^2$

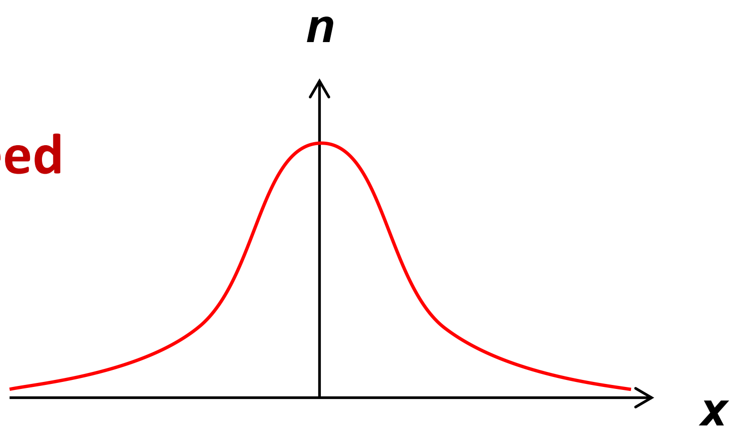
The initial data $n(0, x)$ is zero for $x \neq 0$

For any small time $t > 0$, $n(t, x)$ is nonzero everywhere



$n(t=0, x) = \delta(x)$

Infinite propagation speed

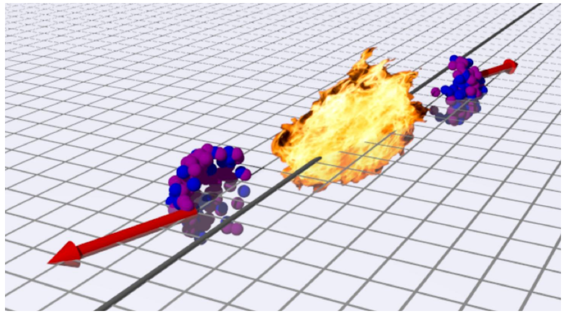


$$n(t>0, x) = \frac{1}{2\sqrt{\pi D_n t}} \exp\left(-\frac{x^2}{4D_n t}\right)$$

Analytic solutions in Gubser Expansion

DLW, Xie, Fang, Pu, PRD 105, 114050 (2022)

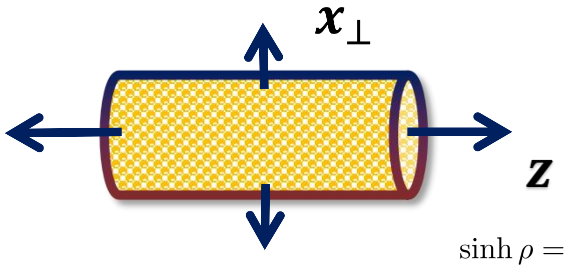
picture from F. Becattini, M. A. Lisa, ARNPS 70 (2020)



- **Nonzero spin components**

$$S^{\tau x_\perp}, S^{\varphi\eta} \neq 0 \qquad (\tau, x_\perp, \varphi, \eta)$$

- **Self-consistent velocity profile in spin hydrodynamics (without Gubser symmetry)**



$$\sinh\rho = -\frac{L^2 - \tau^2 + x_\perp^2}{2L\tau}$$

- **Self-consistent solution**

$$\boxed{\begin{aligned} S^{\tau x_\perp} &= c_1 \frac{4L^2}{\tau} G(L,\tau,x_\perp)^{-1} \\ S^{\varphi\eta} &= c_2 \frac{4L^2}{x_\perp \tau^2} G(L,\tau,x_\perp)^{-1} \underline{A(\rho)} \end{aligned}}$$

$$u_\mu = \left(-\frac{1}{\cosh\rho} \frac{L^2 + \tau^2 + x_\perp^2}{2L\tau}, \frac{1}{\cosh\rho}\frac{x_\perp}{L}, 0, 0 \right)$$

$G(L,\tau,x_\perp) \equiv 4L^2\tau^2 + (L^2 - \tau^2 + x_\perp^2)^2$

$A(\rho)$: exponential decaying factor

Gubser, PRD 82, 085027 (2010)
Gubser, Yarom, NPB 846, 469 (2011)