Casimir Effect in Kinetic Theory

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Quantum Kinetic Theory

- Widely studied, not only in heavy-ion physics.
- Covers both equilibrium and non-equilibrium situations.
- Kinetic theory for gauge field relatively less studied (see Hattori, Hidaka, Yamamoto, Yang, JHEP 2021, 1; Huang, Mitten, Sadofyev, Speranza, JHEP 2020 117; Lin, PRD 105 7 076017; Chen, Lin, PRD 105 1 014015…)
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- Finite-size effect could be important.

- What is the finite-size quantum effect of a gauge field? Casimir effect

Casimir Effect

- Shifted QED vcuum energy between two conductive plate.
- Infinitely large, but its variation (Casimir force) is finite and measurable.
- Recent advances:
  - Repulsive Casimir force [Munday, Capasso, Parsegian, Nature 457 170]
  - Lateral Casimir force [Manjavacas, Rodriguez-Fortuno, Garcia de Abajo, Zayats, PRL 118 133605; Bao, Shi, Cao, Evans, He, PRL 121 130401]
  - Casimir torque [P. K. M. J. Somers DAT, Garrett JL, Nature 564 386]
  - Tailoring of Casimir force [K. S. J. I. e. A. Intravaia F., Nat Commnun 4]
  - …

Casimir Effect: Conventional Calculation

- Electromagnetic standing waves

$$\phi_n = e^{-i\omega_n t + ik_x x + ik_y y} \sin(k_n z), k_n = \frac{n\pi}{a}$$

$$\mathscr{E} = \int \frac{\mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^2} \sum_{n=0}^{\infty} \omega_n$$

$$= -\frac{\hbar\pi^2}{6a^3}\zeta(-3) = -\frac{\hbar\pi^2}{720a^3}$$

Zeta regularization

$$F_0 = -\frac{\mathrm{d}E}{\mathrm{d}a} = -\frac{\hbar\pi^2}{240a^4}$$

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- What if there is not vacuum between the plates?
  - Finite temperature
  - Non-equilibrium

Pure U(1) Gauge Field

- Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- Field equation:

$$\partial^\nu \partial_\nu A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \xrightarrow[\partial^\mu A_\mu = 0]{\text{Lorenz Gauge}} \partial^2 A_\mu = 0$$

Wigner Function

$$G_{\mu\nu}(x,p) = \int \mathrm{d}^4 y e^{-ip\cdot y} \langle A_\mu(x+\frac{y}{2}) A_\nu(x-\frac{y}{2})\rangle$$

- 'Hermitian':

$$G_{\nu\mu} = G^*_{\mu\nu}$$

- Kinetic equation:

$$(p_\sigma \pm \frac{i}{2}\partial_{x\sigma})(p^\sigma \pm \frac{i}{2}\partial_x^\sigma) G_{\mu\nu} \equiv p^\pm_\sigma p^{\pm\sigma} G_{\mu\nu} = 0$$

- Gauge-fixing condition:

$$p^{\pm\mu} G_{\mu\nu} = 0$$

Kinetic equation

- Separate the symmetric and anti-symmetric part

$$G_{\mu\nu} = G^{+}_{\mu\nu} + G^{-}_{\mu\nu}, G^{\pm}_{\mu\nu} \equiv (G_{\mu\nu} \pm G_{\nu\mu})/2$$

- G^{+} is real and G^{-} is pure imaginary.
- Transport equation

$$p \cdot \partial G^{\pm}_{\mu\nu} = 0$$

- On-shell condition

$$(p^2 - \partial^2/4) G^{\pm}_{\mu\nu} = 0$$

- Gauge-fixing condition

$$p^{\mu} G^{\pm}_{\mu\nu} = \partial^{\mu} G^{\pm}_{\mu\nu} = 0$$

Current and Energy-Momentum Tensor

$$j_\mu(x,p) = \frac{\partial_\mu}{2} G^\nu{}_\nu - \partial^\nu G^+_{\mu\nu}$$

$$j_{5\mu}(x,p) = -\, i\epsilon^{\mu\nu\sigma\rho} p_\nu G^-_{\sigma\rho}$$

$$t_{\mu\nu}(x,p) = p_\mu p_\nu G^\sigma{}_\sigma + p^2 G^+_{\mu\nu} - \frac{g_{\mu\nu}}{2}\left(p^2 G^\sigma{}_\sigma - p^\sigma p^\rho G^+_{\rho\sigma}\right)$$

$$\longrightarrow$$

$$J_\mu = F_{\mu\nu} A^\nu$$

$$J_{5\mu} = \epsilon_{\mu\nu\sigma\rho} A^\nu F^{\sigma\rho}$$

$$T_{\mu\nu} = F_{\mu\sigma} F^\sigma{}_\nu - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}$$

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- G^+: number density

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- G^+: number density
- G^-: chiral imbalance

Free-streaming Solution

- Assumptions:
  - On shell: $p^2 G = 0$
  - Distribution functions: $G^{\pm}_{\mu\nu}(x, p) = C^{\pm}_{\mu\nu}(p) f_{\pm}(x, p)$

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- Solution:

$$C^{+}_{\mu\nu} = \frac{p_\mu p_\nu}{(p\cdot u)^2} - \frac{(p_\mu u_\nu + p_\nu u_\mu)}{p\cdot u} + g_{\mu\nu}$$

$$C^{-}_{\mu\nu} = i\epsilon_{\mu\nu\sigma\rho}\frac{p^\sigma u^\rho}{2p\cdot u}$$

Free-streaming Solution

- Kinetic equations for f_+

$$p \cdot \partial f_+ = 0$$

$$\partial^2 f_+ = 0$$

$$(p_\mu u \cdot \partial - p \cdot u \partial_\mu) f_+ = 0$$

- General solution:

$$f_+(t, \vec{x}, \vec{p}) = f_+(t_0, \vec{x}_0 - \frac{\vec{p}}{E_p}(t - t_0), \vec{p})$$

- Energy:

$$t_{00} = 2p_0^2 f_+$$

Boundary Conditions

- Two parallel conducting plates at $z = 0$ and $z = a$:

$$E_x|_{z=0} = E_y|_{z=0} = E_x|_{z=a} = E_y|_{z=a} = 0$$

$$B_z|_{z=0} = B_z|_{z=a} = 0$$

- Solution:

$$f(x,p) \to \tilde{f}(x,p) \sum_n \delta(p_z - \frac{n\pi}{a})$$

Equilibrium System

- Equilibrium, finite temperature

$$f_+ = \frac{1}{2} + \frac{1}{e^{\epsilon_p/T} - 1}$$

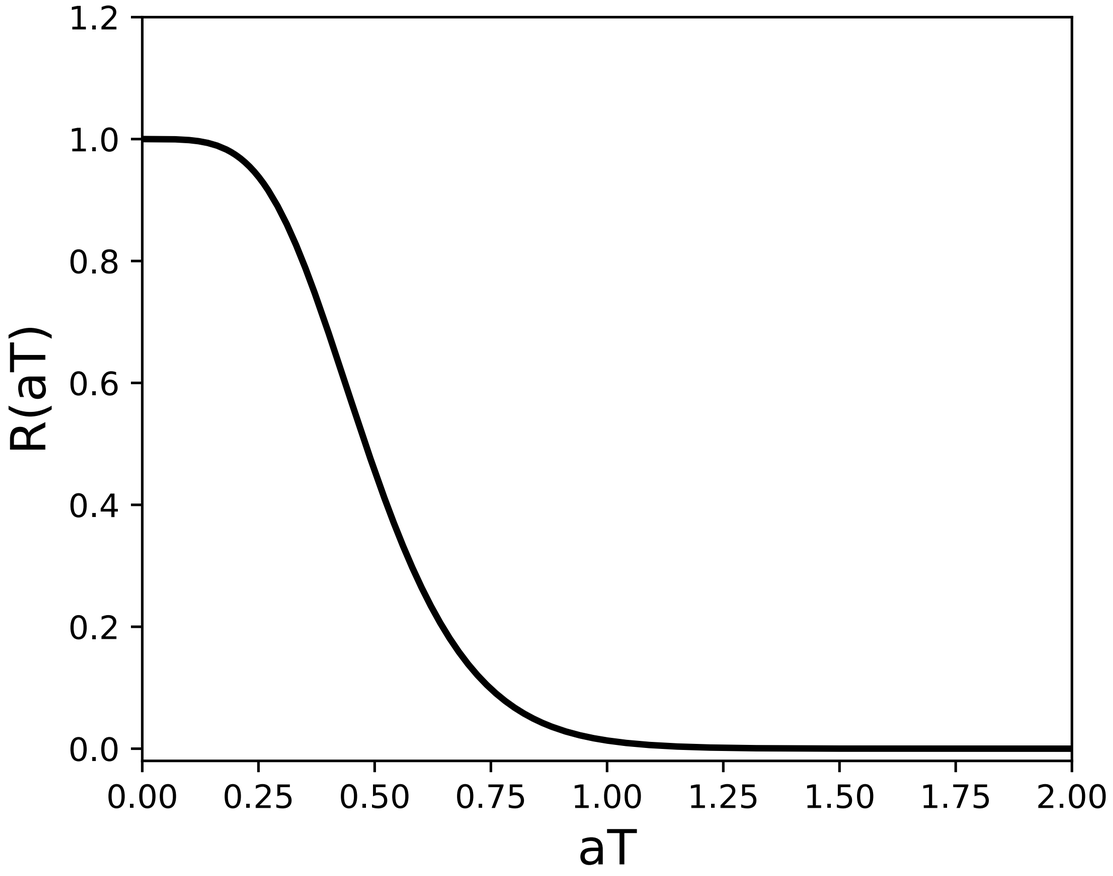
$$\Delta\mathscr{E} = 2\left(\sum_{n=0}^{\infty} - \int_0^{\infty} dn\right)\int \frac{d^2p_\perp}{(2\pi)^2}\epsilon_p f_+(p)$$

$$F = -\frac{\partial\Delta\mathscr{E}}{\partial a} = -\frac{\hbar\pi^2}{240a^4} + \frac{\pi^2}{15}T^4 - \frac{\pi^2}{a^4}\sum_{n=0}^{\infty}\frac{n^3}{e^{n\pi/(aT)} - 1}$$

$$F/F_0 = 1 + 240\sum_{n=0}^{\infty}\frac{n^3}{e^{n\pi/(aT)} - 1} - 16(aT)^4 \equiv R(aT)$$

Equilibrium System

- Casimir force decreases with temperature.
- Vacuum effect is gradually cancelled by the thermal motion.



Internal Energy vs. Free Energy

- We use internal energy to calculate Casimir force.
- Thermodynamic relation

$$\mathscr{E} = \mathscr{F} + \beta \frac{\partial \mathscr{F}}{\partial \beta}$$

- Consistent with previous free energy calculation.
- Adiabatic vs. thermalized —— J/ψ dissociation
- Internal energy more suited for the non-equilibrium case.

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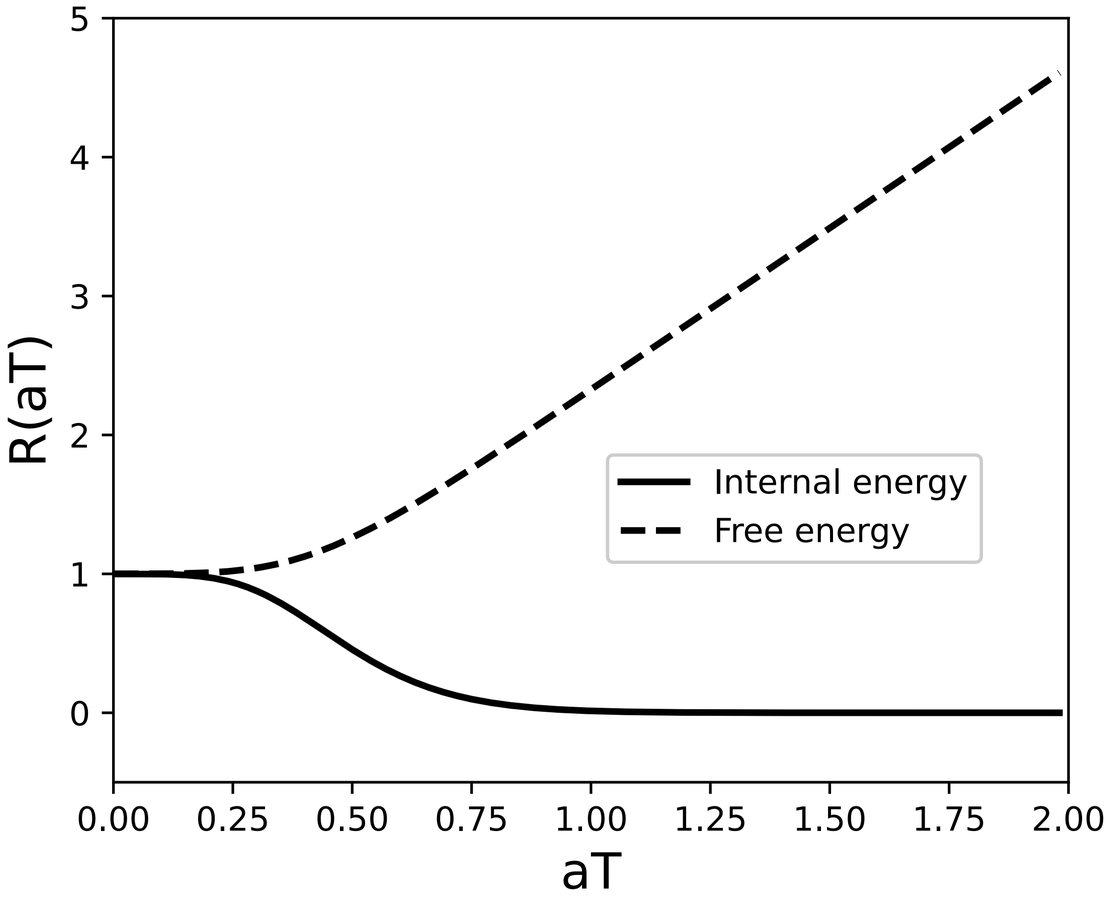


Non-equilibrium System

- Inhomogeneous initial state

$$f_+(t=0) = \frac{1}{2} + e^{-\epsilon_p |x|}$$

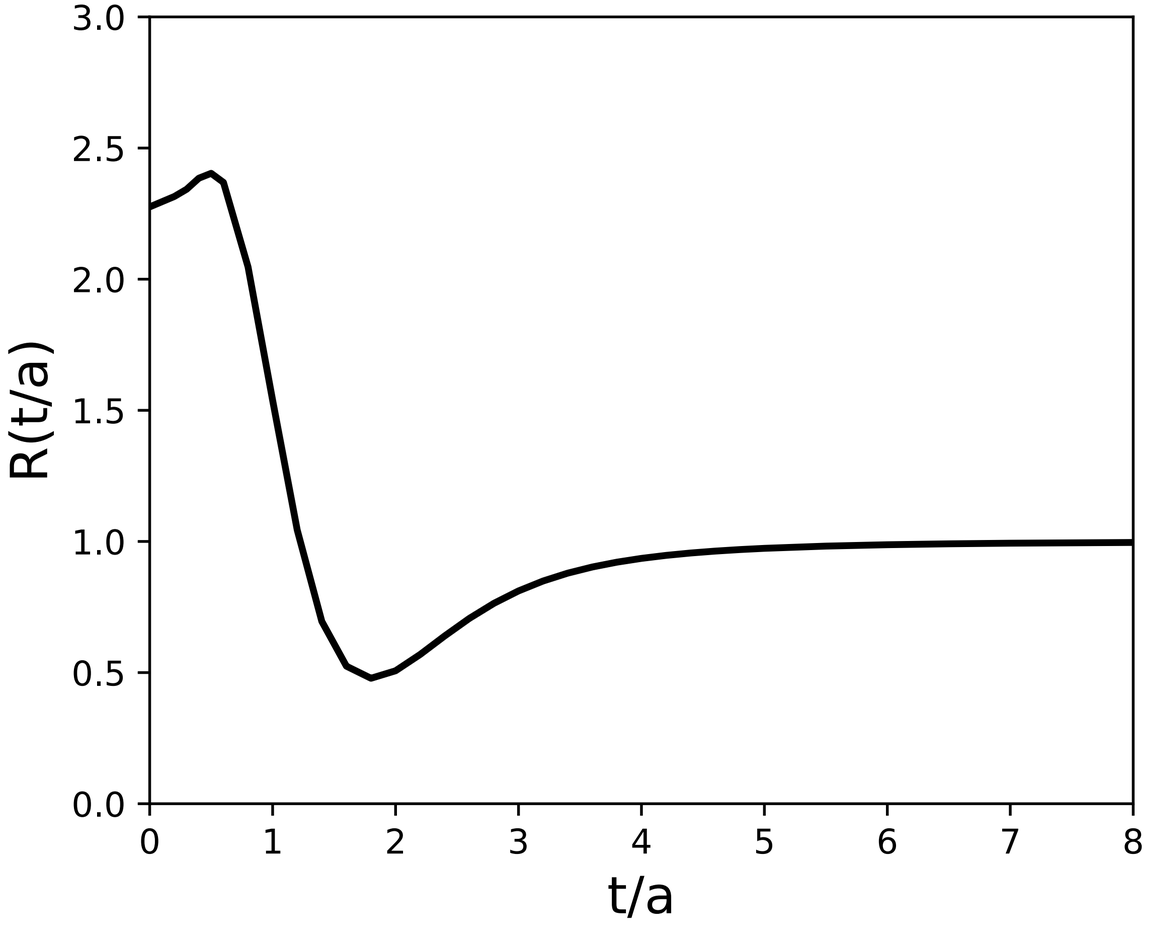
- Time evolution

$$f_+(t, \vec{x}, p) = f(t=0, \vec{x} - \vec{p}/\epsilon_p t, p) = \frac{1}{2} + e^{-\sqrt{(\epsilon_p \vec{x} - \vec{p} t)^2}}$$

$$F/F_0 = 1 - \frac{240}{\pi^2}\left(3\Delta\mathscr{E}(t/a) + \frac{\partial \Delta\mathscr{E}(t/a)}{\partial(t/a)} t/a\right) \equiv R(t/a)$$

Non-equilibrium System

- Strong non-equilibrium enhancement at the beginning.
- Photon gas is diluted by expansion.
- Decay and oscillation.



Summary

- We derived the kinetic theory for $U(1)$ gauge theory and used it to calculate the Casimir force in equilibrium and non-equilibrium systems.
- At equilibrium, the Casimir force decreases with temperature.
- The non-equilibrium Casimir force is affected by initial distribution, the evolution of photon gas and has oscillation behavior.