Exact Polarization in Relativistic Fluids at Global Equilibrium

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Main results

Exact spin density matrix for spin-S particles at general global equilibrium:

$$\Theta(p) = \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb\cdot p} D^{(S)}(\Lambda^n)}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb\cdot p} \mathrm{tr}\left(D^{(S)}(\Lambda^n)\right)}$$

Exact spin vector for Dirac field at global equilibrium

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma \qquad S^\mu(p) = \frac{1}{2}\,\frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p+\zeta}}$$

Including **all quantum corrections** in vorticity

Pauli-Lubanski Vector

The Hilbert space of relativistic particles is built using the four-momentum and the Pauli-Lubanski vector

$$\widehat{\Pi}^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}\widehat{P}_{\sigma} \qquad [\widehat{\Pi}^{\mu}, \widehat{P}^{\nu}] = 0$$

For states with definite momentum p

$$\widehat{\Pi}^{\mu}|p\rangle = \widehat{\Pi}^{\mu}(p)|p\rangle \qquad \widehat{\Pi}^{\mu}(p) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_{\sigma}$$

One has

$$p_{\mu}\widehat{\Pi}^{\mu}(p) = 0$$

If we consider **massive** particles the Pauli-Lubanski vector is connected to the generators of rotations

$$\widehat{S}^\mu(p) = \frac{\widehat{\Pi}^\mu(p)}{m} = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma$$

$$\widehat{S}^\mu(p) = \sum_{i=1}^{3}\widehat{S}_i(p)n_i^\mu(p) \qquad [\widehat{S}_i(p), \widehat{S}_j(p)] = i\epsilon_{ijk}\widehat{S}_k(p)$$

In the **massless** case:

$$\widehat{\Pi}^\mu(p) = \widehat{h}(p)p^\mu + \widehat{\Pi}_1(p)n_1^\mu(p) + \widehat{\Pi}_2(p)n_2^\mu(p) \qquad \widehat{\Pi}_{1,2}(p)|p,h\rangle = 0$$

$$\widehat{h}(p) = \frac{\widehat{\Pi}(p)\cdot q}{p\cdot q} = -\frac{1}{2p\cdot q}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma q_\mu$$

$$q\cdot p \neq 0, \qquad q^2 = 0, \qquad q\cdot n_i(p) = 0$$

Helicity is the only physical degree of freedom

Expectation values

To compute mean values we need the spin density matrix:

$$\Theta_{sr}(p) = \frac{\mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\right)}{\sum_l \mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\right)} = \frac{\langle\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\rangle}{\sum_l\langle\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\rangle}$$

Once we are given a spin density matrix, expectation values are computed as
[F. Becattini, Lect.Notes Phys. 987 (2021) 15-52, AP, F. Becattini Eur.Phys.J.Plus 138 (2023) 6, 547]:

$$S^\mu(p) = \sum_{i=1}^{3}[p]^\mu{}_i \mathrm{tr}\left(\Theta(p)D^S(\mathrm{J}^i)\right), \qquad m \neq 0$$

$$\Pi^\mu(p) = p^\mu \sum_{h=\pm S} h\Theta_{hh}(p), \qquad m = 0$$

Polarization and the Wigner function

To compute expectation values it is useful to use the Wigner function.

$$W(x,k)_{ab} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4 y e^{-ik\cdot y}\langle :\bar{\Psi}_b(x+y/2)\Psi_a(x-y/2):\rangle$$

Massive Dirac fermions [F. Becattini, Lect.Notes Phys. 987 (2021) 15-52]:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\,\mathrm{tr}(\gamma^\mu\gamma_5 W_+(x,p))}{\int \mathrm{d}\Sigma\cdot p\,\mathrm{tr}(W_+(x,p))}$$

Massless Dirac fermions

$$\Pi^\mu(p) = \frac{p^\mu}{2}\frac{\int \mathrm{d}\Sigma\cdot p\,\mathrm{tr}(\not{a}\gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma\cdot p\,\mathrm{tr}(W_+(x,p)\not{a})}$$

In the massless case the mean spin always points in the direction of momentum.
See also *[Y.-C. Liu, K. Mameda, X.-G. Huang, Chin. Phys. C 44(9), 094101]*

Global equilibrium

Density operator at global equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}\right] \qquad \langle\widehat{O}\rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

The vector b is constant and the thermal vorticity ϖ is a constant antisymmetric tensor. The four-temperature β vector is a Killing vector:

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu}x_\nu \equiv \frac{u^\mu}{T}$$

At global equilibrium:

$$\frac{A^\mu}{T} = \varpi^{\mu\nu}u_\nu$$

Acceleration

$$\frac{\omega^\mu}{T} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma$$

Angular velocity

The generators of the Poincaré group appear in the density operator.

Analytic continuation of the thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J$\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z}\exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\dots\phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k\ \text{times}}b^{\alpha_k} \qquad \widehat{\Lambda} \equiv \mathrm{e}^{-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}}$$

We can use **group theory** to calculate **thermal expectation values.**

[AP, F. Becattini, M. Buzzegoli JHEP 10 (2021) 077]

The number operator at (imaginary) global equilibrium:

$$\langle \widehat{a}^{\dagger}_s(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p}') D^S(W(\Lambda^n, p))_{ts} \mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n} \Lambda^k p}$$

$D(W) = [\Lambda p]^{-1}\Lambda[p]$ is the Wigner rotation in the spin-S representation of the rotation group.

For vanishing vorticity (i.e. $\Lambda = \mathrm{I}$) we recover Bose and Fermi statistics:

$$\langle \widehat{a}^{\dagger}_s(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \delta_{ts}\, \mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\, \delta^3(\boldsymbol{p} - \boldsymbol{p}')\delta_{ts}}{\mathrm{e}^{b\cdot p} + (-1)^{2S+1}}$$

Exact Wigner function for free fermions at global equilibrium:

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-(\Lambda^n p+p)/2\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+(\Lambda^n p+p)/2\right)\right]$$

Where $\Lambda = \mathrm{e}^{-i\frac{\phi}{2}:J}$ is in the four-vector representation.
Solves the Wigner equation! Full summation of the "$\hbar$ expansion".

Can be used to compute expectation values:

$$\langle :\{\bar{\Psi}\Psi, j^\mu, T^{\mu\nu}\}:\rangle = \int \mathrm{d}^4 k\,\{\mathrm{tr}\,(W)\,,\mathrm{tr}\,(\gamma^\mu W)\,,k^\mu \mathrm{tr}\,(\gamma^\nu W)\}$$

The Dirac delta is integrated out and results are expressed as series of functions that can be regularized using the **analytic distillation**.

Spin vector

Spin vector of massive Dirac fermions:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(\gamma^\mu \gamma_5 W_+(x,p)\right)}{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(W_+(x,p)\right)}$$

Exact spin vector at global equilibrium:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\gamma^\mu\gamma_5 \mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right)\delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right)\delta^3(\Lambda^n p - p)}$$

How to handle a ratio of series of δ-functions? Where does it come from?

In quantum field theory the spin density matrix is defined:

$$\Theta(p)_{rs} = \frac{\langle \widehat{a}_s^\dagger(p)\widehat{a}_r(p)\rangle}{\sum_t \langle \widehat{a}_t^\dagger(p)\widehat{a}_t(p)\rangle}$$

From the **analytic continuation** of the density operator:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle = 2\varepsilon \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p}) D^S(W(\Lambda^n, p))_{ts} \mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n} \Lambda^k p}$$

The spin density matrix is singular unless $\Lambda \mathrm{p} = \mathrm{p}$. The analytic continuation of the density operator is forced to be in the little group of p.

Massive particles

The constraint equation:

$$\Lambda p = p \implies \phi^{\mu\nu} p_\nu = 0$$

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \xi_\rho \frac{p_\sigma}{m} \qquad \xi^\rho = -\frac{1}{2m} \epsilon^{\rho\mu\nu\sigma} \phi_{\mu\nu} p_\sigma$$

Thanks to the constraint it is possible to make more simplifications

$$\mathrm{tr}\left(\exp[-in\phi : \Sigma/2]\right) = 4\cos\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right)$$

$$\mathrm{tr}(\gamma^\nu \gamma^\mu \gamma_5 \exp[-in\phi : \Sigma/2]) = 4i\tilde{\phi}^{\mu\nu} \sin\left(\frac{n}{2}\sqrt{\frac{\phi:\phi}{2}}\right) / \sqrt{\frac{\phi:\phi}{2}}$$

$$n\widetilde{b}(in\phi) \cdot p = \widetilde{b}(\phi) \cdot \sum_{k=1}^{n} \Lambda^k p = nb \cdot p$$

The series in the polarization vector formula can be summed for imaginary vorticity.

$$S^\mu(p) = \frac{-i\xi^\mu}{2\sqrt{-\xi^2}} \frac{\sin\left(\sqrt{-\xi^2}/2\right)}{\cos\left(\sqrt{-\xi^2}/2\right) + \mathrm{e}^{-b\cdot p+\zeta}}$$

The result is later continued to *any* real vorticity $\phi \to \mathrm{i}\varpi$

$$\xi^\mu \mapsto \theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma$$

$$S^\mu_E(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}} \frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p+\zeta}}$$

Corrections to **all orders in vorticity**

Similar arguments can be repeated for a generic spin-S field

$$\Theta(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta}\mathrm{e}^{-n\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\zeta}\mathrm{tr}\left(\mathrm{e}^{-n\boldsymbol{\xi}_0\cdot D^S(\mathbf{J})}\right)} \qquad \xi_0^\mu = [p]^{-1\mu}_{\ \ \nu}\xi^\nu = (0,\boldsymbol{\xi}_0)$$

From here we can compute the spin-vector and alignment, and anything else.

Mean spin vector for any spin

$$\boxed{S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sum_{k=-S}^{S} k\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}}-(-1)^{2S}\right]^{-1}}{\sum_{k=-S}^{S}\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}}-(-1)^{2S}\right]^{-1}}}$$

The formula:

- Reproduces linear results

$$S^{\mu}(p)_{\theta\to 0} \sim \theta^{\mu}\frac{S(S+1)}{3}(1+(-1)^{2S}n_{F/B}(b\cdot p-\zeta))$$

- Reproduces exact Boltzmann calculation

$$S_B^{\mu} = \frac{\theta^{\mu}}{\sqrt{-\theta^2}}\frac{\chi'(\sqrt{-\theta^2})}{\chi(\sqrt{-\theta^2})} \qquad \chi\left(\sqrt{-\theta^2}\right) = \sum_{k=-S}^{S} e^{k\sqrt{-\theta^2}}$$

- Is unitary

$$S^{\mu}(p)_{\theta\to\infty} \sim S\frac{\theta^{\mu}}{\sqrt{-\theta^2}}$$

Massless particles

For massless particles of arbitrary helicity S:

$$\Theta_{hk}(p) = \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} \mathrm{e}^{n\eta h} \delta_{hk}}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} \mathrm{e}^{-nb\cdot p} 2\cos n\eta}$$

$$\boxed{\Pi^{\mu}(p) = -p^{\mu} S \frac{\sinh(S\,H)}{\cosh(S\,H) - (-1)^{2S} \mathrm{e}^{-b\cdot p}}}$$

$$H = \frac{1}{2(p\cdot q)} \epsilon^{\mu\nu\alpha\beta} \varpi_{\alpha\beta} p_{\nu} q_{\mu} = -\frac{1}{2\varepsilon} \epsilon^{0\nu\alpha\beta} \varpi_{\alpha\beta} p_{\nu}$$

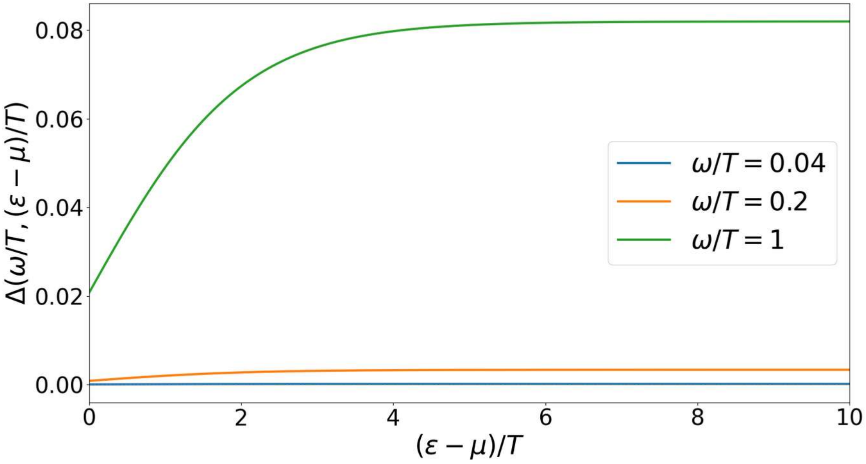
The reason for this simplified structure is that massless particles only have +S and -S helicity and that

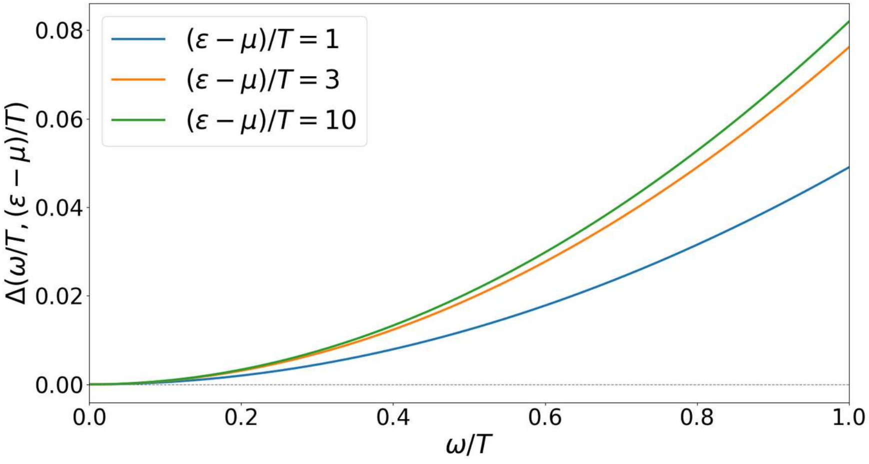
$$D^{S}(W(\Lambda^{n}, p))_{ts} = \mathrm{e}^{i\eta r} \delta_{hk} \qquad \widehat{\Pi}_{1,2}(p)|p, h\rangle = 0$$

Exact polarization in heavy-ion collisions

Consider the Λ polarization in relativistic heavy ion collisions $\omega \sim 10^{22}$ s^{-1} and $\omega/T \sim 0.04$.

$$\Delta = \left| \frac{S_E - S_L}{S_E} \right|$$

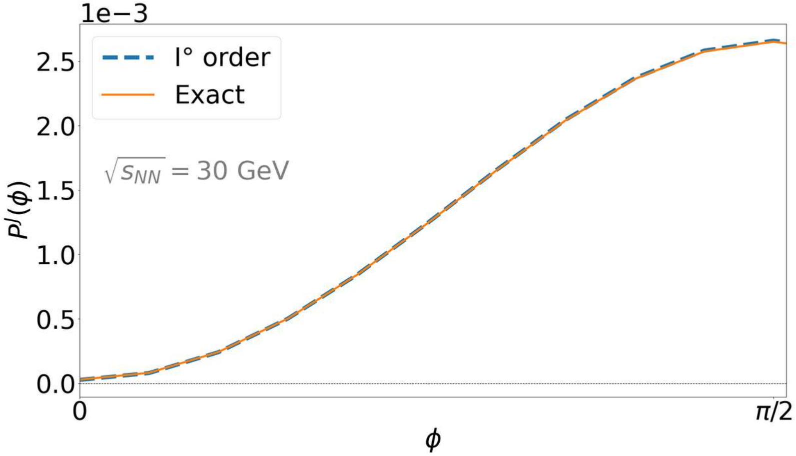


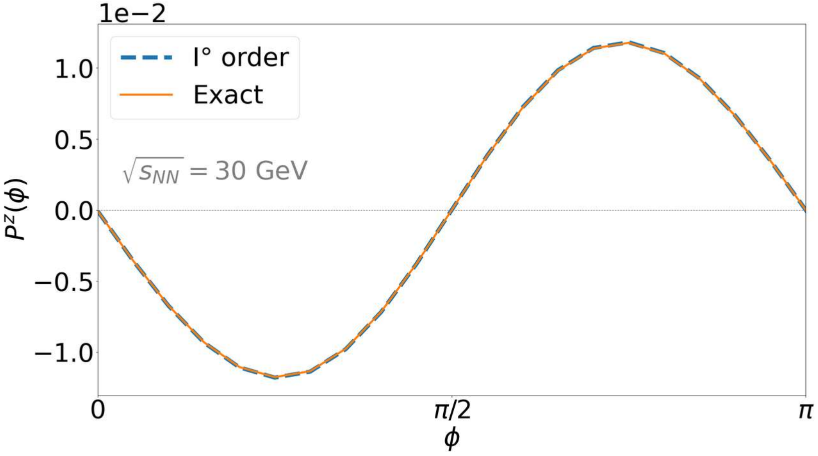


The difference is very small in most physical cases.

Extending the formula to local equilibrium with $\varpi(x)$

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int \mathrm{d}\Sigma\cdot p\, n_F \frac{\varpi_{\nu\rho}}{\sqrt{-\theta^2}}\frac{\sinh\left(\sqrt{-\theta^2}/2\right)}{\cosh\left(\sqrt{-\theta^2}/2\right)+\mathrm{e}^{-b\cdot p+\zeta}}}{\int \mathrm{d}\Sigma\cdot p\, n_F}$$





Conclusions

We reviewed the Pauli-Lubanski vector for massive and massless particles.

Formula for **polarization of massless particles.**

Exact Wigner function at general global equilibrium with thermal vorticity.

Exact spin polarization vector and **spin density matrix** for massive and massless particles at global equilibrium.

Higher order corrections in vorticity are negligible, at least in high energy collisions.

Thanks for the attention!

Backup

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\tilde{b}_\mu(\phi)\widehat{P}^\mu]\widehat{\Lambda}\ \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\right)$$

$$[\widehat{a}_s^\dagger(p), \widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin S):

$$\begin{aligned}\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle =&(-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle + \\ &+ 2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')\end{aligned}$$

$D(W) = [\Lambda p]^{-1}\Lambda[p]$ is the “Wigner rotation” in the S-spin representation.

We find a solution by iteration:

I

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

II

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\,(-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot(\Lambda p+\Lambda^2 p)}\delta^3(\Lambda^2\boldsymbol{p}-\boldsymbol{p}')+ \\ +\,2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

∞

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\boldsymbol{p}-\boldsymbol{p}')D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^n \Lambda^k p}$$

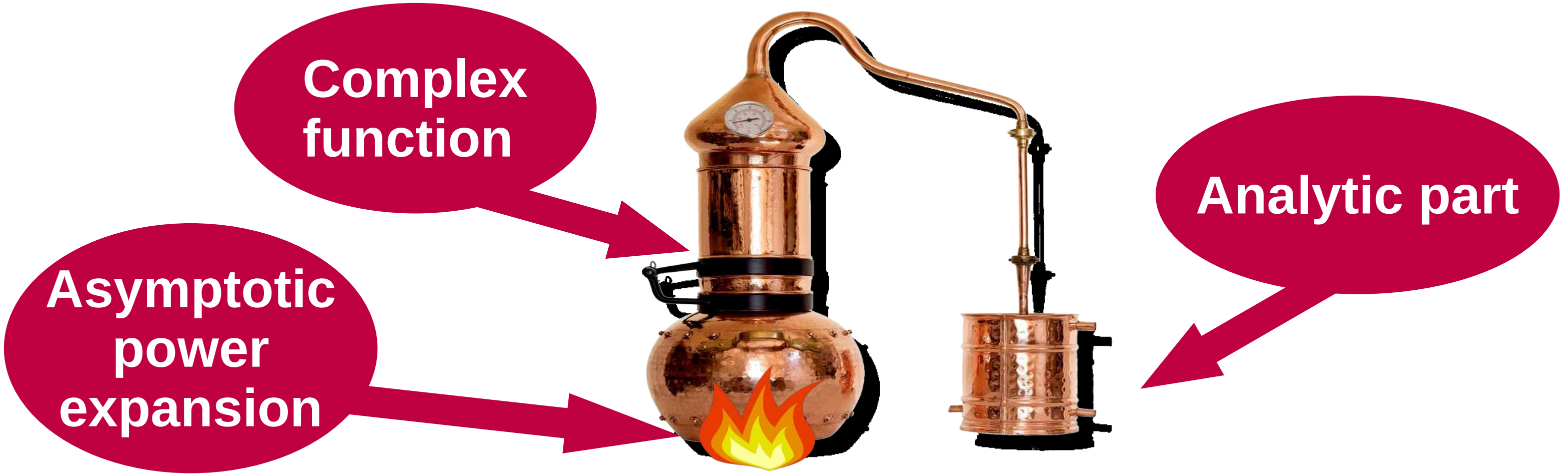
Energy density for massless fermions, equilibrium with acceleration ($\phi = ia/T$)

$$\rho = \frac{3T^4}{8\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5(n\phi/2)}$$

The series is finite as long as ϕ is real. For real thermal vorticity it diverges!

The series includes terms which are non analytic at $\phi = 0$.

Analytic distillation:



The series boils down to polynomials: $\alpha^\mu = \frac{A^\mu}{T}$ $w^\mu = \frac{\omega^\mu}{T}$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}$$

Expectation values vanish at the Unruh temperature $T_U = \sqrt{-A\cdot A}/2\pi$
[G.Prokhorov, O. Teryaev, V. Zakharov, JHEP03(2020)137]

Axial current under rotation: [V. Ambrus, E. Winstanley, 1908.10244]

$$j_A^\mu = T^2\left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2}\right)\frac{w^\mu}{\sqrt{\beta^2}}$$

First exact results at equilibrium with **both rotation and acceleration.**
[V. Ambrus, E. Winstanley Symmetry 2021, 13(11)]

$$\rho = T^4\left(\frac{7\pi^2}{60} - \frac{\alpha^2}{24} - \frac{w^2}{8} - \frac{17\alpha^4}{960\pi^2} + \frac{w^4}{64\pi^2} + \frac{23\alpha^2w^2}{1440\pi^2} + \frac{11(\alpha\cdot w)^2}{720\pi^2}\right)$$

Massless Dirac field

$$\Pi^\mu = \frac{p^\mu}{2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n} \Lambda^k p} \mathrm{tr}\left(\not{q}\gamma_5 \exp[-in\phi : \Sigma/2]\not{p}\right) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n} \Lambda^k p} \mathrm{tr}\left(\not{q} \exp[-in\phi : \Sigma/2]\not{p}\right) \delta^3(\Lambda^n p - p)}$$

We can deal with the series as we did before

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \frac{h_\rho p_\sigma}{p \cdot q} \qquad\qquad h^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\nu\rho} q_\sigma$$

The series reduces to

$$\eta = \frac{h \cdot p}{q \cdot p} = \frac{1}{2(p \cdot q)} \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta} p_\nu q_\mu$$

$$\Pi^\mu(p) = \frac{p^\mu}{2} \frac{i \sin(\eta/2)}{\cos(\eta/2) + e^{-b\cdot p}}$$

Dirac fermions:

$$\Pi^\mu(p) = -\frac{p^\mu}{2} \frac{\sinh(H/2)}{\cosh(H/2) + \mathrm{e}^{-b \cdot p}}$$

$$H = \frac{1}{2(p \cdot q)} \epsilon^{\mu\nu\alpha\beta} \varpi_{\alpha\beta} p_\nu q_\mu = -\frac{1}{2\varepsilon} \epsilon^{0\nu\alpha\beta} \varpi_{\alpha\beta} p_\nu$$

The mean Pauli-Lubanski vector depends on the orientation of the momentum

$$\varphi^\nu = -\frac{1}{2} \epsilon^{\nu\alpha\beta 0} \varpi_{\alpha\beta} \qquad\qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{\varepsilon}$$