Sound of rigidly moving fluids: on linear waves in inhomogeneous backgrounds

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Introduction

A dissipative hydro theory must predict that the equilibrium state is stable

https://acrossthemargin.com/skipping-stones/

How to study stability?

- Information current method:
  The **equilibrium** state must have the maximum entropy between the solutions with a shared initial state
  [Hiscock and Lindblom (1983)] - [Olson (1990)] - [Gavassino et al. (2022)]
- Mode stability analysis:
  Plane wave solutions of linearized hydrodynamics equations of motion around an **equlibrium** state may not grow with time
  [Hiscock and Lindblom (1985)]
- Equilibrium state is defined by

$$\overbrace{\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0}^{\text{a Killing vector}} \quad \text{and} \quad \overbrace{\beta \cdot \beta > 0}^{\text{that is timelike}}$$

- Geometry $\rightarrow$ Physics in equilibrium (see, e.g., [Becattini (2016)])

$$u^\mu = \beta^\mu / \sqrt{\beta \cdot \beta} \qquad T = 1/\sqrt{\beta \cdot \beta} \qquad \mathcal{L}_\beta \text{Phys.} = 0$$

In flat spacetime using thermal vorticity, we can categorize equilibrium configurations

- Homogenous configurations $\varpi_{\mu\nu} = 0$: hydrostatic and uniformly moving fluids
- Inhomogenous configurations $\varpi_{\mu\nu} \neq 0$: pure acceleration and rigid rotation (see e.g. [Becattini (2018)])
- To keep β timelike we need to enforce a boundary that introduces a length scale ℓ_{vort}

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* Thermal vorticity $\varpi_{\mu\nu} \equiv -\nabla_{[\mu}\beta_{\nu]} = \frac{2}{T} a_{[\mu}u_{\nu]} + \frac{1}{T}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha}u^{\beta}$
* Hydrostatic (fluid at rest with constant temperature) $\beta = \frac{1}{T_0}\frac{\partial}{\partial t}$
* Uniformly moving fluid with constant temperature $\beta = \frac{1}{T_0}\left(\frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i}\right)$
* Uniformly accelerating fluid $\beta = \frac{1}{T_0}\left[\frac{\partial}{\partial t} + a_0\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right)\right]$
* Rigidly rotating fluid $\beta = \frac{1}{T_0}\left[\frac{\partial}{\partial t} + \Omega_0\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]$

Information current method has pros

- + Doesn't assume a homogenous configuration
- + Is more fundamental in some sense: *proves* that u and T must be related to the thermal Killing vector, leads to some important thermodynamic inequalities ...
- + *Can* be easier to apply
- + Is independent of the equations of motion for dissipative fluxes
- + Recently applied to electromagnetic fields and charged equilibria:
  The electromagnetic part of the information current is stable and causal by construction and, therefore, the stability criteria found for Israel-Stewart theories of hydrodynamics automatically extend to similar formulations of magnetohydrodynamics. **L. Gavassino and MS, to be appeared soon**

. . . and cons

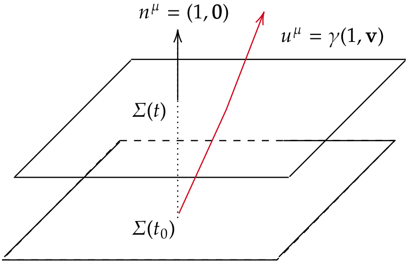
- Neglects the existence of boundaries
- Works only for certain types of theories
- Doesn't tell us much about the nature of the solutions

Linearized equations of hydrodynamics in a homogenous equilibrium configuration have linear wave solutions which reveal the nature of the theory in the linear regime and can be used to investigate linear stability

- We perturb our around a homogenous equilibrium $X_0 \to X_0 + \delta X$ $(X = \varepsilon, u, \ldots)$ with Fourier modes $\delta X(x) \to \delta X(k) \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$
- Insert these into the EOM $\partial_\mu \delta T^{\mu\nu} = \mathcal{O}\big(\delta^2\big)$
- Find the matrix form of the EOM $M^{AB}\delta X^B = 0$
- This has solutions if $\det(M) = 0 \implies$ dispersion relations $\omega = \omega(\mathbf{k})$

Sound waves in a perfect fluid

$$\underbrace{\begin{pmatrix} \omega & -h_{\rm eq}k \\ -\frac{\partial p}{\partial \varepsilon}k & h_{\rm eq}\omega \end{pmatrix}}_{M^{AB}} \underbrace{\begin{pmatrix} \delta\varepsilon(k) \\ \delta u^x(k) \end{pmatrix}}_{\delta X^B} = 0 \qquad \det(M) = 0 \implies \omega^2 - \frac{\partial p}{\partial \varepsilon}k^2 = 0$$



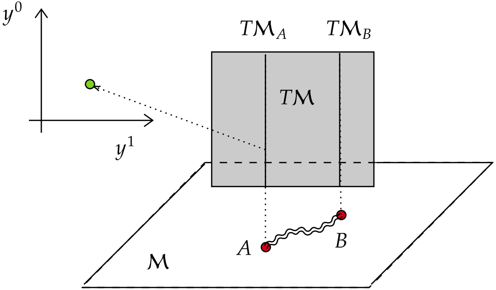
- Dissipative hydrodynamics $\rightarrow$ complex ω
- Linear stability requires $\mathrm{Im}\,\omega \leq 0$ **[Hiscock and Lindblom (1985)]**
- If $\mathrm{Im}\,\omega > 0$ for some domain of $\mathbf{k}$ the norm of δX over subsequent spacelike hypersurfaces grows without a bound

Linear stability analysis in inhomogeneous configurations

- Naive Fourier modes do not work $\omega = \omega(x, \mathbf{k})$
- (Q.1) Can we find linear wave solutions in inhomogeneous configurations?
- (Q.2) How are they related to stability?
- (Q.2.a) ... How do the known stability criteria in homogeneous configurations generalize to inhomogeneous ones?

Hydrodynamics in the tangent bundle

- The idea: plane waves in an infinitesimal neighborhood
- Tangent space $\mathbb{T}_x\mathrm{M}$ as a local infinitesimal homogeneous configuration



Wigner transform extends a tensor to the tangent bundle (Inspired by [Fonarev (1994)])

$$F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x,y) = \left(1 + y^\alpha \nabla_\alpha + \frac{1}{2!} y^\alpha y^\beta \nabla_\alpha \nabla_\beta + \cdots \right) F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x)$$

It knows all the local information about the base tensor

$$F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x) = \int_{\mathbb{T}_x \mathrm{M}} \mathrm{d}^4 y\, \delta^4(y) F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x,y)$$

$$\nabla_\mu F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x) = \int_{\mathbb{T}_x \mathrm{M}} \mathrm{d}^4 y\, \delta^4(y) \partial^y_\mu F^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}(x,y)$$

∇ is the covariant derivative

- (S1.a) We extend the EOM to the whole tangent space

$$\partial^y_\mu \delta T^{\mu\nu}(x,y) = 0$$

- (S1.b) ... and Fourier transform using the cotangent space

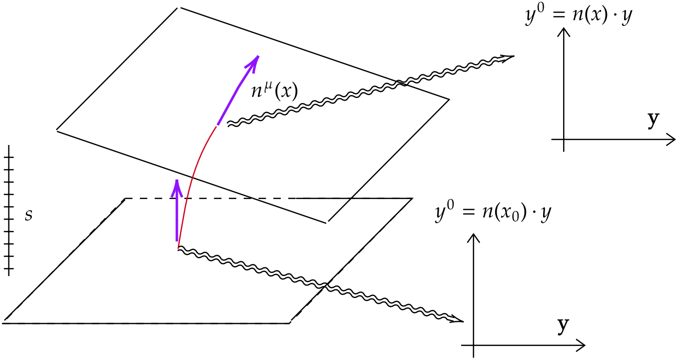
$$\delta T^{\mu\nu}(x,y) = \int_k \delta T^{\mu\nu}(x,k) e^{-ik\cdot y}$$

- ... therefore

$$k_\mu \delta T^{\mu\nu}(x,k) = 0$$

- (S2) We choose a future-directed timelike $n^\mu(x)$ normalized as $n \cdot n = 1$
- ...to find $\omega = n \cdot k$ in terms of $k_\perp$

We will work in the LRF $n^\mu(x) = u^\mu(x)$



What is $\delta T^{\mu\nu}(x,k)$?

- This does not work

$$\delta T^{\mu\nu}(x) \to \text{Decompose w.r.t } u^\mu_{\text{eq}}(x) \to \delta T^{\mu\nu}(x,k)$$

- This works

$$\delta T^{\mu\nu}(x) \to \delta T^{\mu\nu}(x,k) \to \text{Decompose w.r.t } u^\mu_{\text{eq}}(x)$$

- (S3) Decompose $\delta T^{\mu\nu}(x,k)$ with $u^{\mu}_{\rm eq}(x)$

$$\begin{aligned}\delta T^{\mu\nu}(x,k) \;=\;& \delta\mathcal{E}(x,k)u^{\mu}_{\rm eq}(x)u^{\nu}_{\rm eq}(x) - \delta\mathcal{P}(x,k)\Delta^{\mu\nu}_{\rm eq}(x)\\ &+h_{\rm eq}(x)\left[u^{\mu}_{\rm eq}(x)\delta u^{\nu}(x,k) + u^{\nu}_{\rm eq}(x)\delta u^{\mu}(x,k)\right]\\ &+\delta\mathcal{Q}(x)^{\mu}(x,k)u^{\nu}_{\rm eq}(x) + \delta\mathcal{Q}^{\nu}(x,k)u^{\mu}_{\rm eq}(x)\\ &+\delta\pi^{\mu\nu}(x,k)\end{aligned}$$

- Equilibrium quantities are not Wigner transformed

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* We will work in the local rest frame $n^{\mu}(x) = u^{\mu}(x)$
* In our mostly minus metric sign convention $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}_{\rm eq}u^{\nu}_{\rm eq}$
* For example

$$\delta\mathcal{E}(x,k) = u^{\alpha}_{\rm eq}(x)u^{\beta}_{\rm eq}(x)\delta T_{\alpha\beta}(x,k) \qquad \delta\mathcal{P}(x,k) = -\frac{1}{3}\Delta^{\alpha\beta}_{\rm eq}(x)\delta T_{\alpha\beta}(x,k)$$

- The resulting dispersion relations are valid for any fixed background metric

- (S4) Now we can write $k_\mu \delta T^{\mu\nu}(x,k)$ in matrix form and find $\omega_a(x,k)$
- Applying to perfect fluids we find $\omega_\pm(x,\mathbf{k}) = \pm v_s(x)\mathbf{k}$
- But for dissipative fluids we need derivatives of $\delta X(x,k)$
- ...which are found by taking the derivative of the definition
- For example

$$\nabla_\mu \delta\mathcal{E}(x) \to -ik_\mu \delta\mathcal{E}(x,k) - 2T_{\text{eq}}(x)\varpi_{\mu\nu}(x)\delta\tilde{\mathcal{Q}}^\nu(x,k)$$

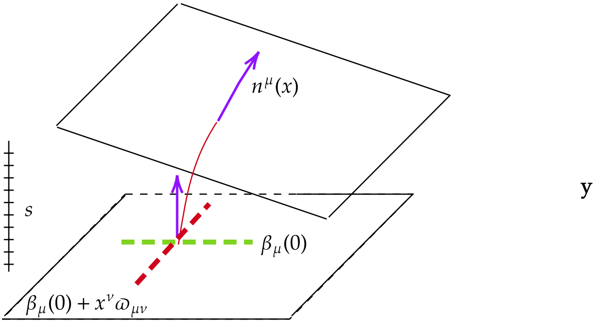
$$\delta\tilde{\mathcal{Q}}^\mu(x,k) = \delta\mathcal{Q}^\mu(x,k) + h_{\text{eq}}(x)\delta u^\mu(x,k)$$

What if $\mathrm{Im}\,\omega > 0$?

- Equilibrium-preserving directions

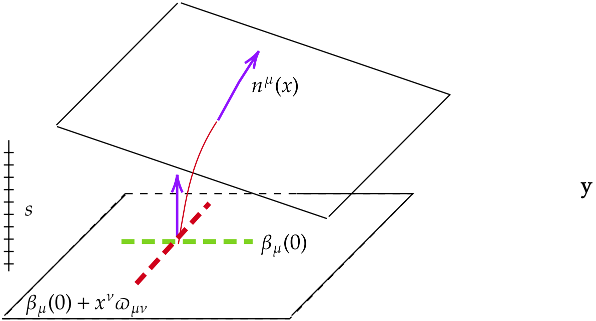
$$\beta_\mu(x, y_\mathrm{e}) = \beta_\mu(x)$$

- ... exist if the spacetime is flat and $\omega_\mu a^\mu = 0$



They are given by $y_{\mathrm{e}}^{\mu}\varpi_{\mu\nu}(x) = 0$

$$\nabla_{\mu}\varpi_{\alpha\beta} = R_{\alpha\beta\mu\sigma}\beta^{\sigma}$$



- (S6) Restrict $\mathbf{k}$ via ($k^\mu \varpi_{\mu\nu}(x) = 0$) to $\mathbf{k}_\mathrm{e}$ in the dispersion relations
- If $\mathrm{Im}\,\omega_a > 0$ in this case
- ... provided that

$$\ell_\mathrm{micro} \ll \ell_\mathrm{vort}$$

- ... instability is proved
- But hydro is applicable if $\ell_\mathrm{micro} \ll \ell_\mathrm{macro} \sim \ell_\mathrm{vort}$
- If $\mathrm{Im}\,\omega_a > 0$ for k in NEQP directions $\rightarrow$ inconclusive

Application to MIS hydrodynamics

- According to the info-current method: **same** stability criteria for homogeneous/accelerating/rotating/non-self-gravitating equilibria [Hiscock and Lindblom (1983)]

Linearized MIS hydrodynamics

$$\delta T^{\mu\nu} = \delta\mathcal{E} u^\mu_{\rm eq} u^\nu_{\rm eq} - \left(v_s^2 \delta\mathcal{E} + \delta\Pi\right) \Delta^{\mu\nu}_{\rm eq} + h_{\rm eq} \left(u^\mu_{\rm eq}\delta u^\nu + u^\nu_{\rm eq}\delta u^\mu\right) + \delta\pi^{\mu\nu}$$

$$\tau_\Pi u_{\rm eq} \cdot \nabla\delta\Pi + \delta\Pi + \zeta\nabla\cdot\delta u = 0$$

$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta\,\rm eq} \left(u_{\rm eq}\cdot\nabla\delta\pi^{\alpha\beta} - 2\delta\pi^\alpha_\lambda \Omega^{\beta\lambda}_{\rm eq}\right) + \delta\pi^{\mu\nu} - 2\eta\,\delta\sigma^{\mu\nu} = 0$$

Recall

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} \qquad \sigma_{\mu\nu} \equiv \Delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha u_\beta$$

MIS hydrodynamics with bulk viscosity alone

- The sound modes are modified in the direction of the **acceleration** $(\alpha \equiv a/T_{\rm eq})$

$$\Omega_{\rm sound} = \pm\sqrt{v_s^2\kappa_t^2 + \alpha^2\mathcal{V}_\zeta^2\kappa_\ell^2} - \alpha\mathcal{V}_\zeta\kappa_\ell + \cdots$$

- Decomposition of k (a generalization of [Brito and Denicol (2020)])

$$k^\mu = T_{\rm eq}\left(\Omega u_{\rm eq}^\mu + \kappa_\ell \ell^\mu + \kappa^\mu\right) \qquad \omega = T_{\rm eq}\Omega \qquad \kappa_\ell = k\cdot\ell$$

* Tetrad of orthonormal vectors $\{u, \ell, \tilde{\kappa}, \chi\}$

$$\ell_\mu = a_\mu/\sqrt{-a\cdot a} \qquad \tilde{\kappa}_\mu = \kappa_\mu/\sqrt{-\kappa\cdot\kappa} \qquad \chi^\mu \equiv \epsilon^{\mu\nu\alpha\beta}u_\nu^{\rm eq}\ell_\alpha\tilde{\kappa}_\beta$$

* Auxiliary parameter

$$\mathcal{V}_\zeta = \left(\frac{3}{2} + \frac{1}{v_s^2}\right)C_\zeta - \left(\frac{1}{3} - v_s^2\right)R_\zeta \qquad R_\zeta = \tau_\Pi T_{\rm eq} \qquad C_\zeta = T_{\rm eq}\zeta/h_{\rm eq}$$

MIS hydrodynamics with bulk viscosity alone

- The nonhdyro mode receives linear contribution $\sim \kappa_\ell$

$$\Omega_{\text{gapped}} = -\frac{i}{R_\zeta} + 2\alpha\mathcal{V}_\zeta\kappa_\ell + \cdots$$

- There is no novel contribution in EQP directions
- The acceleration-induced terms disappear in $\mathbf{k} \to \infty$: standard causality/stability criteria [Pu et al. (2010)]

$$R_\zeta > C_\zeta \; , \qquad \frac{C_\zeta}{R_\zeta} < 1 - v_s^2$$

- But $\operatorname{Im}\omega$ can be positive in ℓ direction if $\alpha > \alpha_c$

Is this physically relevant?

- Assume a cylinder of QGP rotating with $\Omega_0 \sim 10^{22}\mathrm{s}^{-1}$ and $T_0 \sim 200\mathrm{MeV}$
- Then $\alpha \sim 0.01$ while $\alpha_c \sim 0.1$
- The unknown effects of a positive $\mathrm{Im}\,\omega$ don't seem to be physically relevant in the domain of applicability of vanilla MIS

Conformal MIS hydrodynamics

- Modes are modified by **acceleration** and **rotation**
- ...not only in EP directions
- $\mathrm{Im}\,\omega$ becomes positive for some modes if (1) a and/or ω are large enough or (2) we are very close to the causal boundary
- ...not only in EP directions!!
- (1) requires $\alpha > 1 \to \ell_{\mathrm{micro}} \sim$ Maximum size of the system!
- Homogeneous modes are recovered in $k \to \infty$ limit
- In the domain of applicability of MIS hydrodynamics stability requires

$$T\tau_\pi > 2\eta/s > 0$$

- We numerically investigated the full MIS and ended up with similar results

Summary and outlook

- We extended the equations to the tangent bundle to find linear wave solutions in inhomogeneous equilibrium configurations
- This machinery can be consistently applied to hydrodynamics
- Novel modes are found in MIS theory arising from coupling between dissipative fluxes and thermal vorticity
- Such modes are only present in the **long wavelength regime**
- The bulk viscous pressure couples only to the acceleration
- Shear stress tensor couples both to acceleration and kinematic vorticity
- MIS theory **in its domain of validity and far from the boundary** remains linearly stable in purely accelerating and rigidly rotating configurations, with the standard stability and causality conditions.
- In **agreement** with the info-current method

Outlook

- Applications to hydro theories with the explicit presence of thermal vorticity in fluxes (Spin hydrodynamics, hydrodynamic theories with quantum corrections arising from acceleration and rotation, . . .)
- Boundary effects

Backup

Information current

- One defines

$$\phi^\mu = S^\mu + \alpha_\star N^\mu - \beta^\star_\nu T^{\nu\mu}$$

- A common perturbation parameter λ, with $\lambda = 0$ denoting the equilibrium
- (1) In equilibrium

$$\frac{\mathrm{d}\phi^\mu(0)}{\mathrm{d}\lambda} = 0$$

- (2) The information current must be future-directed non-spacelike:

$$E^\mu = -\frac{1}{2}\frac{\mathrm{d}^2\phi^\mu(0)}{\mathrm{d}\lambda^2}$$

Uniform acceleration

- We can add the generator of boost along z-direction (see for example [Becattini (2018)])

$$\beta = \frac{1}{T_0}\left[\frac{\partial}{\partial t} + a_0\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right)\right]$$

- To keep β timelike we need to a enforce boundary $|1 + a_0 z| > |a_0 t|$
- Thermal vorticity scale $\ell_{\text{vort}} \sim a_0^{-1}$
- In Rindler coordinates (τ, x, y, ξ)

$$u^\mu = e^{-a_0\xi}\,(1, \mathbf{0}) \qquad T = e^{-a_0\xi}\,T_0 \qquad a^\mu = a_0 e^{-2a_0\xi}\,(0,0,0,1),$$

$$\tau = \frac{1}{2a_0}\log\left[\frac{1 + a_0\,(z+t)}{1 + a_0\,(z-t)}\right] \qquad \xi = \frac{1}{2a_0}\log\left[(1 + a_0 z)^2 - a_0^2 t^2\right]$$

Rigid rotation

- ...and/or we can the generator of rotation around z-direction (see for example [Palermo et al. (2021)])

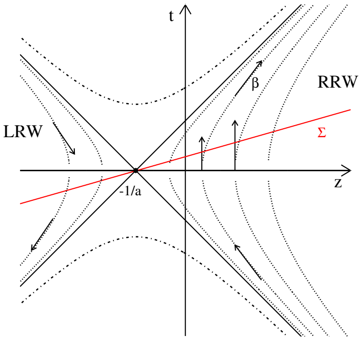
$$\beta = \frac{1}{T_0}\left[\frac{\partial}{\partial t} + \Omega_0\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]$$

- Again, we have a boundary $\Omega_0^2\left(x^2+y^2\right) < 1$
- Thermal vorticity length scale $\ell_{\rm vort} \sim \Omega_0^{-1}$
- In cylindrical coordinates (t, ρ, φ, z)

$$u^\mu = \gamma(\rho)\ (1, 0, \Omega_0, 0)\ , \quad T = \gamma(\rho) T_0 \quad \gamma(\rho) = \frac{1}{\sqrt{1-\rho^2\Omega_0^2}}$$
$$a^\mu = -\gamma^2(\rho)\,\rho\Omega_0^2\,(0,1,0,0)\,, \qquad \omega^\mu = \gamma^2(\rho)\,\Omega_0\,(0,0,0,1)$$

Equilibrium-preserving directions

- In pure accelerating equilibrium T changes in ξ-direction while u^μ changes in τ-direction (Figure from [Becattini (2018)])
- x and y are EP directions
- In the cylindrical rotation z is the only EP directions



Wave equation in the tangent bundle

- Let's assume a toy model (f and m are functions of T_{eq})

$$\left(\Box - \frac{f(x)}{T_{\text{eq}}(x)} u_{\text{eq}}(x) \cdot \partial + m(x)^2\right) \phi(x) = 0$$

- Wave equation in the tangent space

$$\left[\Box_y^2 - f(x)\beta(x) \cdot \partial_y + m^2(x)\right] \phi(x, y) = 0$$

- Characteristic equation at x in the LRF

$$\omega(x, \mathbf{k})^2 - \mathbf{k}^2 - i\frac{f(x)}{T(x)}\omega(x, \mathbf{k}) - m^2(x) = 0$$

- The base solution

$$\phi(x) = \int_k \sum_{a=\pm} \phi_a(x, k)\ \delta(u \cdot k - \omega_a)$$

Frame Title

- The amplitudes fulfill

$$\tilde{\mathcal{D}}_\mu \left[\phi_a(x,k)\delta(u\cdot k-\omega_a)\right] = -ik_\mu\phi_a(x,k)\delta(u\cdot k-\omega_a) + \text{curvature terms.}$$

- Horizontal lift in the cotangent bundle

$$\tilde{\mathcal{D}}_\mu\phi(x,k) = \nabla_\mu\phi(x,k) + \Gamma^\rho_{\mu\sigma}k_\rho\partial^\sigma_k\phi(x,k)$$

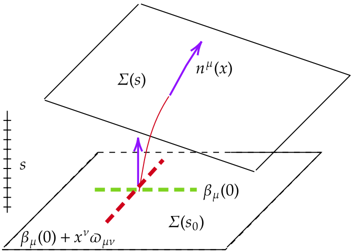
- Separate EQP part

$$\phi(x) = \int \frac{\mathrm{d}^d k_\mathrm{e}}{(2\pi)^d} \sum_{a=\pm} e^{\Gamma_a(x_\mathrm{ne},\mathbf{k}_\mathrm{e}) + i\mathbf{k}_\mathrm{e}\cdot\mathbf{x}_\mathrm{e}} \phi_a(x_\mathrm{ne},\mathbf{k}_\mathrm{e})$$

- Frequencies depend on $\mathbf{k}$ and equilibrium quantities

$$\Gamma_a(x_\mathrm{ne}, k) = -i\int_0^{\mathfrak{s}} \mathrm{d}\mathfrak{s}'\, \omega_a(x_\mathrm{ne}, k)$$

- $f(T) > 0 \implies \Gamma_+(x,\mathbf{k}) > \Lambda\mathfrak{s} > 0$ the norm grows without a bound



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Full decomposition