



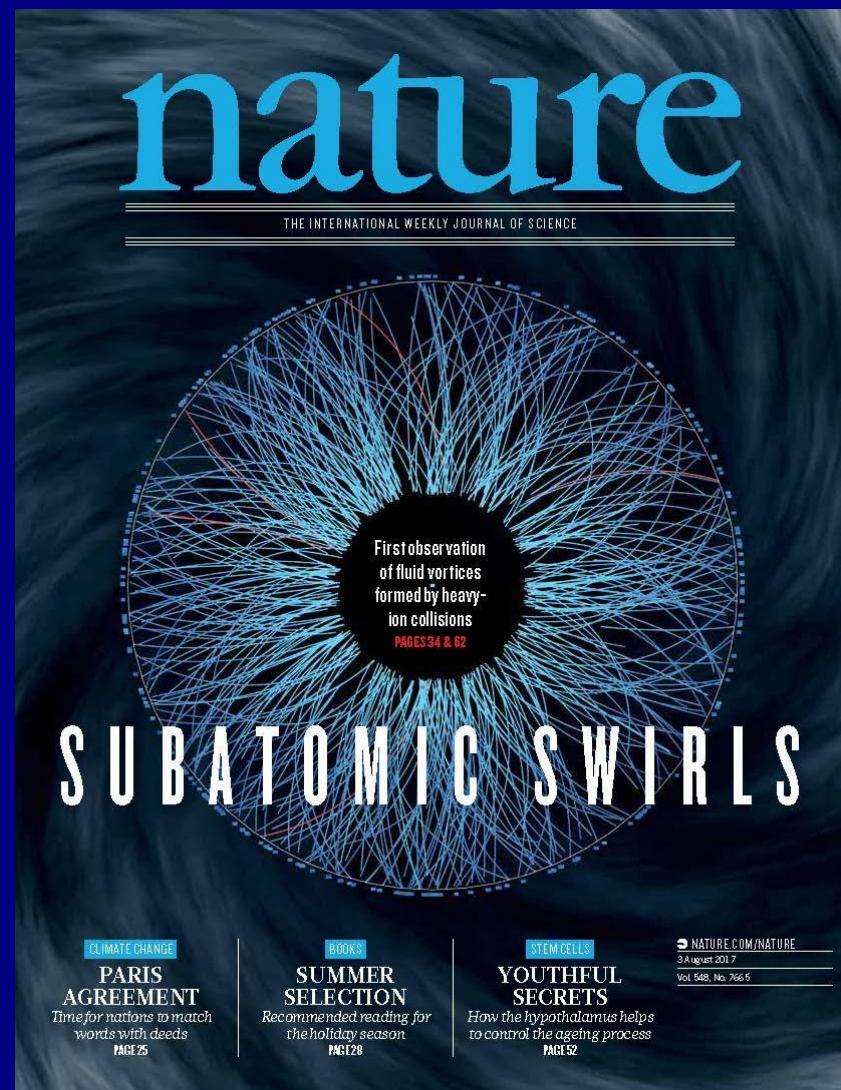
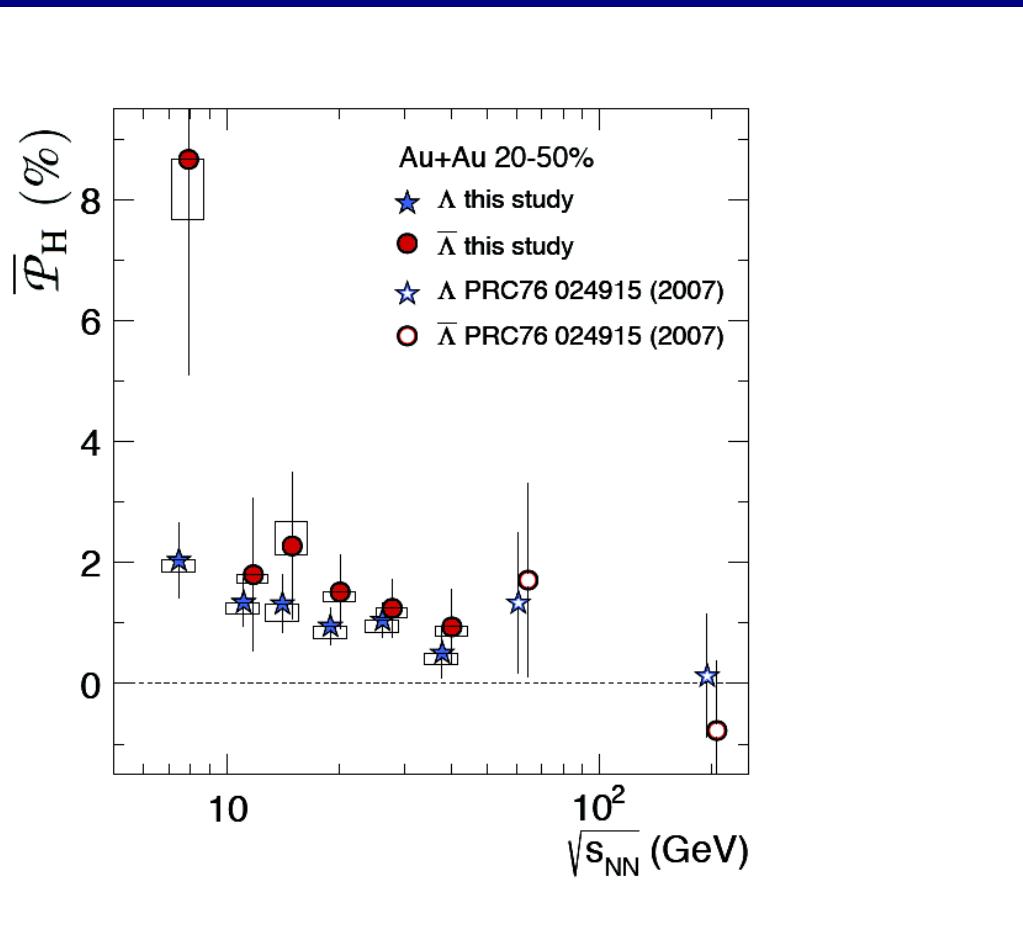
Spin polarization and thermal shear

OUTLINE

- Introduction
- Theory
- Phenomenology
- Conclusions

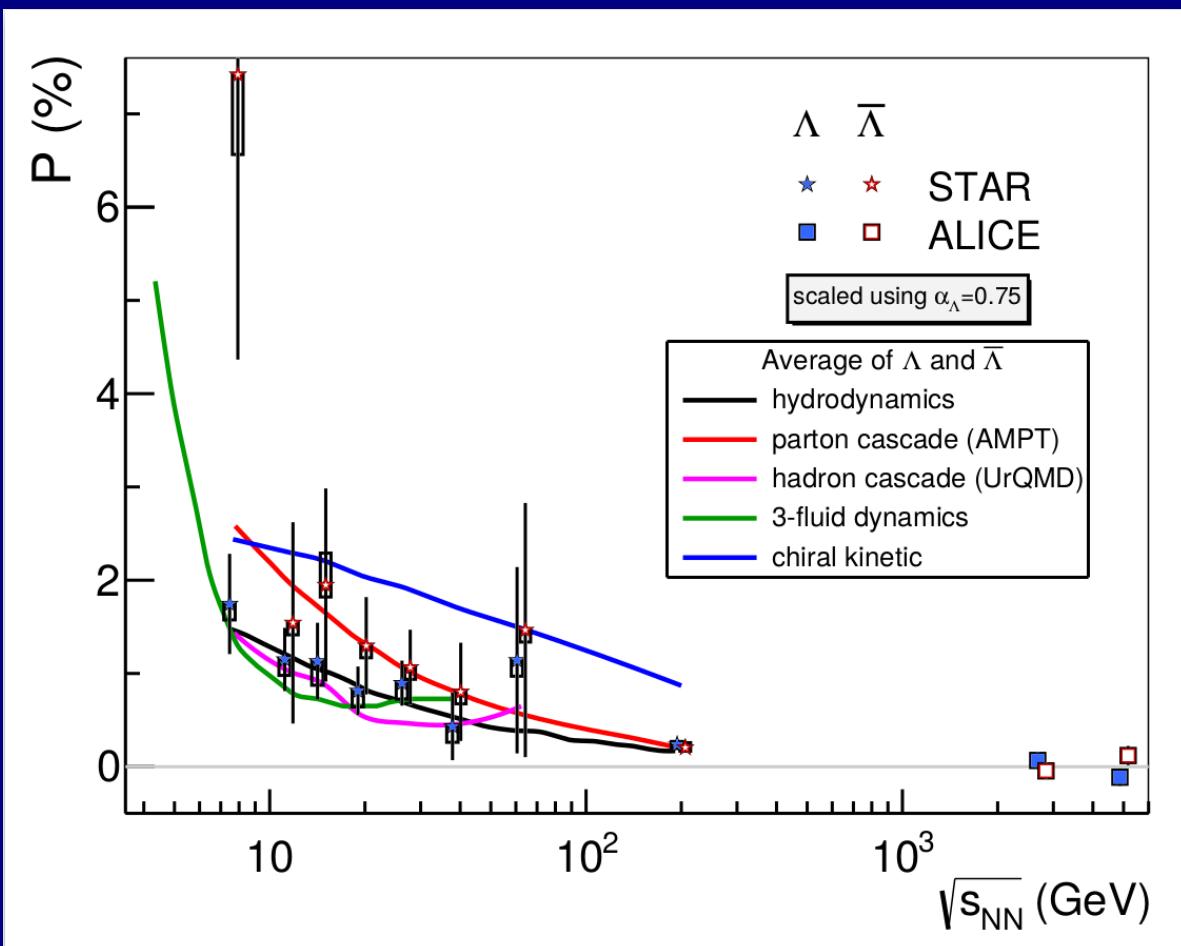
Introduction

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions*, Nature 548 62-65, 2017



Particle and antiparticle have the same polarization sign.
This shows that the phenomenon cannot be driven
by a mean field (such as EM) whose coupling is *C-odd*.
Definitely favours the thermodynamic (equipartition) interpretation

Comparison with the data (date Jan 2020)

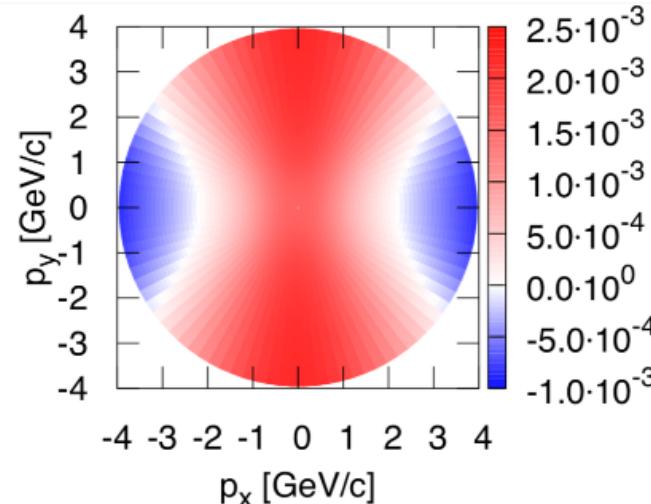
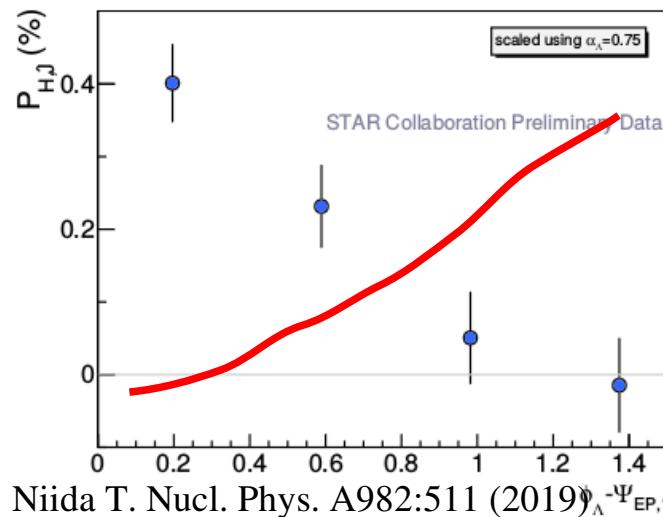


F. B., M. Lisa,
Ann. Rev. Part. Nucl. Sc. 70, 395 (2020)

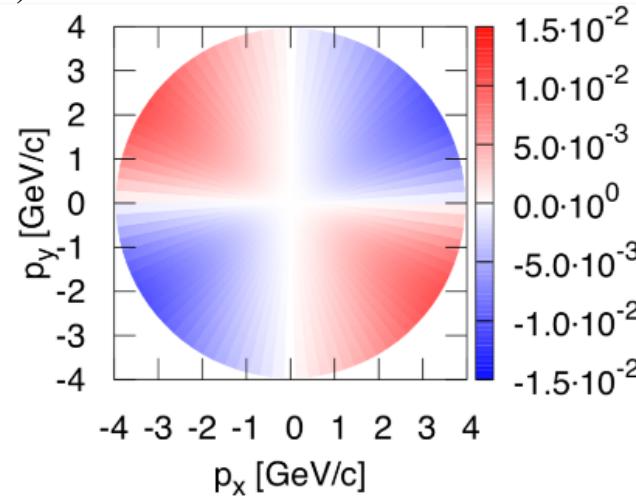
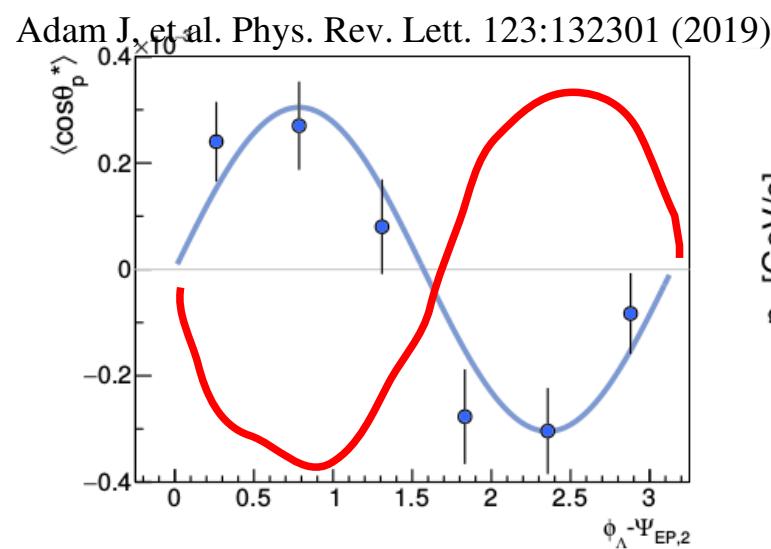
$$S^\mu(p) = \frac{1}{8m} \epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int_{\Sigma} d\Sigma_\tau p^\tau n_F (1 - n_F) \partial_\nu \beta_\rho}{\int_{\Sigma} d\Sigma_\tau p^\tau n_F}$$

Different models of the collision, same formula for polarization

Puzzles: momentum dependence of polarization (until march 2021)



Theory prediction



Not the effect of decays:

X. L. Xia, H. Li, X.G. Huang and H. Z. Huang,
Phys. Rev. C 100 (2019), 014913

F. B., G. Cao and E. Speranza,
Eur. Phys. J. C 79 (2019) 741

Theory

Spin polarization vector for spin $\frac{1}{2}$ particles:

$$S^\mu(p) = \frac{1}{2} \frac{\int d\Sigma \cdot p \operatorname{tr}_4(\gamma^\mu \gamma^5 W_+(x, p))}{\int d\Sigma \cdot p \operatorname{tr}_4 W_+(x, p)}$$

Wigner function is an expectation value of an integrated two-point function of the Dirac field

$$W(x, k) = \operatorname{Tr}(\hat{\rho} \hat{W}(x, k))$$

One needs to know the statistical operator to calculate mean values

$$\langle \hat{X} \rangle = \operatorname{tr}(\hat{\rho} \hat{X})$$

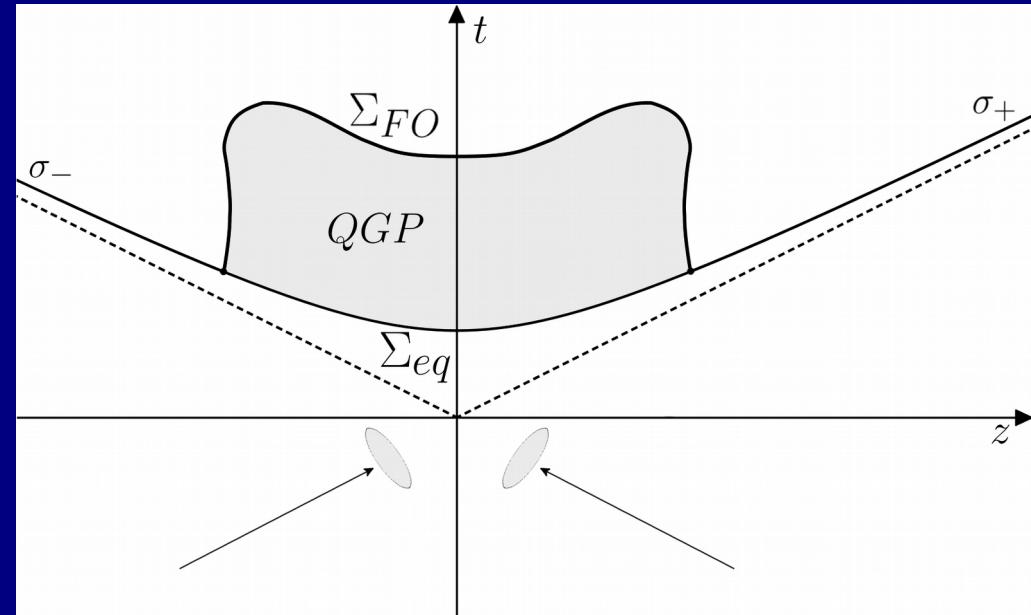
The actual statistical operator

(covariant Zubarev theory)

F.B., M. Buzzegoli and E. Grossi, *Reworking the Zubarev's approach to non-equilibrium quantum statistical mechanics*, Particles 2 (2019) 197

In the covariant Zubarev theory, this is the LTE at some initial space-like hypersurface:

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma(\tau_0)} d\Sigma_\mu \left(\hat{T}_B^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right].$$



With the Gauss theorem

NOTE: T_B stands for the symmetrized Belinfante stress-energy tensor

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma(\tau)} d\Sigma_\mu \left(\hat{T}_B^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) + \int_{\Theta} d\Theta \left(\hat{T}_B^{\mu\nu} \nabla_\mu \beta_\nu - \hat{j}^\mu \nabla_\mu \zeta \right) \right],$$



Local equilibrium, non-dissipative terms



Dissipative terms

$$\widehat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma(\tau)} d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) + \int_{\Theta} d\Theta \left(\widehat{T}_B^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta \right) \right],$$

\widehat{A} = local equilibrium \widehat{B} = dissipation

$$\exp[\widehat{A} + \widehat{B}] = \exp[\widehat{A}] + \int_0^1 dz \exp[z(\widehat{A} + \widehat{B})] \widehat{B} \exp[-z\widehat{A}] \exp[\widehat{A}]$$

Kubo identitty

$$\widehat{\rho} \simeq \widehat{\rho}_{\text{LE}} + \int_0^1 dz \exp[z\widehat{A}] \widehat{B} \exp[-z\widehat{A}] \widehat{\rho}_{\text{LE}} - \langle \widehat{B} \rangle_{\text{LE}} \widehat{\rho}_{\text{LE}}$$

Linear response

This is the method to generate the so-called Kubo formulae

A. Hosoya, M. Sakagami and M. Takao, Annals Phys. 154 (1984), 229.

Local thermodynamic equilibrium approximation

$$\begin{aligned}\hat{\rho} &\simeq \hat{\rho}_{\text{LE}} = \frac{1}{Z} \exp \left[- \int_{\Sigma(\tau)} d\Sigma_\mu \hat{T}_B^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right] \\ &= \frac{1}{Z} \exp \left[- \int_{\Sigma_{FO}} d\Sigma_\mu \hat{T}_B^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right]\end{aligned}$$

Corresponds to the ideal fluid approximation.

Neglecting dissipative term in the exponent of the density operator

$$W(x, k) \simeq W(x, k)_{\text{LE}} = \text{Tr}(\hat{\rho}_{\text{LE}} \hat{W}(x, k))$$

$$W(x, k)_{\text{LE}} = \frac{1}{Z} \text{Tr} \left(\exp \left[- \int_{\Sigma_{FO}} d\Sigma_\mu(y) \hat{T}_B^{\mu\nu}(y) \beta_\nu(y) - \zeta(y) \hat{j}^\mu(y) \right] \hat{W}(x, k) \right)$$

Even the local equilibrium value of the Wigner function is hard to calculate for general four-temperature and chemical potential/T fields.

Hydrodynamic limit: Taylor expansion

Expand the β and ζ fields from the point x where the Wigner operator is to be evaluated

$$\beta_\nu(y) = \beta_\nu(x) + \partial_\lambda \beta_\nu(x)(y - x)^\lambda + \dots$$

$$\widehat{\rho}_{\text{LE}} \simeq \frac{1}{Z} \exp \left[-\beta_\mu(x) \widehat{P}^\mu - \frac{1}{2} (\partial_\mu \beta_\nu(x) - \partial_\nu \beta_\mu(x)) \widehat{J}_x^{\mu\nu} - \frac{1}{2} (\partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x)) \widehat{Q}_x^{\mu\nu} + \dots \right]$$

$$\widehat{J}_x^{\mu\nu} = \int d\Sigma_\lambda (y-x)^\mu \widehat{T}_B^{\lambda\nu}(y) - (y-x)^\nu \widehat{T}_B^{\lambda\mu}(y)$$

$$\widehat{Q}_x^{\mu\nu} = \int d\Sigma_\lambda (y-x)^\mu \widehat{T}_B^{\lambda\nu}(y) + (y-x)^\nu \widehat{T}_B^{\lambda\mu}(y)$$

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

Thermal vorticity

Adimensional in natural units

$$\xi_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu)$$

Thermal shear

Adimensional in natural units

At global equilibrium the thermal shear vanishes because the four-temperature fulfills the Killing equation

Linear response theory for the LE operator

$$\hat{\rho}_{\text{LE}} \simeq \frac{1}{Z} \exp[-\beta_\mu(x) \hat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu}(x) \hat{J}_x^{\mu\nu} - \frac{1}{2} \xi_{\mu\nu}(x) \hat{Q}_x^{\mu\nu} + \dots]$$



$$n_F = (\mathrm{e}^{\beta \cdot p - \xi} + 1)^{-1}$$

$$S_\xi^\mu(p) = -\frac{1}{4m} \epsilon^{\mu\nu\sigma\tau} \frac{p_\tau p^\rho}{\varepsilon} \frac{\int_\Sigma d\Sigma \cdot p n_F (1 - n_F) \hat{t}_\nu \xi_{\sigma\rho}}{\int_\Sigma d\Sigma \cdot p n_F},$$

F. B., M. Buzzegoli, A. Palermo, Phys. Lett. B 820 (2021) 136519

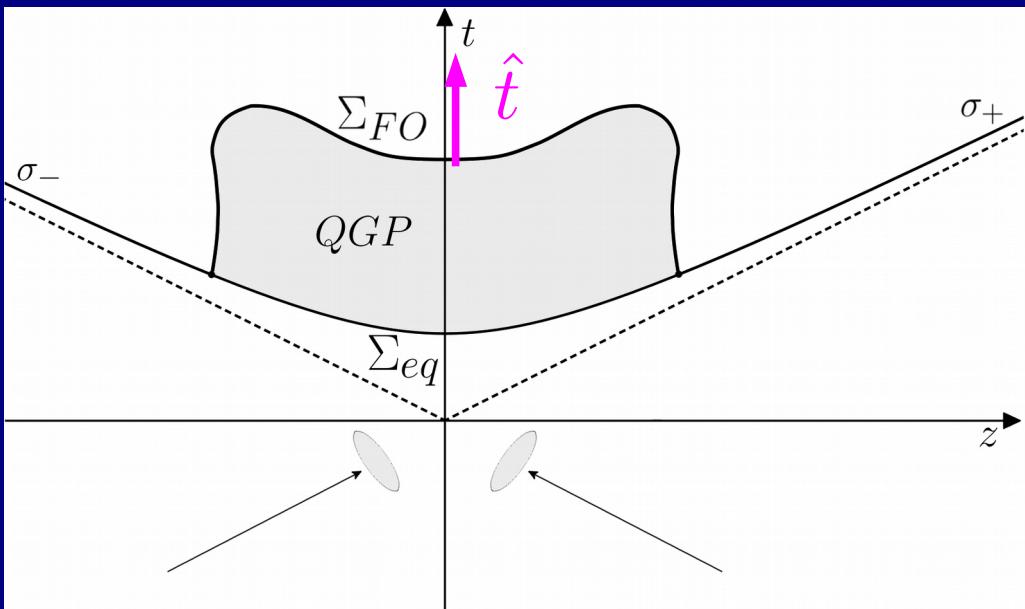
Very similar formula (see later) obtained at the same time in: S. Liu, Y. Yin, JHEP 07 (2021) 188

The above formula confirmed in:

Y. C. Liu, X. G. Huang, arXiv 2109.15301, Sci. China Phys. Mech. Astron. 65 (2022) 7, 272011

both from a local equilibrium *ansatz* of some distribution function and the expansion of the local equilibrium density operator.

Why do we have a dependence on Σ ?



The thermal shear term depends on the correlator:

$$\langle \hat{Q}_x^{\mu\nu} \hat{W}(x, p) \rangle$$

$$\begin{aligned}\hat{J}_x^{\mu\nu} &= \int d\Sigma_\lambda (y-x)^\mu \hat{T}_B^{\lambda\nu}(y) - (y-x)^\nu \hat{T}_B^{\lambda\mu}(y) \\ \hat{Q}_x^{\mu\nu} &= \int_{\Sigma_{FO}} d\Sigma_\lambda (y-x)^\mu \hat{T}_B^{\lambda\nu}(y) + (y-x)^\nu \hat{T}_B^{\lambda\mu}(y)\end{aligned}$$

The divergence of the integrand of $J^{\mu\nu}$ vanishes, therefore it does not depend on the integration hypersurface (it is a constant of motion) and J is thus a tensor operator:

$$\hat{\Lambda} \hat{J}_x^{\mu\nu} \hat{\Lambda}^{-1} = \Lambda_\alpha^{-1\mu} \Lambda_\beta^{-1\nu} \hat{J}_x^{\alpha\beta}$$

The divergence of the integrand of $Q^{\mu\nu}$ does not vanish, therefore it does depend on the integration hypersurface and Q is NOT a tensor operator

$$\hat{\Lambda} \hat{Q}^{\mu\nu} \hat{\Lambda}^{-1} \neq \Lambda_\alpha^{-1\mu} \Lambda_\beta^{-1\nu} \hat{Q}^{\alpha\beta}$$

Dissipative vs non-dissipative

$$S_\xi^\mu(p) = -\frac{1}{4m} \epsilon^{\mu\nu\sigma\tau} \frac{p_\tau p^\rho}{\varepsilon} \frac{\int_\Sigma d\Sigma \cdot p n_F (1 - n_F) \hat{t}_\nu \xi_{\sigma\rho}}{\int_\Sigma d\Sigma \cdot p n_F}, \quad \text{is a non-dissipative term}$$

Gradient corrections, even those proportional to the shear tensor, may be non-dissipative if they stem from the *gradient expansion of the LE operator*.

Two important signatures:

- 1) they do not involve any unknown, dynamical, transport coefficient (fulfilled)
- 2) the two sides have the same time-reversal transformation (fulfilled)

Dissipative: correlators of operators at different time (e.g. Kubo formula):

$$(\widehat{X}, \widehat{Y}) \equiv n^\alpha \frac{\partial}{\partial K^\alpha} \Big|_{n \cdot k=0} \lim_{k_T \rightarrow 0} \text{Im } iT \int_{-\infty}^t d^4x' \langle [\widehat{X}(x), \widehat{Y}(x')] \rangle_{\beta(x)} e^{-iK \cdot (x' - x)} \quad \text{From operator } \widehat{B}$$

Non-dissipative: correlators of operator on the SAME hypersurface (or same time)

$$\int_0^1 dz \int_\Sigma d\Sigma_\mu(y) \left(\langle \widehat{O}(x) \widehat{T}^{\mu\nu}(y + iz\beta(x)) \rangle_{\beta(x)} - \langle \widehat{O}(x) \rangle_{\beta(x)} \langle \widehat{T}^{\mu\nu}(y + iz\beta(x)) \rangle_{\beta(x)} \right) \delta\beta_\nu \quad \text{From operator } \widehat{A}$$

Spin-thermal shear in kinetic theory

Derivations by using expansions of the Wigner functions in h :

C. Yi, S. Pu, D. L. Yang, Phys.Rev.C 104 (2021) 6, 064901

Y. C. Liu, X. G. Huang, arXiv 2109.15301, Sci. China Phys.Mech.Astron. 65 (2022) 7, 272011

Need of a local equilibrium ansatz

From relativistic kinetic theory:

N. Weickgenannt, D. Wagner, E. Speranza and D. H. Rischke, Phys. Rev. D 106 (2022), L091901

$$\begin{aligned}\Pi_{\text{NS}}^{\mu}(p) = & \int d\Sigma \cdot p \frac{f_{0p}}{2N} \left\{ -\frac{\hbar}{2m} \tilde{\Omega}^{\mu\nu} p_{\nu} + \left(g_{\nu}^{\mu} - \frac{u^{\mu} p_{\langle\nu\rangle}}{E_p} \right) \right. \\ & \times \left. \left[\mathfrak{e} \chi_{\mathfrak{p}} \left(\tilde{\Omega}^{\nu\rho} - \tilde{\omega}^{\nu\rho} \right) u_{\rho} - \chi_{\mathfrak{q}} \mathfrak{d} \beta_0 \sigma_{\rho}^{\langle\alpha} \epsilon^{\beta\rangle} u_{\sigma} p_{\langle\alpha} p_{\beta\rangle} \right] \right\},\end{aligned}$$



dissipative?

Spin-thermal shear for vector bosons

Can the spin-thermal shear coupling account for the spin alignment of vector mesons?

Not at local equilibrium, the tensor part of the spin density matrix vanishes at the linear order of the LE gradient expansion (F.B., M. Buzzegoli, A. Palermo unpublished)

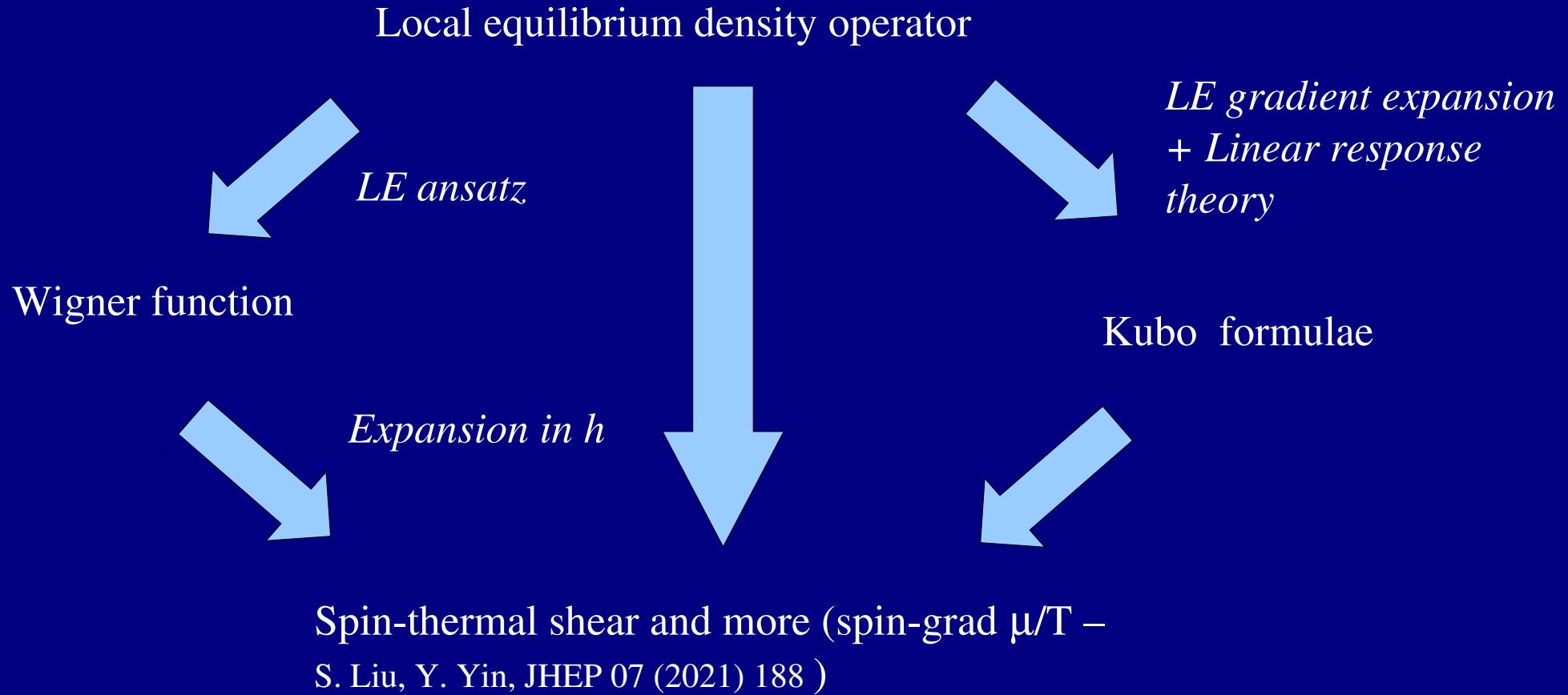
Not at local equilibrium, but a dissipative term is found with the kinetic theory of vector bosons (D. Wagner, N. Weickgenannt and E. Speranza, Phys. Rev. Res. 5 (2023), 013187)

$$\rho_{00}(k) = \frac{1}{3} - \frac{4}{15} \frac{\int d\Sigma_\alpha k^\alpha \xi \beta_0 f_{0\mathbf{k}} \mathcal{H}_{\mathbf{k}1}^{(2,0)} \epsilon_\alpha^{(0)} \epsilon_\beta^{(0)} K_{\mu\nu}^{\alpha\beta} \Xi_{\rho\sigma}^{\mu\nu} \pi^{\rho\sigma}}{\int d\Sigma_\alpha k^\alpha f_{0\mathbf{k}} \left(1 - 3\mathcal{H}_{\mathbf{k}0}^{(0,0)} \Pi/m^2 + \mathcal{H}_{\mathbf{k}0}^{(0,2)} \pi^{\mu\nu} k_{\langle\mu} k_{\nu\rangle} \right)},$$

dissipative?

At local equilibrium including interactions? (F. Li and S.Y.F. Liu, arXiv:2206.11890).

Theory summary



Note: there is no principle distinction between “statistical method” or “Quantum field theory method” or “Kubo-formula method” or “Wigner function method”

Phenomenology: application to relativistic heavy ion collisions

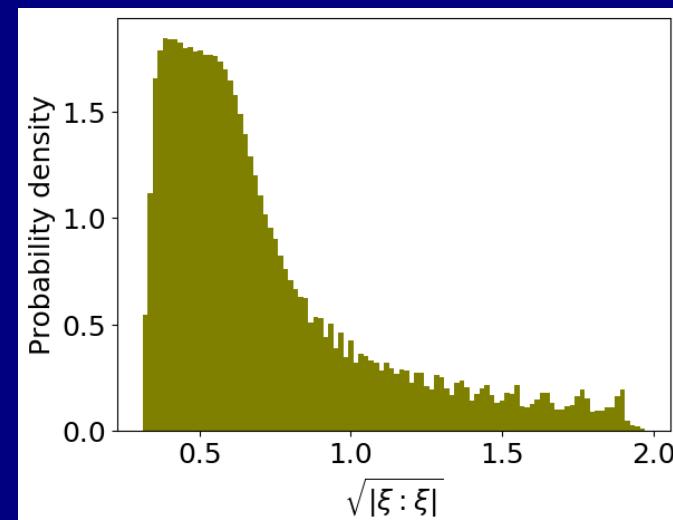
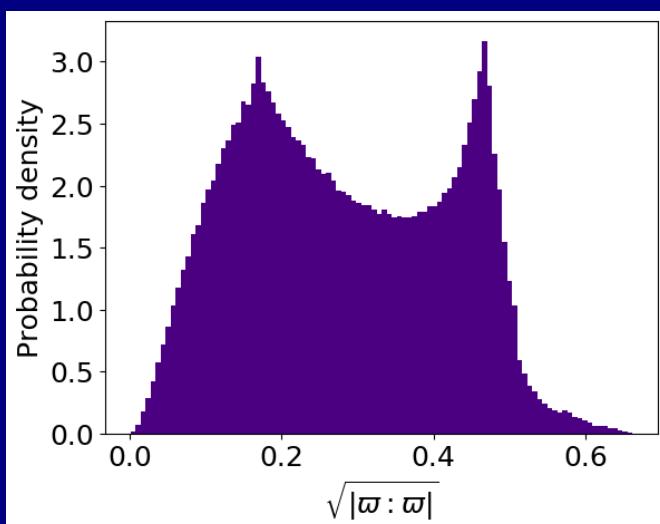
$$S^\mu = S_{\varpi}^\mu + S_\xi^\mu$$

$$S_{\varpi}^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\int_\Sigma d\Sigma \cdot p n_F(1-n_F)\varpi_{\rho\sigma}}{\int_\Sigma d\Sigma \cdot p n_F}$$

$$S_\xi^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\rho\sigma\tau} \frac{p_\tau p^\lambda}{\varepsilon} \frac{\int_\Sigma d\Sigma \cdot p n_F(1-n_F)\hat{t}_\rho\xi_{\sigma\lambda}}{\int_\Sigma d\Sigma \cdot p n_F}$$

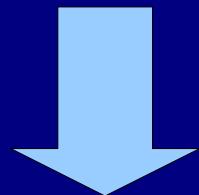
Is linear response theory adequate?

F. B., M. Buzzegoli, A. Palermo, G. Inghirami and I. Karpenko, Phys. Rev. Lett. 127 (2021) 27, 272302



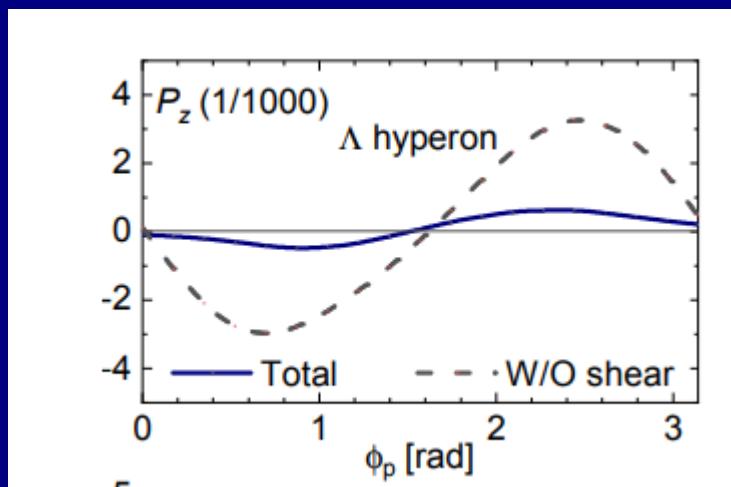
Comparison between different calculations -I

$$S^\mu(p^\alpha) = \frac{1}{4m} \frac{\int d\Sigma \cdot p n_0 (1 - n_0) \mathcal{A}^\mu}{\int d\Sigma \cdot p n_0},$$



$$\begin{aligned}\mathcal{A}_{LY}^\mu = -\varepsilon^{\mu\rho\sigma\tau} & \left[\frac{1}{2} \omega_{\rho\sigma} p_\tau + \frac{1}{E} u_\rho \xi_{\sigma\lambda} p_\perp^\lambda p_\tau \right. \\ & \left. + \frac{b_i}{\beta E} u_\rho p_\sigma^\perp \partial_\tau^\perp (\beta \mu_B) \right].\end{aligned}$$

S. Liu, Y. Yin, JHEP 07 (2021) 188

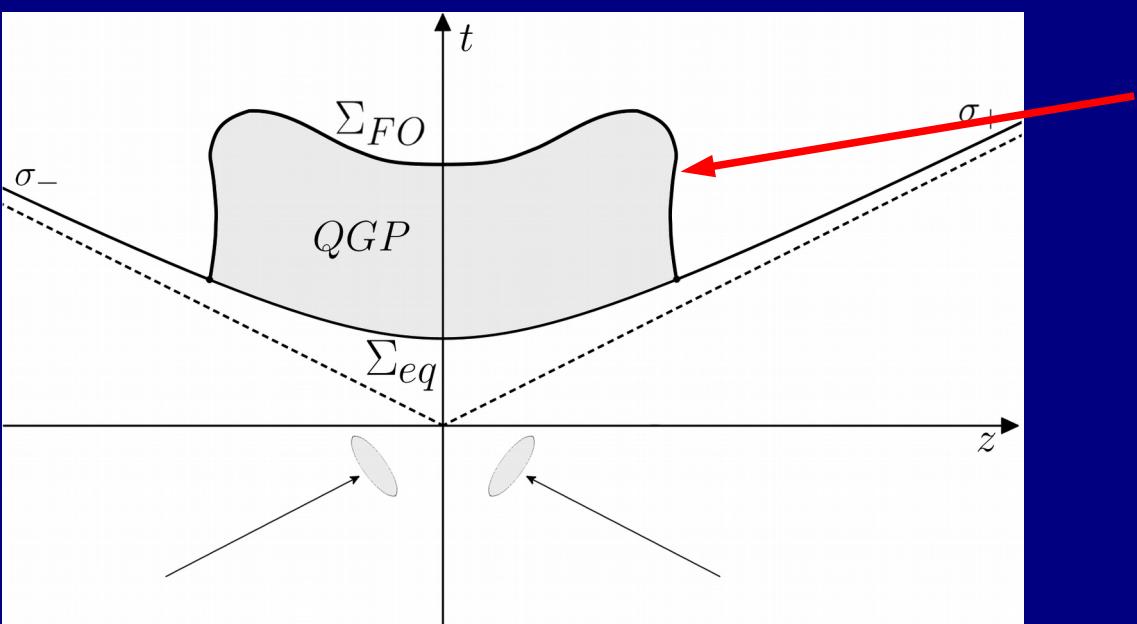


B. Fu, S.Y.F. Liu, L. Pang, H. Song and Y. Yin,
Phys. Rev. Lett. 127 (2021), 142301

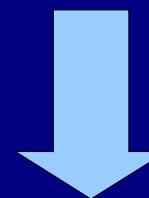
See more in Baochi Fu's talk

Comparison between different calculations -II

F. B., M. Buzzegoli, A. Palermo, G. Inghirami and I. Karpenko, Phys. Rev. Lett. 127 (2021) 27, 272302



At high energy, Σ_{FO} expected to be $T = \text{constant!}$



$$\beta^\mu = (1/T)u^\mu$$

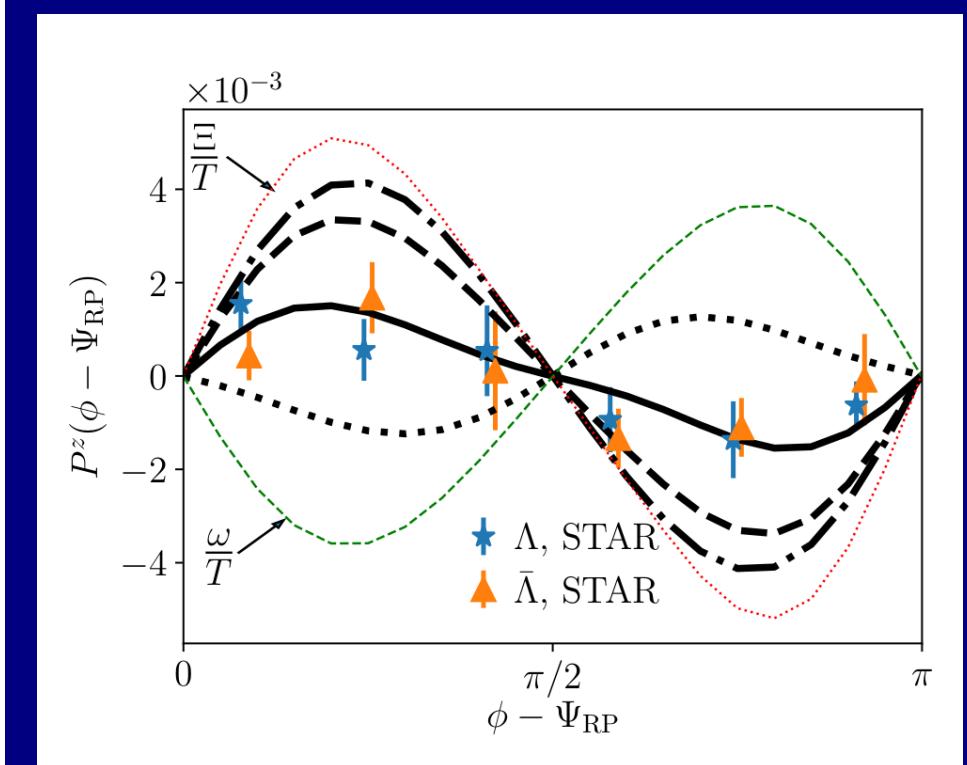
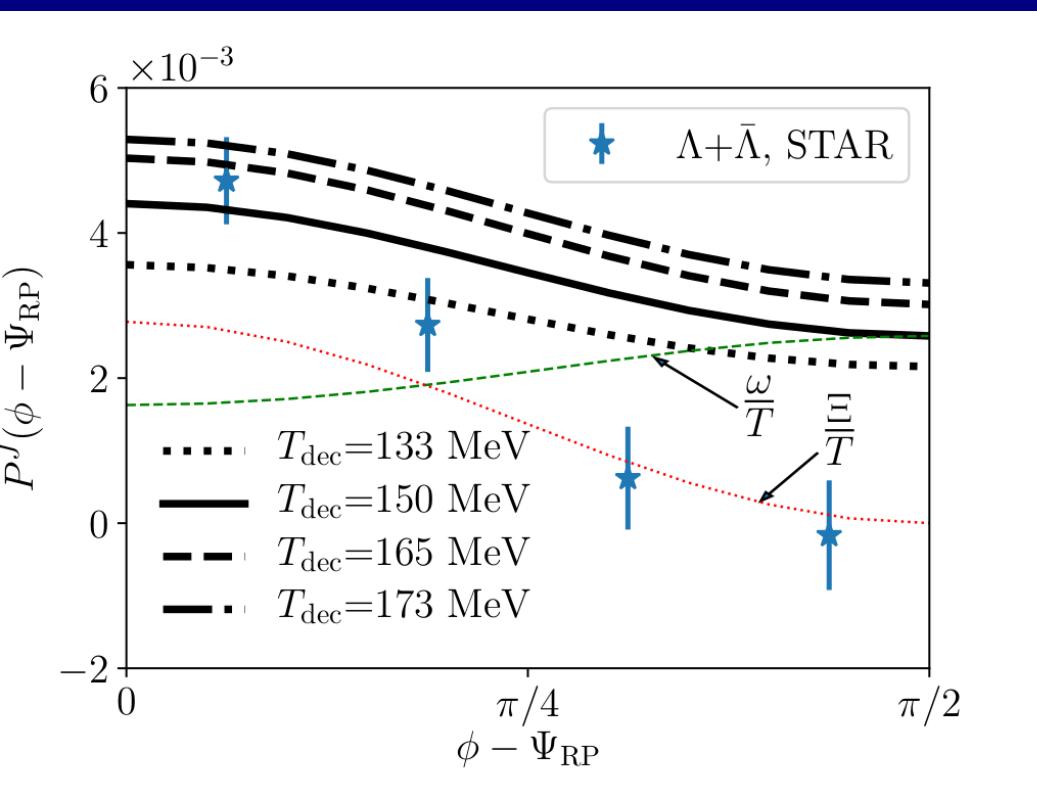
$$\hat{\rho}_{\text{LE}} = \frac{1}{Z} \exp \left[- \int_{\Sigma_{FO}} d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu \right] = \frac{1}{Z} \exp \left[- \frac{1}{T} \int_{\Sigma_{FO}} d\Sigma_\mu \hat{T}^{\mu\nu} u_\nu \right]$$

$$S^\mu(p^\alpha) = \frac{1}{4m} \frac{\int d\Sigma \cdot p n_0 (1 - n_0) \mathcal{A}^\mu}{\int d\Sigma \cdot p n_0},$$

$$\mathcal{A}_{\text{BBP}}^\mu = -\varepsilon^{\mu\rho\sigma\tau} \left(\frac{1}{2} \omega_{\rho\sigma} p_\tau + \frac{1}{E} \hat{t}_\rho \xi_{\sigma\lambda} p^\lambda p_\tau \right)$$

Comparison between different calculations -II

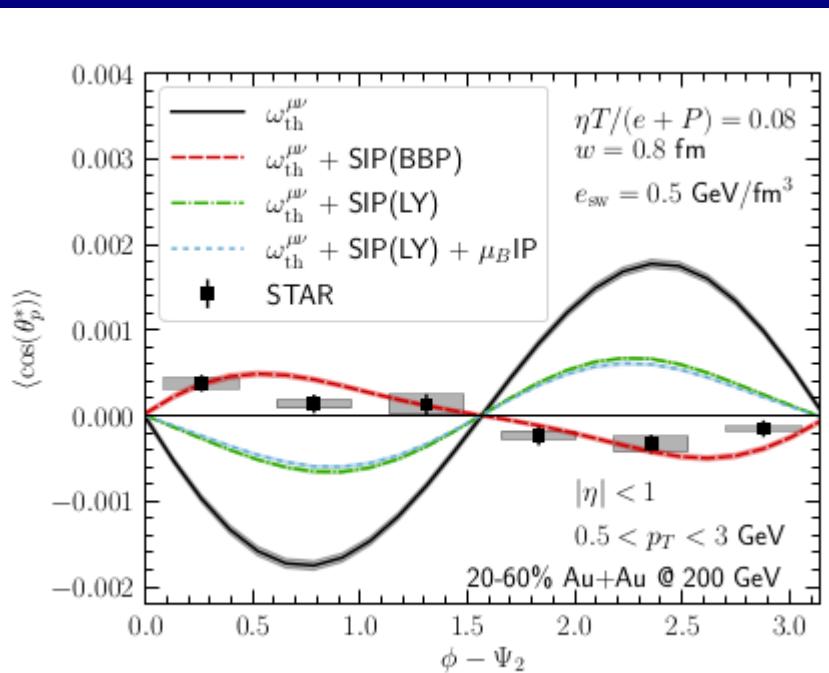
$$S_{\text{ILE}}^\mu(p) = \left(-\epsilon^{\mu\rho\sigma\tau} p_\tau \frac{\int_\Sigma d\Sigma \cdot p n_F (1 - n_F) \left[\omega_{\rho\sigma} + 2 \hat{t}_\rho \frac{p^\lambda}{\varepsilon} \Xi_{\lambda\sigma} \right]}{8m T_{\text{dec}} \int_\Sigma d\Sigma \cdot p n_F} \right)$$



Note: in the LY formula, both the thermal gradient and acceleration terms are killed by the four-velocity u replacing t and the p_\perp replacing p

Comparison between different calculations -III

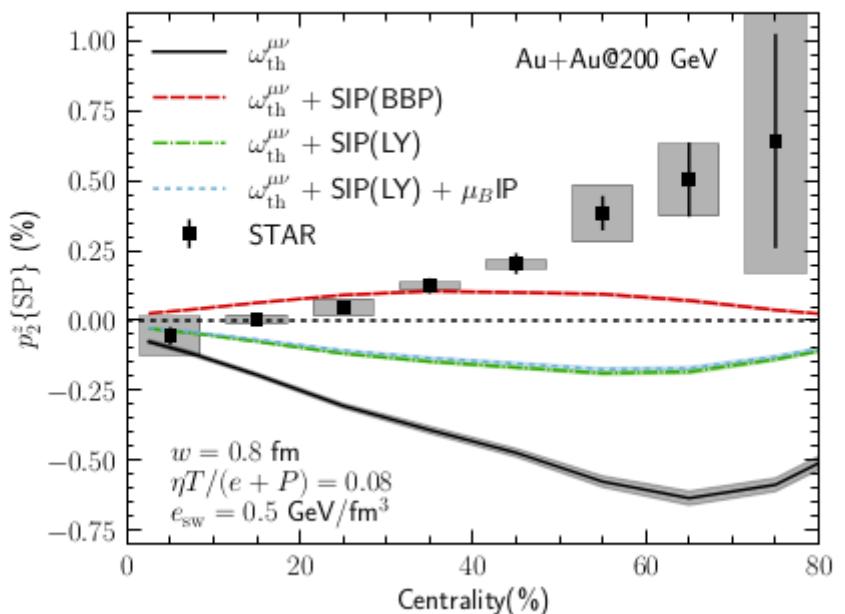
S. Alzhrani, S. Ryu, C. Shen, Phys.Rev.C 106 (2022) 1, 014905,



Dependence of the dominant Fourier component ($\sin 2\phi$) on centrality

Agreement with the BBP formula
WITH thermal gradients!

Attributed to different initial conditions...



Consistency check

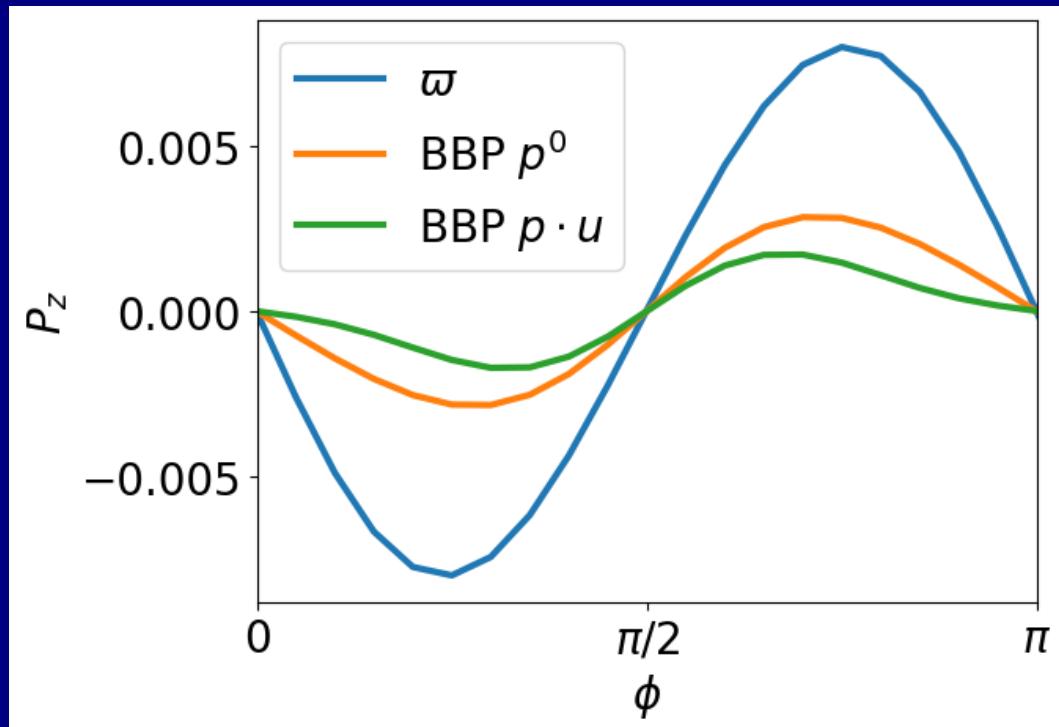
It is necessary to test the consistency of different codes before assessing the impact of different initial conditions.

Same initial hydro conditions, same EOS and transport coefficients



Same results
with the same
formula

ONGOING: Study of the consistency between VHLLE (I. Karpenko) and C. Shen's code.



EXAMPLE (A. Palermo courtesy)

VHLLE (I. Karpenko) run
With C. Shen's initial conditions

Summary and outlook

- Spin-thermal shear coupling (as well as spin-grad μ/T) is a new non-dissipative phenomenon, not dependent on unknown dynamical transport coefficient. Also a dissipative contribution?
- Different forms owing to different approximations. Yet, all different methods are ultimately based on quantum statistical mechanics.
- This new term can solve the puzzles of the hydrodynamic predictions of local spin polarization of particles in heavy ion collisions.
- Ongoing work to understand the difference in numerical calculations: need of a common reference.

MUCH WORK TO BE DONE!

Polarization has a great potential to pin down the initial conditions and the QGP evolution which is yet unexploited to a large extent

What is this new term?

It is a quantum, *non-dissipative*,
correction to local equilibrium

$$S_\xi^\mu(p) = -\frac{1}{4m} \epsilon^{\mu\nu\sigma\tau} \frac{p_\tau p^\rho}{\varepsilon} \frac{\int_\Sigma d\Sigma \cdot p \ n_F (1 - n_F) \hat{t}_\nu \xi_{\sigma\rho}}{\int_\Sigma d\Sigma \cdot p \ n_F},$$

Does it have a non-relativistic limit?

$$\xi_{\sigma\rho} = \frac{1}{2} \partial_\sigma \left(\frac{1}{T} \right) u_\rho + \frac{1}{2} \partial_\rho \left(\frac{1}{T} \right) u_\sigma + \frac{1}{2T} (A_\rho u_\sigma + A_\sigma u_\rho) + \frac{1}{T} \sigma_{\rho\sigma} + \frac{1}{3T} \theta \Delta_{\rho\sigma}$$

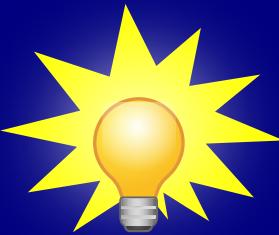
A is the acceleration field

$$\sigma_{\mu\nu} = \frac{1}{2} (\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \frac{1}{3} \Delta_{\mu\nu} \theta$$

All terms are relativistic (they vanish in the infinite c limit) EXCEPT grad T terms, which give rise to:

$$\mathbf{S}_\xi = \frac{1}{8} \mathbf{v} \times \frac{\int d^3x \ n_F (1 - n_F) \nabla \left(\frac{1}{T} \right)}{\int d^3x \ n_F}$$

There is an equal contribution in the NR limit from thermal vorticity



Isothermal local equilibrium

*The most appropriate setting for relativistic heavy ion collisions
at very high energy!*

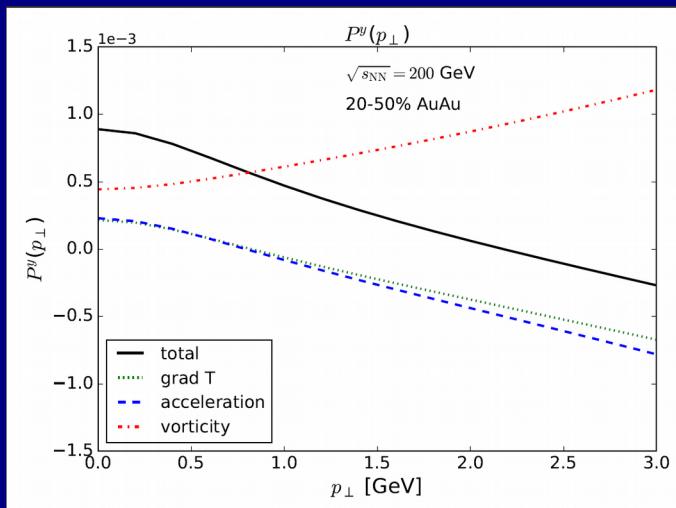
Both thermal shear and thermal vorticity include temperature gradients

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$$

$$\xi_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu)$$

$$\beta^\mu = (1/T)u^\mu$$

Thermal gradients do contribute to the polarization



$$\mathbf{S}^* \propto \frac{\hbar}{KT^2} \mathbf{u} \times \nabla T + \frac{\hbar}{KT} (\boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{v} \mathbf{u}/c^2) + \frac{\hbar}{KT} \mathbf{A} \times \mathbf{u}/c^2$$

Is it the best thing to do?

The formulae of the spin vector are based on a Taylor expansion of the density operator

$$W(x, k)_{\text{LE}} = \frac{1}{Z} \text{Tr} \left(\exp \left[- \int_{\Sigma_{FO}} d\Sigma_\mu(y) \hat{T}^{\mu\nu}(y) \beta_\nu(y) - \zeta(y) \hat{j}^\mu(y) \right] \hat{W}(x, k) \right)$$

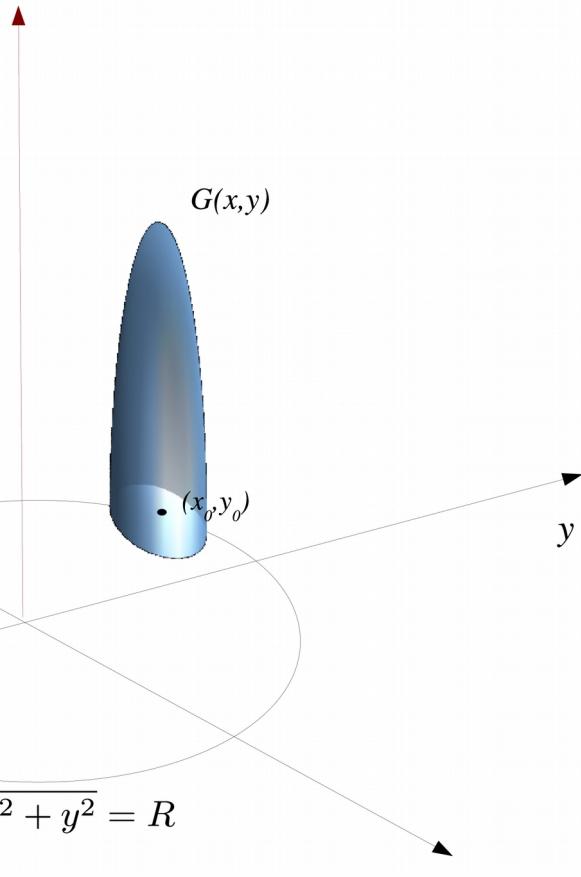
$$\beta_\nu(y) = \beta_\nu(x) + \partial_\lambda \beta_\nu(x) (y - x)^\lambda + \dots$$

$$\hat{\rho}_{\text{LE}} \simeq \frac{1}{Z} \exp \left[-\beta_\mu(x) \hat{P}^\mu - \frac{1}{2} (\partial_\mu \beta_\nu(x) - \partial_\nu \beta_\mu(x)) \hat{J}_x^{\mu\nu} - \frac{1}{2} (\partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x)) \hat{Q}_x^{\mu\nu} + \dots \right]$$

This is generally correct, but it is an approximation after all.

Can we find a better approximation for a special case?

Why isothermal matters: a simple example



Task: approximate the integral

$$W = \int_{\Gamma} e^{\sqrt{x^2+y^2}} G(x, y) ds$$

where $G(x, y)$ is a peaked function around the point (x_0, y_0) on the circle.

Since G is peaked, one can Taylor expand the exponent about (x_0, y_0)

$$\begin{aligned} W &\simeq e^{\sqrt{x_0^2+y_0^2}} \int_{\Gamma} e^{x_0(x-x_0)/R+y_0(y-y_0)/R} G(x, y) ds \\ &= e^R \int_{\Gamma} e^{x_0(x-x_0)/R+y_0(y-y_0)/R} G(x, y) ds \\ &= e^R \int_{\Gamma} e^{\nabla r|_{(x_0, y_0)} \cdot (\mathbf{x}-\mathbf{x}_0)} G(x, y) ds \end{aligned}$$

But it is just pointless if we integrate over the circle!

$$W = e^R \int_{\Gamma} G(x, y) ds$$

In the previous example, the Taylor expansion at first order introduces an undesired term:

$$W = e^R \int_{\Gamma} G(x, y) ds$$

exact

$$W \simeq e^R \int_{\Gamma} e^{\nabla r|_{(x_0, y_0)} \cdot (\mathbf{x} - \mathbf{x}_0)} G(x, y) ds$$

With gradient of r expansion

which is proportional to the gradient of the constant quantity on the circle, perpendicular to the integration line. This term does not vanish in the integration!