N-particle irreducible actions for stochastic fluids

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FLUID DYNAMICS FOR RELATIVISTIC QCD MATTER

Fluid dynamics is a universal effective field theory (EFT) of non-equilibrium many-body systems with a stable equation of state and

- Conservation of charge: $\partial_\mu J^\mu = 0$
- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

$$
\begin{aligned}
J^\mu &= n\, u^\mu + \nu^\mu \\
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \qquad \nu^\mu = -\kappa T \Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) \\
\pi^{ij} &= -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}
\end{aligned}
$$

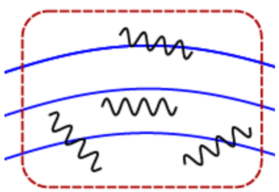
The dissipation terms are described by the shear viscosity η, bulk viscosity ζ and charge conductivity κ

FLUCTUATIONS IN HYDRO

- The deterministic hydro equations do not lead to spontaneous fluctuations
- The occurrence of fluctuations is a consequence of the microscopic dynamics and must persist at the coarse-grained hydro-level

Introducing non-linear dissipation with temperature-dependent transport coefficients and random noises:

$$
\begin{aligned}
J^{\mu} &\rightarrow J^{\mu} + \theta^{\mu} \\
T^{\mu\nu} &\rightarrow T^{\mu\nu} + \theta^{\mu\nu}
\end{aligned}
$$



$$\langle \theta^{\mu} \rangle = 0 \quad \left\langle (\theta^{\mu})^2 \right\rangle \sim \delta(x - x')(t - t')$$

$$\langle \theta^{\mu\nu} \rangle = 0 \quad \left\langle (\theta^{\mu\nu})^2 \right\rangle \sim \delta(x - x')(t - t')$$

Representation in MSRJD Field Theory

In terms of the slow variable (a conserved density), the free energy of the fluid:

$$\mathcal{F}[\psi] = \int d^3x \, \left\{ \frac{1}{2}(\vec{\nabla}\psi)^2 + \frac{r}{2}\,\psi(x,t)^2 + \frac{\lambda}{3!}\,\psi(x,t)^3 + \dots + h(x,t)\psi(x,t) \right\}$$

The diffusion equation:

$$\partial_t \psi(x,t) = \vec{\nabla} \left\{ \kappa(\psi)\, \vec{\nabla} \left(\frac{\delta \mathcal{F}[\psi]}{\delta \psi} \right) \right\} + \theta(x,t)$$

where the Gaussian noise term $\theta(x,t)$ has a distribution

$$P[\theta] \sim \exp\left(-\frac{1}{4} \int d^3x \, dt \, \theta(x,t) L(\psi)^{-1} \theta(x,t) \right)$$

Representation in MSRJD Field Theory, cont.

The conductivity, $\kappa(\psi)$, is field-dependent: $\kappa(\psi) = \kappa_0 (1 + \lambda_D \psi)$

The partition function is given as: MSR, PhysRevA.8:423(1973)

$$
\begin{aligned}
Z &= \int \mathscr{D}\psi \, P[\theta] \exp\left(-i\tilde{\psi}\,(\text{e.o.m}[\psi,\,\theta\,])\right) \\
&= \int \mathscr{D}\psi \, \mathscr{D}\tilde{\psi}\ \exp\left(-\int d^3x\, dt\, \mathscr{L}(\psi, \tilde{\psi})\right)
\end{aligned}
$$

The effective Lagrangian of this theory is:

$$
\mathscr{L}(\psi, \tilde{\psi}) = \tilde{\psi}\left(\partial_t - D_0 \nabla^2\right)\psi - \frac{D_0 \lambda'}{2}\left(\nabla^2 \tilde{\psi}\right)\psi^2 - \tilde{\psi} L(\psi) \tilde{\psi}
$$

Note: $D_0 = r\kappa_0$ and $\lambda' = \lambda/r + \lambda_D$.

The noise kernel is chosen as $L(\psi) = \overleftarrow{\nabla}\left[k_B T \kappa(\psi)\right]\overrightarrow{\nabla}$

Time Reversal Symmetry

Stochastic theories must describe the detailed balance condition:

$$\frac{P\left(\psi_1 \longrightarrow \psi_2\right)}{P\left(\psi_2 \longrightarrow \psi_1\right)} = e^{-\Delta\mathscr{F}/k_B T}$$

which is related to time-reversal symmetry:

$$\Psi(t) \longrightarrow \psi(-t)$$

$$\tilde{\Psi}(t) \longrightarrow -\left[\tilde{\psi}(-t) + \frac{\delta F}{\delta\psi}\right]$$

$$\mathscr{L} \longrightarrow \mathscr{L} + \frac{d}{dt}F$$

Janssen, ZPhyB.23:377(1976)

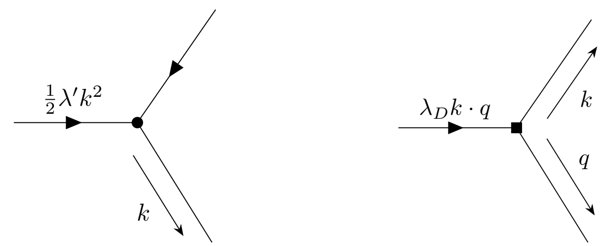
The Ward identity is revised to

$$\left\langle \psi(x_1, t_1)\left[\overleftarrow{\nabla}\kappa(\psi)\overrightarrow{\nabla}\tilde{\psi}\right](x_2, t_2)\right\rangle = \Theta(t_2 - t_1)\left\langle \psi(x_1, t_1)\dot{\psi}(x_2, t_2)\right\rangle$$

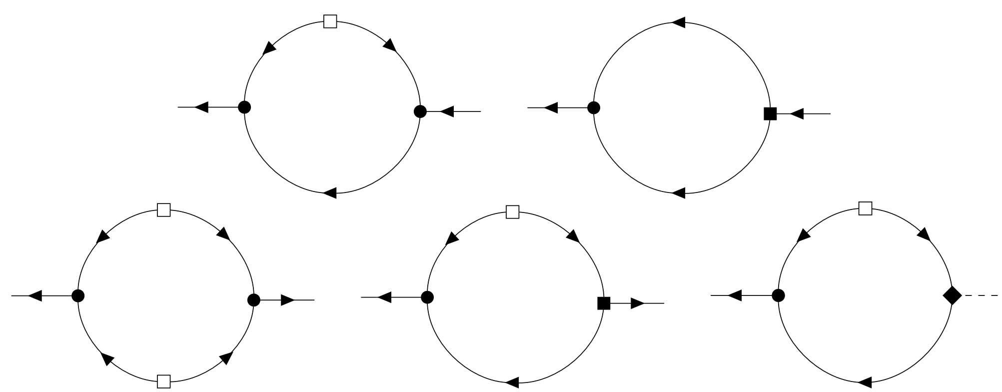
Simpler Example of Model B

- Linearized propagator:

- Vertex and new vertices:



- Loop contributions:



Analytical Results of One-Loop

The retarded function

$$G^{-1}(\omega, k) = \frac{1}{-i\omega + D_0 k^2 + \Sigma(\omega, k)}$$

with
$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi}\left(i\lambda'\omega k^2 + \lambda_D\left[i\omega - D_0 k^2\right] k^2\right)\sqrt{k^2 - \frac{2i\omega}{D_0}}$$

The charge (thermal) conductivity in this system becomes a scale-dependent term in the low-energy effective hydro theory

The vertex function of composite operator $\lambda_D[\psi\vec{\nabla}\tilde{\psi}]$ is given by

$$\Gamma_D(\omega, k) \equiv (-i\omega + D_0 k^2)\left\langle D_0\lambda_D[\psi\vec{\nabla}\tilde{\psi}]\vec{\nabla}\psi\right\rangle_{\omega,k}$$

The field-dependent fluctuation-dissipation relation becomes:

$$2\,\mathrm{Im}\left\{G(\omega, k)\left[D_0 k^2 + \Gamma_D(\omega, k)\right]\right\} = \omega C(\omega, k)$$

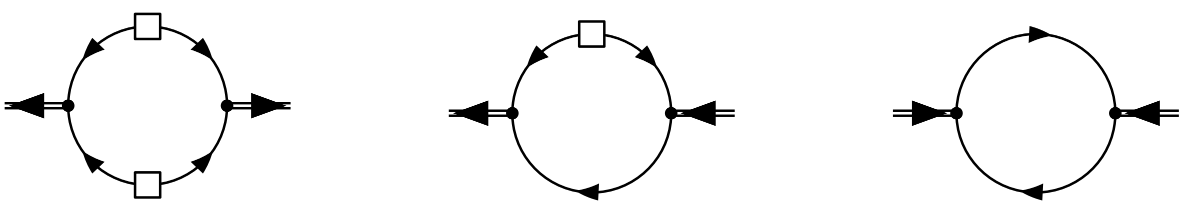
1PI Effective Action

Consider the generating functional with local source J, $\tilde{J}$:

$$W[J,\tilde{J}] = -\ln \int \mathscr{D}\psi \mathscr{D}\tilde{\psi}\; e^{-\int dt\, d^3x \left\{\mathscr{L} + J\psi + \tilde{J}\tilde{\psi}\right\}}$$

Performing a Legendre transform to the 1PI effective action via background field method with $\psi = \Psi + \delta\psi$:

$$\Gamma[\Psi,\tilde{\Psi}] = W[J,\tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi}\right)$$



Taking the derivative of the 1PI effective action w.r.t. the classical field Ψ yields the e.o.m. encoded the fluctuation effects:

$$(\partial_t - D\nabla^2)\Psi - \frac{\kappa\lambda_3^2}{2}\nabla^2\Psi^2 + \int d^3x'\, dt'\, \Psi(x',t')\Sigma(x,t;x',t') = 0$$

Double Legendre Transformation

☞ nPI effective action $\Longrightarrow$ e.o.m. for n-point functions

✓ Couple a bi-local source $\frac{1}{2}\psi_a K_{ab}\psi_b$ to the system Cornwall, Jackiw and Tomboulis, PhysRevD.10:2428 (1974)

✓ Plug in the 1-loop 1PI effective action

✓ Sum beyond 1-loop terms

✓ Apply the stationary conditions:

$$\frac{\delta W}{\delta J_a} = \langle\psi_a\rangle = \Psi_a\,, \qquad \frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle\psi_a\psi_b\rangle = \frac{1}{2}[\Psi_a\Psi_b + G_{ab}]$$

✓ Perform a Legendre transform to yield the 2PI effective action:

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A\Psi_A - \frac{1}{2}K_{AB}[\Psi_A\Psi_B + G_{AB}]$$

2PI EFFECTIVE ACTION

The 2PI effective action is given by:

$$\Gamma[\Psi_a, G_{ab}] = S[\Psi_a] + \frac{1}{2}\frac{\delta^2 S}{\delta\Psi_A \delta\Psi_B} G_{AB} - \frac{1}{2}\mathrm{Tr}\left[\log(G)\right] + \Gamma_F[\Psi_a, G_{ab}]$$

The higher order fluctuations are:

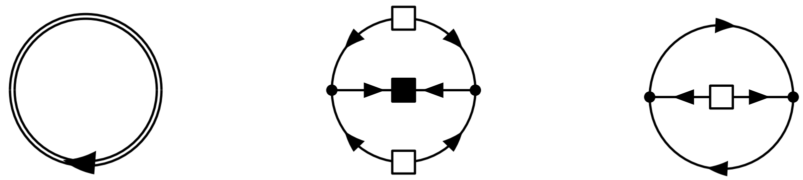
$$\begin{aligned}\exp(-\Gamma_F[\Psi_a, G_{ab}]) \quad = \quad & \frac{1}{\sqrt{\det(G)}}\int D(\delta\psi_a)\ \exp\Big\{ -\frac{1}{2}\delta\psi_A (G^{-1})_{AB}\delta\psi_B \\ & - \Big[S_3[\Psi_a, \delta\psi_a] - \bar{J}_A\delta\psi_A - \bar{K}_{AB}(\delta\psi_A\delta\psi_B - G_{AB})\Big]\Big\}\end{aligned}$$

with

$$\bar{J}_a = \frac{1}{2}\frac{\delta^3 S}{\delta\Psi_a\delta\Psi_B\delta\Psi_C} G_{BC} + \frac{\delta\Gamma_F}{\delta\Psi_a}, \quad \bar{K}_{ab} = \frac{\delta\Gamma_F}{\delta G_{ab}}$$

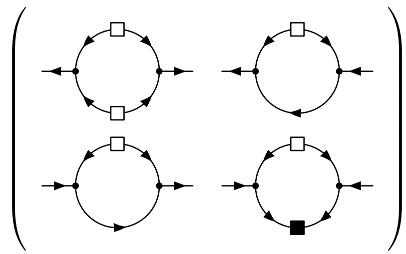
DSE in Mixed Representation

The loop diagrams generated by Γ_F use the full propagator G_{ab}:



Taking the derivative w.r.t G, obtain the Dyson-Schwinger equation:

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$

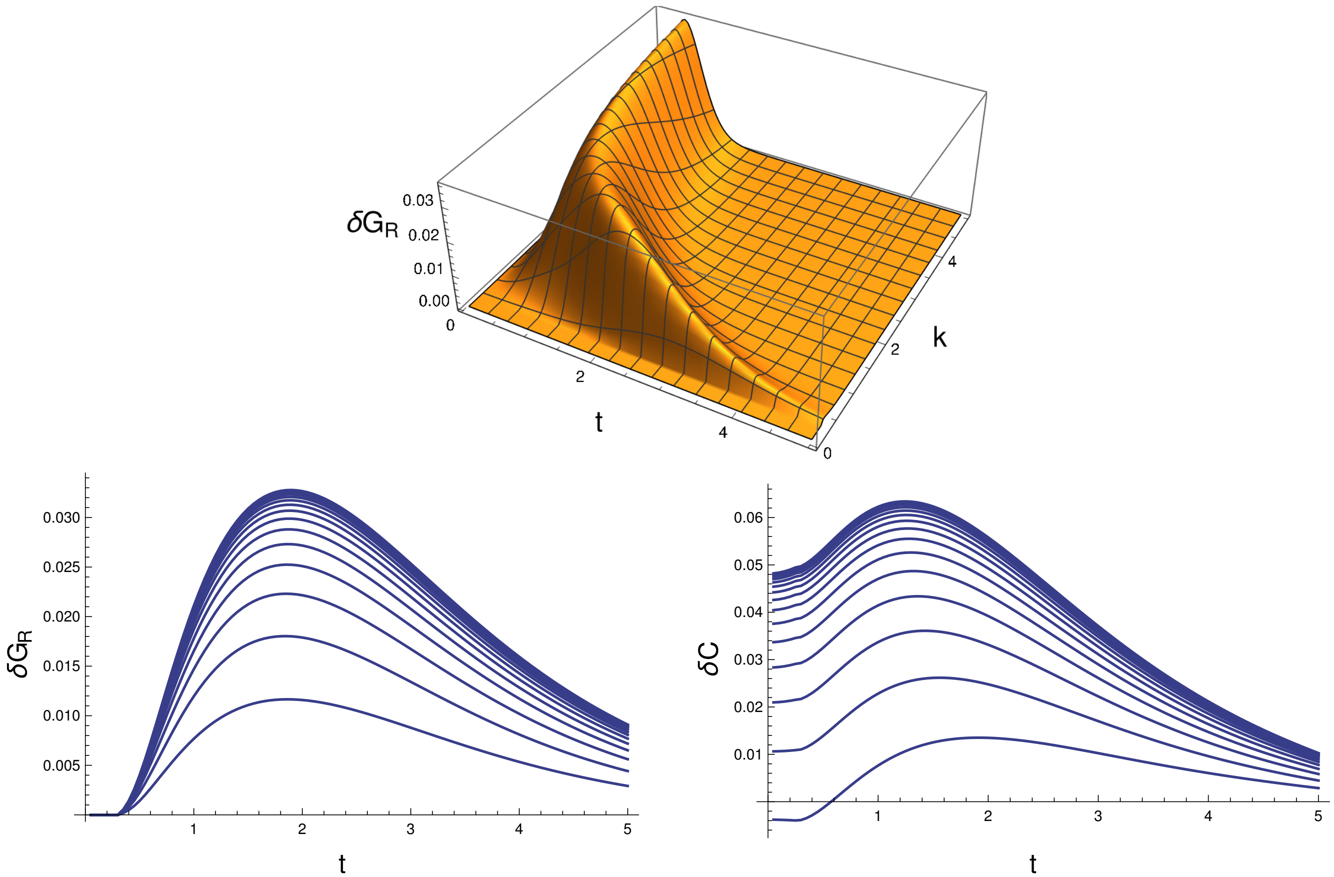


In time-momentum mixed representation

$$\Sigma(t, k^2) = (\kappa\lambda_3)^2 \int d^3k'\, k^2(k + k')^2\, C(t, k')\, G_R(t, k + k')\,,$$

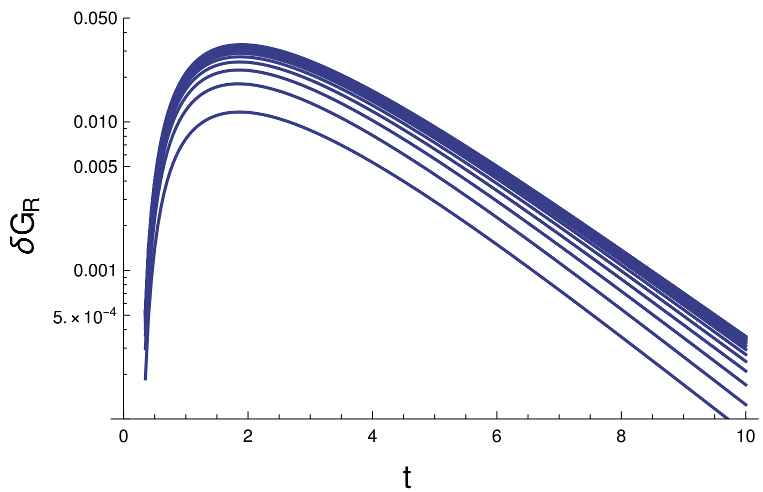
$$\delta D(t, k^2) = \frac{(\kappa\lambda_3)^2}{2} \int d^3k'\, k^4 C(t, k')\, C(t, k + k')$$

Naive Numerical Simulations

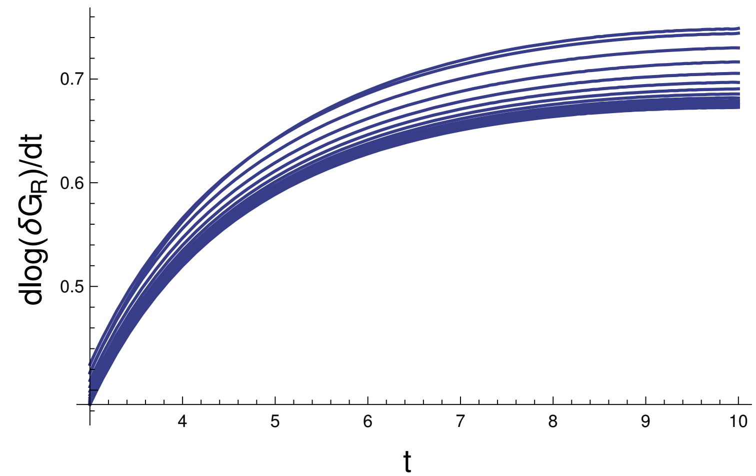


3D curve of $\delta G(t, k)$ and the iterative solutions of DSE taking advantage of convergence

LONG-TIME BEHAVIOR



A logarithmic plot of the loop corrections to the retarded $\delta G_R(k,t)$



The logarithmic derivative of $\delta G_R(k,t)$ w.r.t t

The long-time behavior of the diffusion cascade is conjectured to be $\exp(-\alpha\sqrt{DK^2t})$ with $\alpha\sim 1$ because of the n-loop terms $\sim n!\exp(-Dk^2t/n)$. Delacretaz, SciPostPhys.9:034(2020)

Summary and Outlooks

- To complete the numerical simulation of the mode coupling approximation, as it can provide important insights into the critical region
- To consider extending our model to include expanding systems, which can help us gain a deeper understanding of the dynamical nature of the phase transitions
- To investigate how our approach can be connected to kinetic theory to provide more valuable insights into the microscopic behavior of QCD matter

Thank You for Your Attention!