

# Anomalous transport coefficients from lattice QCD

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- Expectations
- Lattice QCD formulation
- Results
- Summary



# Introduction

- Discuss two anomalous transport coefficients:
- Chiral Magnetic Effect  $\equiv \mathbf{CME}$ ,
  - an **electric current** in the presence of **chiral imbalance** and a **magnetic field**,
  - parallel to the magnetic field:

$$\langle J_z \rangle = \langle \bar{\psi} \gamma_z \psi \rangle = c_{\text{CME}} \mu_5 q B_z .$$

- Chiral Separation Effect  $\equiv \mathbf{CSE}$ ,
  - a **chirality current** in the presence of **charge imbalance** and a **magnetic field**,
  - parallel to the magnetic field:

$$\langle J_z^5 \rangle = \langle \bar{\psi} \gamma_5 \gamma_z \psi \rangle = c_{\text{CSE}} \mu q B_z .$$

- Derivatives of the currents yield the **coefficients**:

$$c_{\text{CME}} = \frac{d^2 \langle J_z \rangle}{d\mu_5 dB_z} \Bigg|_{\mu_5=B=0}, \quad c_{\text{CSE}} = \frac{d^2 \langle J_z^5 \rangle}{d\mu dB_z} \Bigg|_{\mu=B=0} .$$

# Introduction

- Using **lattice QCD**:
- The **partition function** is

$$Z = \int \mathcal{D}\mathcal{U} \bar{\psi} \mathcal{D}\psi e^{iS[\mathcal{U}, \bar{\psi}, \psi]} = \int \mathcal{D}\mathcal{U} \bar{\psi} \mathcal{D}\psi e^{-S_E[\mathcal{U}, \bar{\psi}, \psi]},$$

- where  $S_E$  is the **Wick-rotated, finite temperature** action of QCD

$$S_E = \int_0^{1/T} d\tau \int d^3x \frac{\text{Tr } F^2(x, \tau)}{2g^2} + \sum_f \bar{\psi}^{(f)}(x, \tau) \left( \not{D} + m^{(f)} \right) \psi^{(f)}(x, \tau).$$

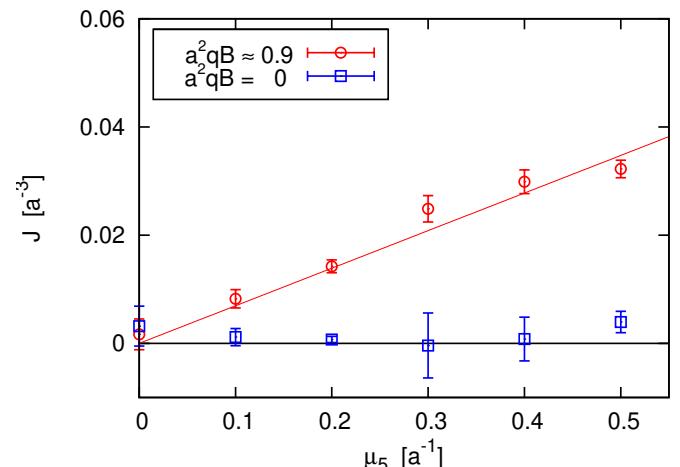
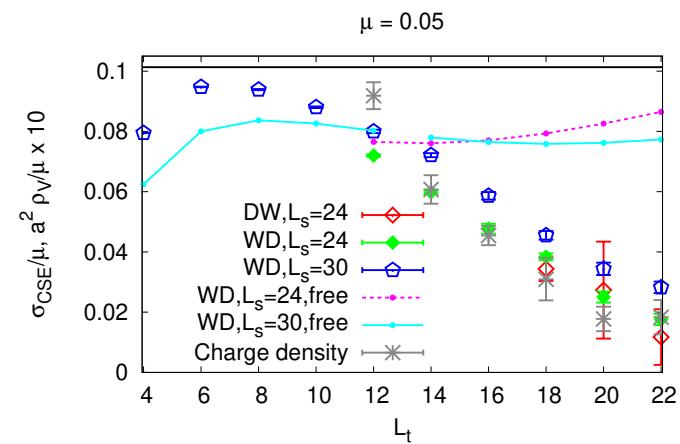
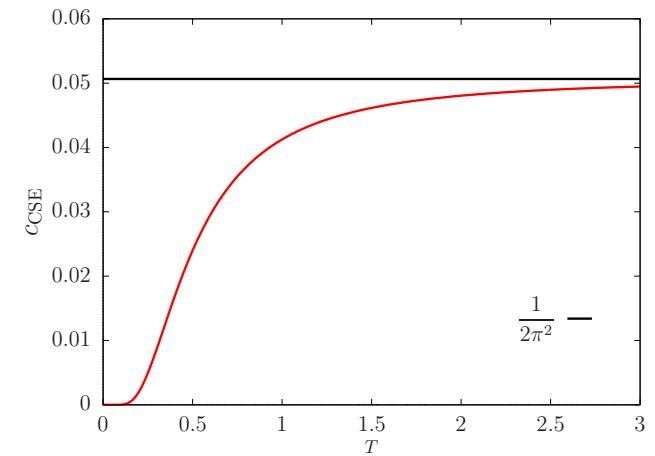
- This is inherently **equilibrium!**
- **Observables** are

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\mathcal{U} \bar{\psi} \mathcal{D}\psi e^{-S_E[\mathcal{U}, \bar{\psi}, \psi]} O[\mathcal{U}, \bar{\psi}, \psi].$$

- Discretize the action, with **lattice spacing**  $a$ , and use **importance sampling Monte Carlo** integration to evaluate the path integral.

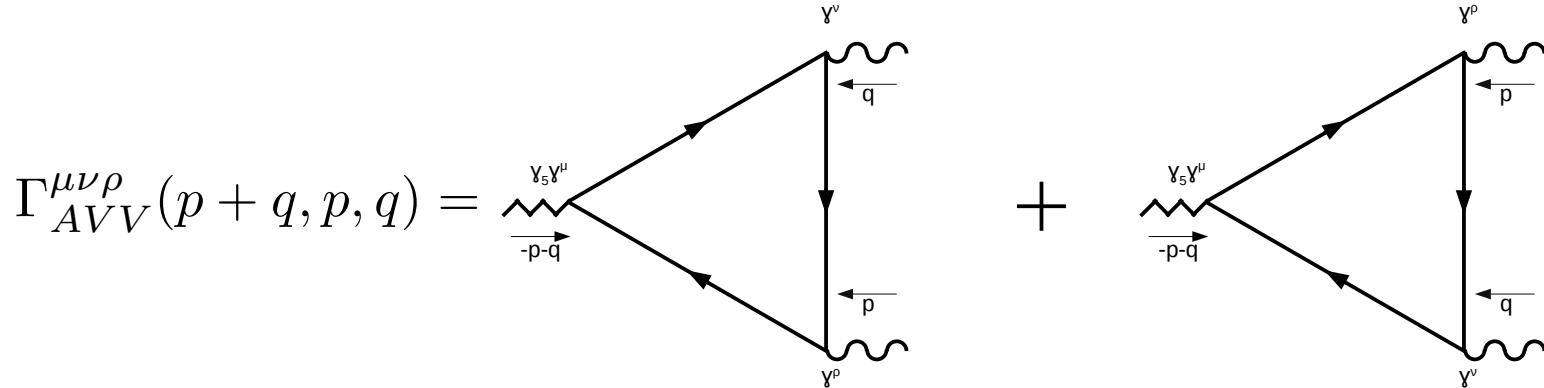
# Expectations

- Perturbative calculations
  - Gluonic **interactions neglected**, except as source of chiral imbalance.
  - $c_{\text{CSE}}$  interpolates between 0 and  $1/(2\pi^2)$  as  $T/m$  goes from  $0 \rightarrow \infty$ . [1]
  - $c_{\text{CME}}$  is calculated in- and out-of-equilibrium, different results 0 or  $1/(2\pi^2)$ . [2, 3, 4]
  - $c_{\text{CME}}$  is sensitive to proper **regularization!** [5]
- Very few lattice results
  - $c_{\text{CSE}}$  in QC<sub>2</sub>D, **compatible** with perturbative results. [6]
  - $c_{\text{CME}}$  QCD with 2, identical flavor, Wilson fermions, **neither compatible** with 0 NOR  $1/(2\pi^2)$ . [7]



# Expectations: regulator sensitivity

Text book example: the triangle anomaly

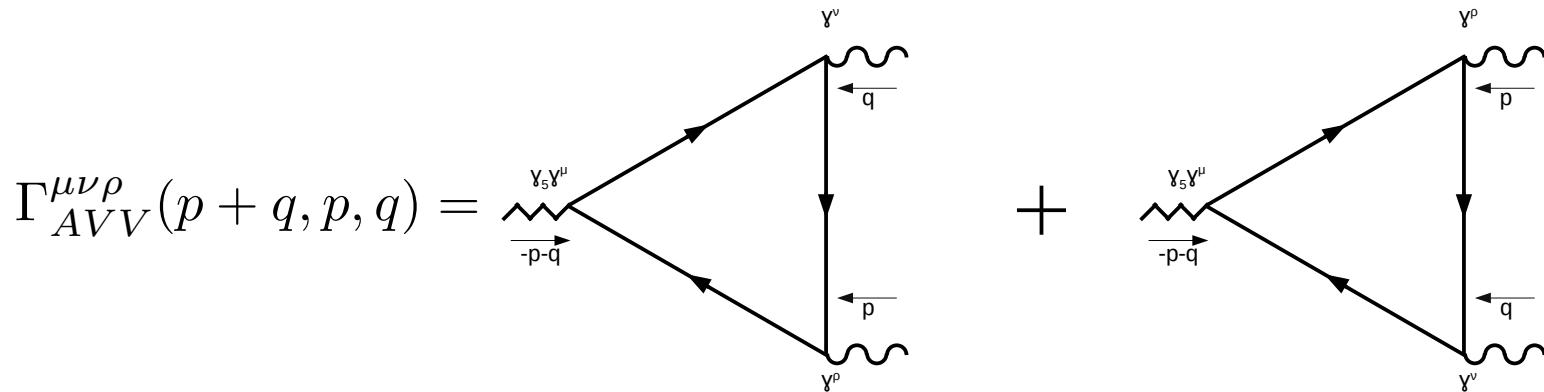


- Massive fermions
- No regularization

$$(p+q)_\mu \Gamma_{AVV}^{\mu\nu\rho}(p+q, p, q) = m P_5^{\nu\rho}(p, q)$$

# Expectations: regulator sensitivity

Text book example: the triangle anomaly



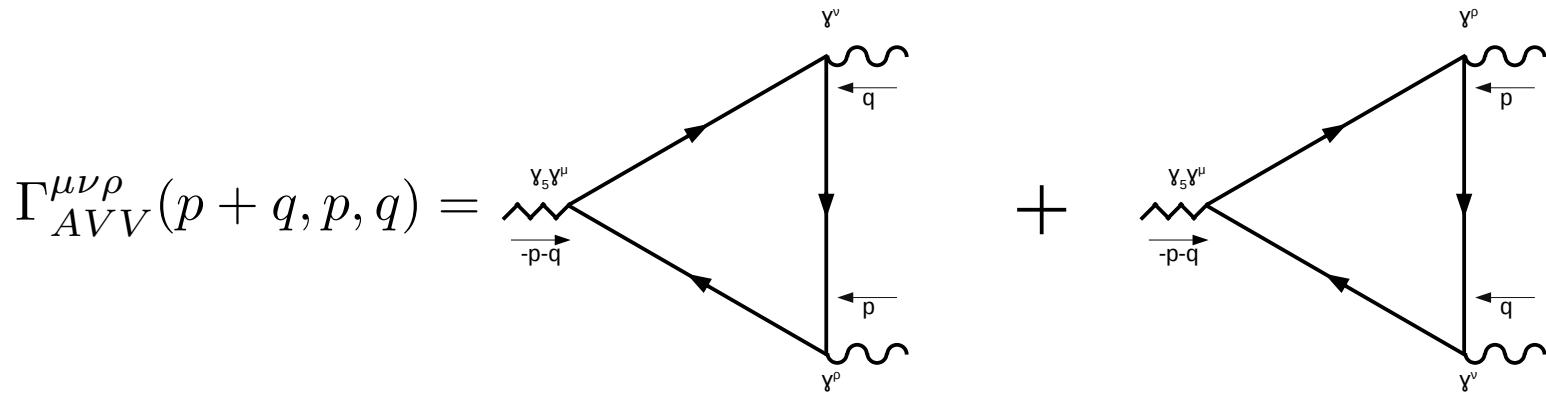
- Massive fermions
- Pauli-Villars regularization
- New particles, with coeffs  $c_s$  and masses  $m_s \rightarrow \infty$ ,  $s = 0, 1, 2, 3$ ,  $s = 0$  the physical fermion.

$$(p+q)_\mu \Gamma_{AVV}^{\mu\nu\rho}(p+q, p, q) = m P_5^{\nu\rho}(p, q) + \sum_{s=1} c_s m_s P_{5,s}^{\nu\rho}(p, q)$$

$$\rightarrow m P_5^{\nu\rho}(p, q) + \frac{\epsilon^{\alpha\beta\nu\rho} q_\alpha p_\beta}{4\pi^2}$$

# Expectations: regulator sensitivity

- $c_{\text{CME/CSE}}$  can also be written with the triangle diagram:



$$J_3 \sim A_3, \quad J_3^5 \sim A_3^5, \quad B_3 = q_1 A_2, \quad \mu = A_0, \quad \mu_5 = A_0^5.$$

$$c_{\text{CME}} = \lim_{p,q,p+q \rightarrow 0} \frac{1}{q_1} \Gamma^{023}(p+q, q, p),$$

$$c_{\text{CSE}} = \lim_{p,q,p+q \rightarrow 0} \frac{1}{q_1} \Gamma^{320}(p+q, q, p).$$

# Expectations: regulator sensitivity

$$\begin{aligned} \Gamma_{AVV}^{\mu\nu\rho}(p+q, p, q) = & -i \sum_{\mathbf{s}=0} \textcolor{blue}{c_s} \int_K \frac{\text{Tr} [\gamma^\mu \gamma_5 (\not{K} + m_s) \gamma^\nu (\not{K} + \not{q} + m_s) \gamma^\rho (\not{K} + \not{q} + \not{p} + m_s)]}{(K^2 - m_s^2)((K+q)^2 - m_s^2)((K+q+p)^2 - m_s^2)} \\ & + (\{\nu, q\} \leftrightarrow \{\rho, p\}) . \end{aligned}$$

$$\mu_5 = A_0^5(p+q=0), \quad A_2(q_0=0, \mathbf{q}=(q_1, 0, 0)) = \frac{B}{q_1} \Rightarrow \Gamma_{AVV}^{023}(0, -q_1, q_1)$$

Evaluating the trace and writing out the Matsubara sum

$$\begin{aligned} \Gamma_{AVV}^{023}(0, -q, q) = & -8\varepsilon^{1230} \sum_{\mathbf{s}=0} \textcolor{blue}{c_s} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{2q_1 m_s^2 + 2k_0(q_0 k_1 - q_1 k_0)}{(K^2 - m_s^2)^2 ((K+q)^2 - m_s^2)} \right. \\ & \left. + \frac{k_1 + q_1}{(K^2 - m_s^2)((K+q)^2 - m_s^2)} \right]_{k_0=i\omega_n} \end{aligned}$$

Take  $q_0 \rightarrow 0$  and evaluate the Matsubara sum as well as the angle integrals

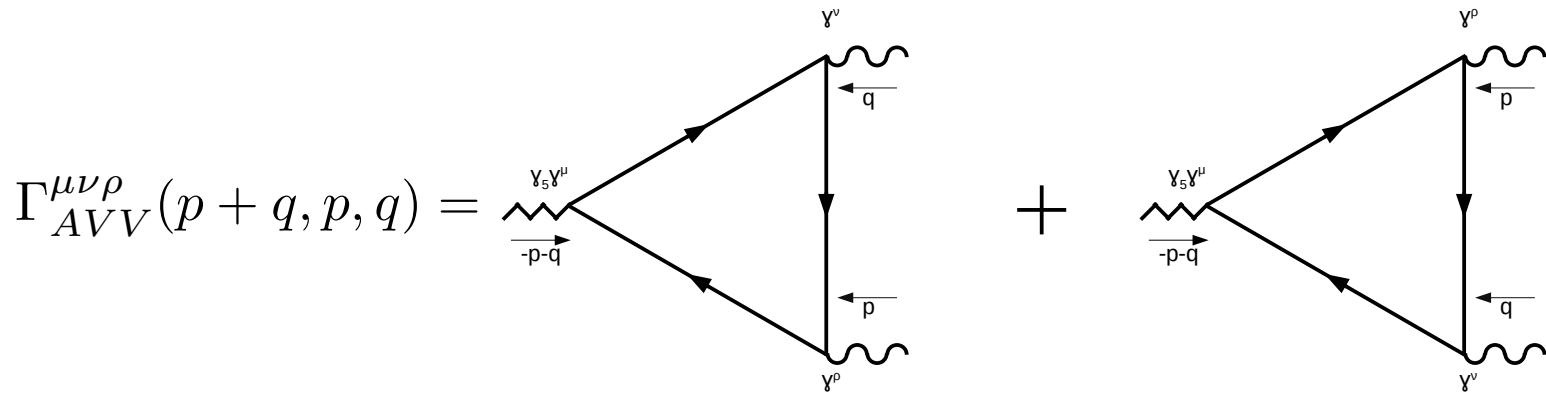
$$\Gamma_{AVV}^{023}(0, -q_1, q_1) = -\frac{1}{2\pi^2} \sum_{\mathbf{s}=0} \textcolor{blue}{c_s} \int_0^\infty dk k \left( \frac{m_s^2(\frac{1}{2} - n_F(E_k))}{E_k^3} + \frac{k^2}{E_k^2} n'_F(E_k) \right) \log \frac{(2k - q_1)^2}{(2k + q_1)^2} .$$

$$c_{\text{CME}} = \lim_{q_1 \rightarrow 0} \frac{\Gamma_{AVV}^{023}(0, -q_1, q_1)}{q_1} = \frac{1}{2\pi^2} \sum_{\mathbf{s}=0} c_s m_s^2 \underbrace{\int_0^\infty dk \frac{1}{(k^2 + m_s^2)^{3/2}}}_{1/m_s^2} + \underbrace{(T \neq 0)}_{\text{cancels!}}$$

$$= \frac{1}{2\pi^2} \left( 1 + \underbrace{\sum_{s=1}^{\infty} \textcolor{blue}{c_s}}_{-1} \right) = 0 , \quad \text{in agreement with [4].}$$

# Expectations: regulator sensitivity

- $c_{\text{CME/CSE}}$  can also be written with the triangle diagram:



$$J_3 \sim A_3, \quad J_3^5 \sim A_3^5, \quad B_3 = q_1 A_2, \quad \mu = A_0, \quad \mu_5 = A_0^5.$$

$$c_{\text{CME}} = \lim_{p,q,p+q \rightarrow 0} \frac{1}{q_1} \Gamma^{023}(p+q, q, p) = \frac{1}{2\pi^2} + \sum_{s=1} \frac{c_s}{2\pi^2} = 0,$$

$$c_{\text{CSE}} = \lim_{p,q,p+q \rightarrow 0} \frac{1}{q_1} \Gamma^{320}(p+q, q, p) = -\frac{1}{\pi^2} \int_0^\infty dk n'_F(E_k).$$

- $c_{\text{CME}}$  is zero due to anomalous contribution!
- $c_{\text{CSE}}$  agrees with known results [1].

# Lattice QCD formulation

- We can simulate in homogeneous  $B$  background, but not at finite  $\mu$ .
- Measure derivatives of the currents, at different  $B$ -s and read off linear coefficient.
- For completeness, at small  $B$

$$c_{\text{CME}}B = \frac{d\langle J_z \rangle}{d\mu_5} \Bigg|_{\mu_5=0} = \langle J_z J_0^5 \rangle_{\mu_5=0} + \left\langle \frac{\partial J_z}{\partial \mu_5} \right\rangle_{\mu_5=0},$$
$$c_{\text{CSE}}B = \frac{d\langle J_z^5 \rangle}{d\mu} \Bigg|_{\mu=0} = \langle J_z^5 J_0 \rangle_{\mu=0} + \left\langle \frac{\partial J_z^5}{\partial \mu} \right\rangle_{\mu=0}.$$

- First, clarify that definitions are correct in the free case.
- Then, turn on gluonic interactions.

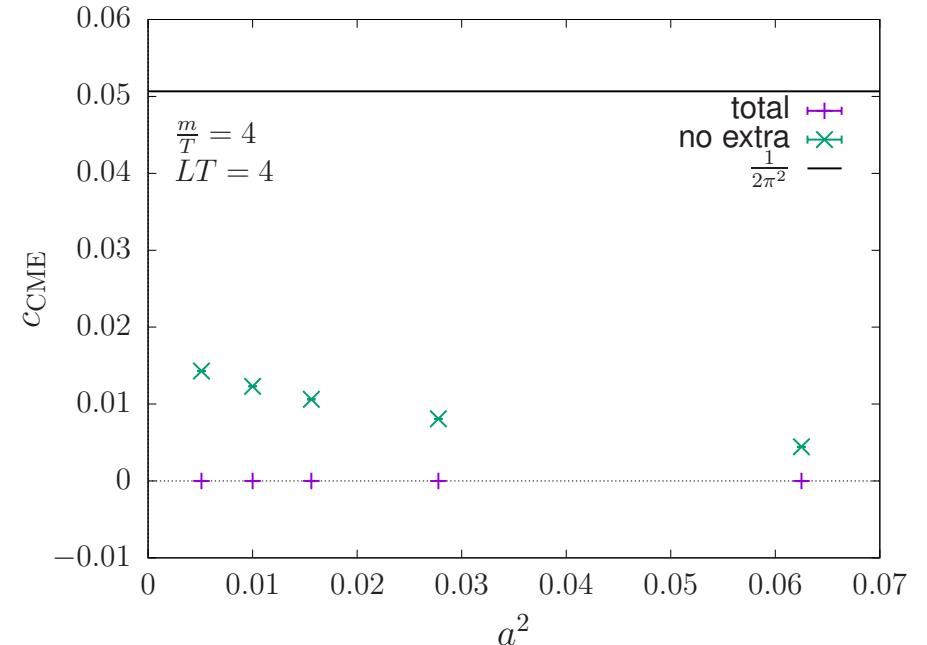
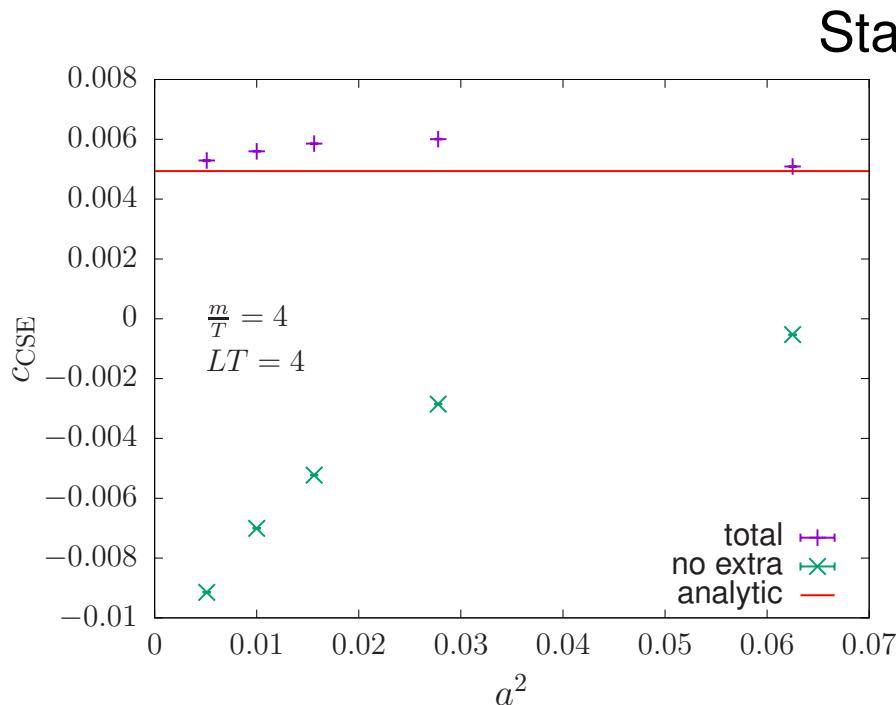
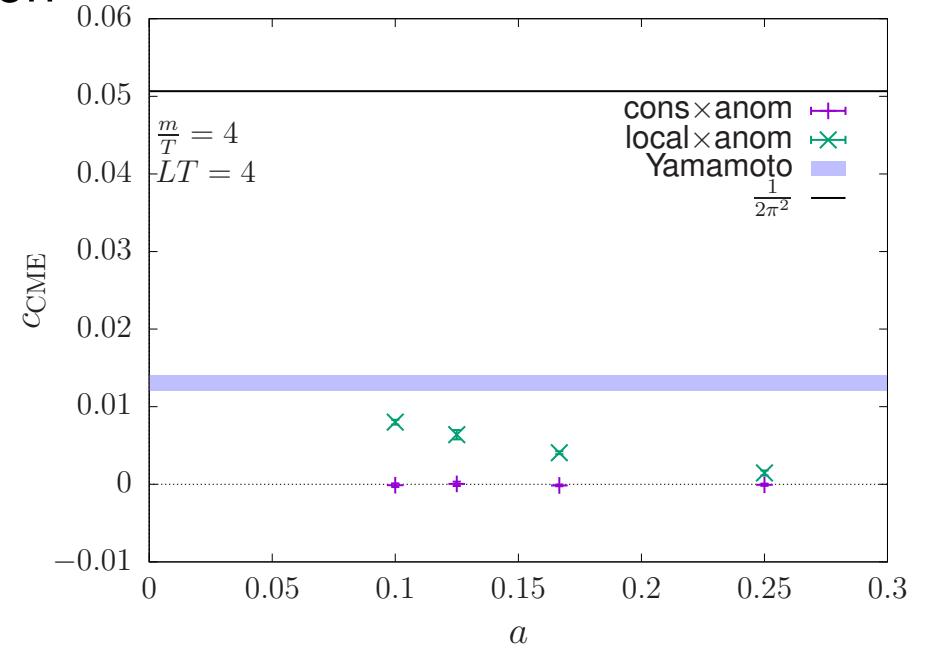
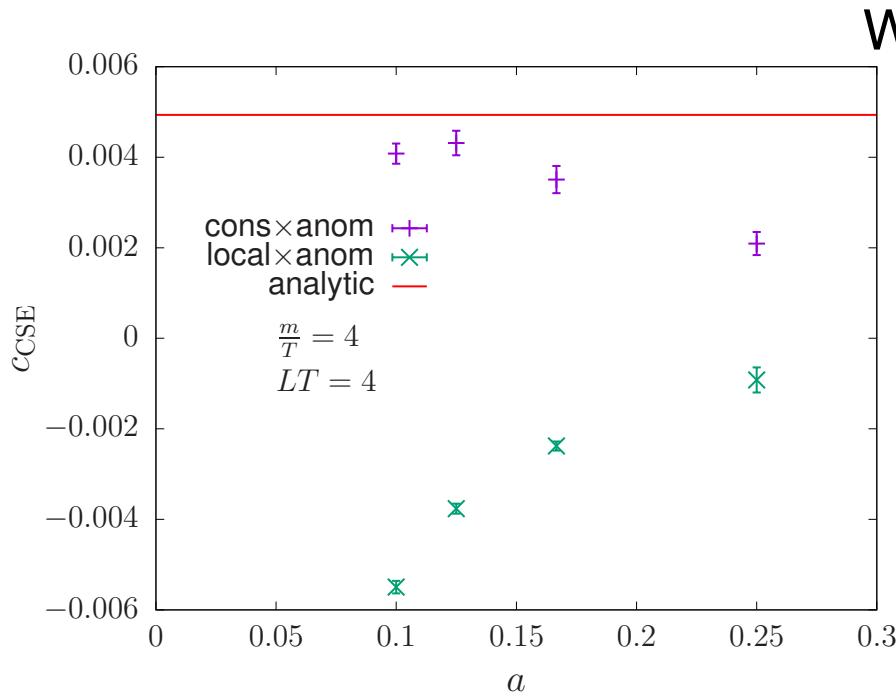
# Lattice QCD formulation

- Regularization is important, **anomalous contributions**:

$$\text{divergence} \times (\text{cutoff} - \text{suppressed}) = \text{finite}$$

- Appears differently in different fermion discretizations.
- Wilson fermions
  - So-called doublers are given a **cutoff dependent mass** to decouple in the continuum limit ( $\leftrightarrow$  PV).
  - Point-split currents are defined so **(anomalous) Ward identities are fulfilled**.
  - Local currents **violate** the (A)WI. (Used e.g. in spectroscopy, where it does not matter.)
- Staggered fermions
  - Dirac and flavor structure is **mixed with coordinate dependence** to reduce doubling problem.
  - Conserved (anomalous) currents pick up **explicit (chiral) chemical potential dependence**.
  - Extra  $\left\langle \frac{\partial J_z}{\partial \mu_5} \right\rangle$ ,  $\left\langle \frac{\partial J_z^5}{\partial \mu} \right\rangle$  terms appear!

# Lattice QCD formulation: free results



# Lattice QCD formulation: chirality

Naturally

$$n_5 \Big|_{\mu_5=0} = \frac{T}{V} \frac{d \log Z}{d \mu_5} \Big|_{\mu_5=0} = 0.$$

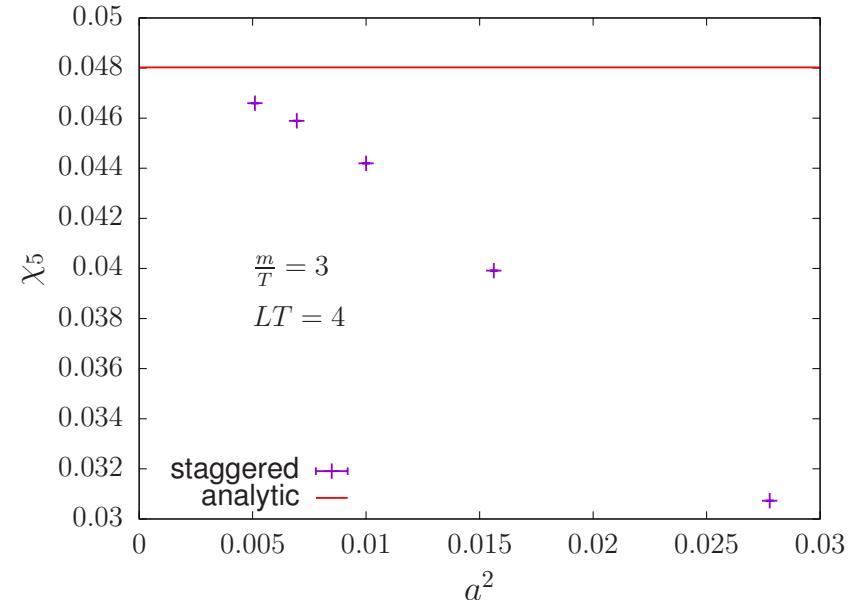
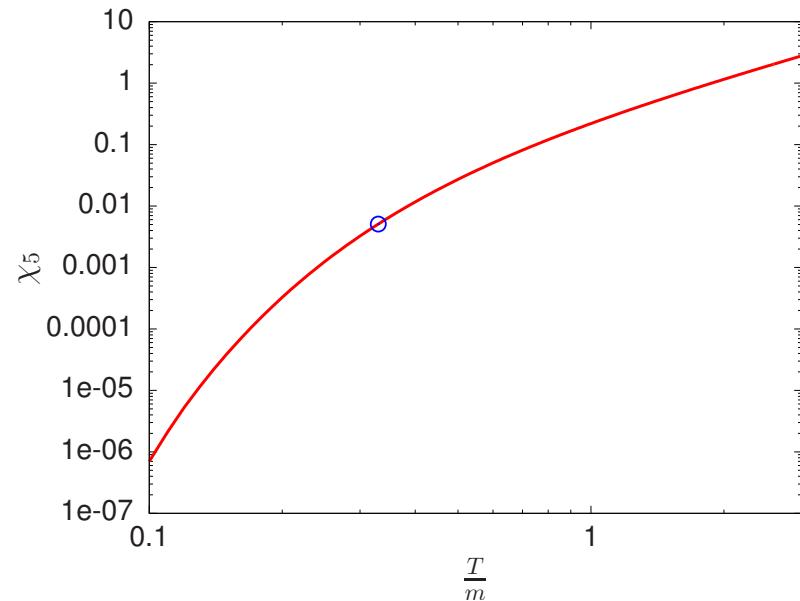
But

$$n_5(\mu_5) = \frac{T}{V} \frac{d^2 \log Z}{d \mu_5^2} \Big|_{\mu_5=0} \mu_5 + \mathcal{O}(\mu_5^2) = \chi_5 \mu_5 + \mathcal{O}(\mu_5^2).$$

In PV

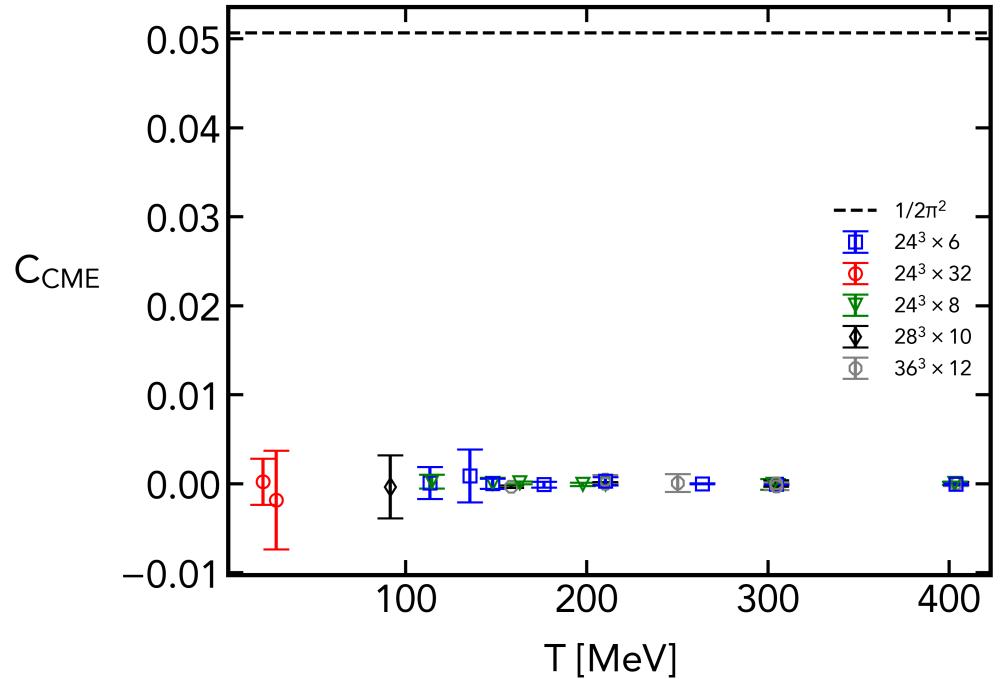
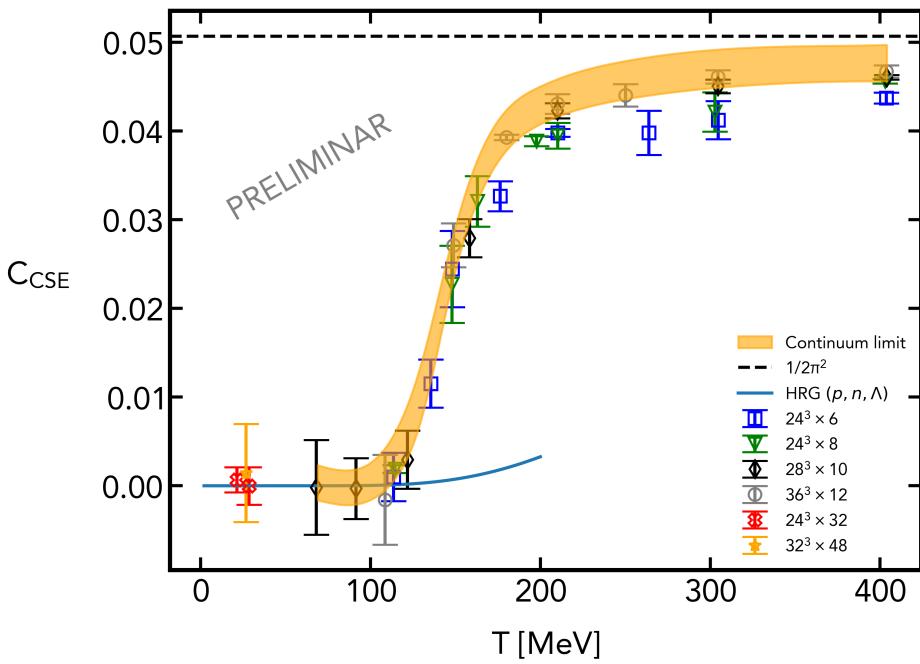
$$\chi_5 = -\frac{1}{32\pi^2} \sum_{s=0} c_s m_s^2 \left( 2 \log \frac{m_s^2}{m^2} - 8 \right) + \frac{4}{\pi^2} \int_0^\infty dk \frac{k^2 n_F(E_k)}{E_k}.$$

**Divergent**, renormalization needed, here by  $T = 0$  subtraction for easy comparison to lattice



# Lattice QCD formulation: interacting results

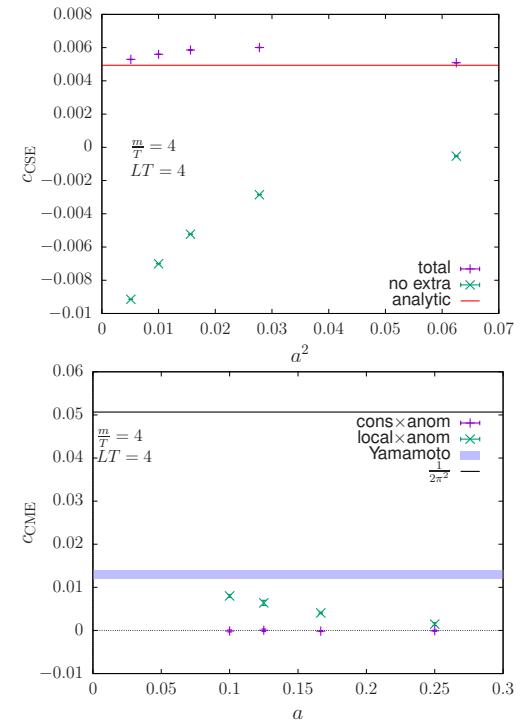
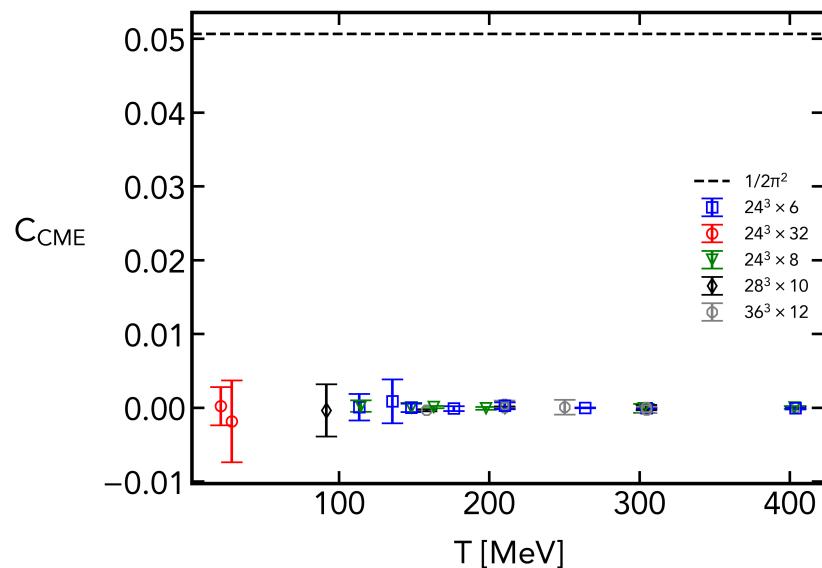
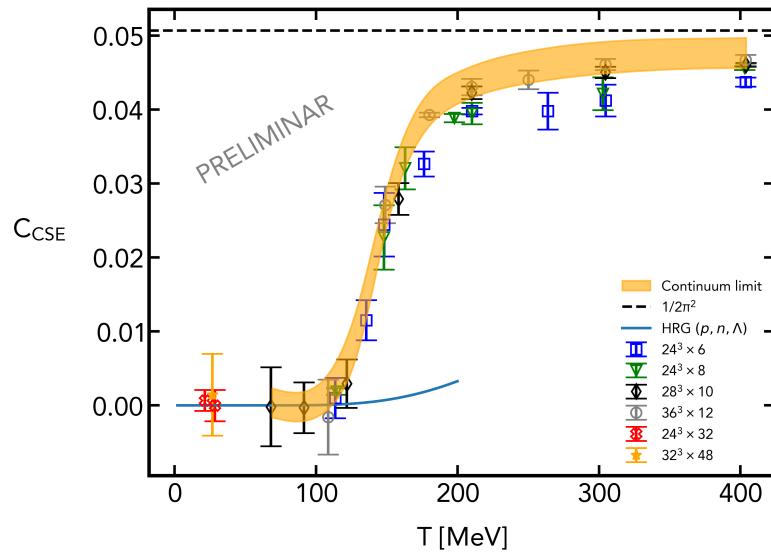
2+1 flavor, staggered, physical pion mass



- $c_{\text{CSE}}$  changes quickly around the transition temperature.
- Approaches the free limit at high temperature.
- $c_{\text{CME}}$  temperature independently vanishes.

# Summary

- Free results show sensitivity to proper regularization.
- It looks plausible that interacting results also suffer from this.
- We presented the first 2+1 flavor, physical point, full QCD results for  $c_{\text{CSE}}$  and  $c_{\text{CME}}$ .



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