



Spin Alignment Formula for Vector Bosons at Local Equilibrium

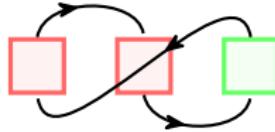
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What is Spin Alignment?

Spin alignment ($\rho_{00} \neq 1/3$) \subset Tensor polarization

$$\rho_{00} - 1/3 = -\sqrt{2/3}T_{2,0}$$

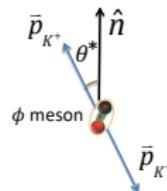
- A massive vector boson's spin density matrix $\rho_{rs} = \begin{pmatrix} \rho_{1,1} & \rho_{1,0} & \rho_{1,-1} \\ \rho_{0,1} & \textcolor{red}{\rho_{0,0}} & \rho_{0,-1} \\ \rho_{-1,1} & \rho_{-1,0} & \rho_{-1,-1} \end{pmatrix}$

$$\rho = \frac{1}{3}\mathbb{1} + \frac{1}{2} \sum_{i=1}^3 \textcolor{red}{P_i} S_i + \sum_{m=-2}^2 (-1)^m \textcolor{red}{T_{2,m}} S_{2,-m} \quad (1)$$

Vector polarization (3 DoFs) Tensor polarization (5 DoFs)

with $S_{2,m} = \sum_{m1,m2} \langle 1, m1; 1, m2 | 2, m \rangle S_{1,m1} S_{1,m2}$

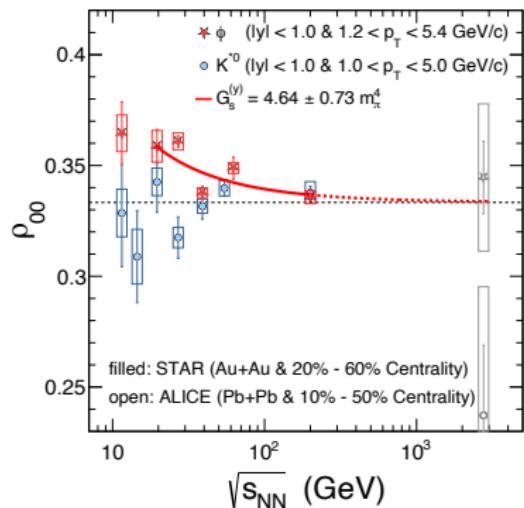
- Strong decay, parity even $\Rightarrow T_{2,m}$ only.



Graph: STAR, Nature.614.244 (2023)

Why Hydrodynamics?

Spin alignment

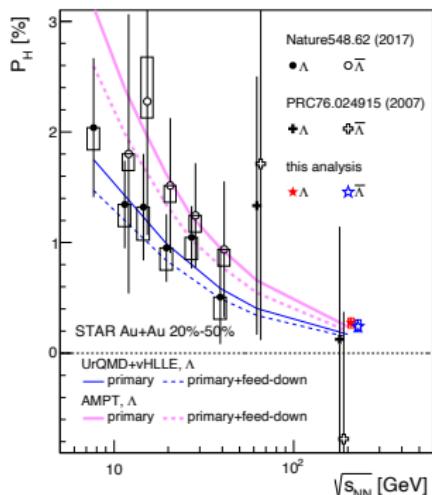


Graph: STAR, Nature.614.244 (2023)

ϕ mean field: X. Sheng *et al.*, PRD.101.096005 (2020)

Glasma: A. Kumar *et al.*, PRD.107.076025 (2023)

Λ 's global polarization



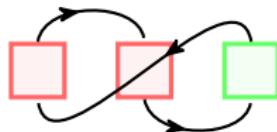
Graph: STAR, PRC.98.014910 (2018)

I. Karpenko, F. Becattini, EPJC.77.213 (2017)

H. Li *et al.*, PRC.96.054908 (2017)

Overview

- Diagram scheme



- Leading order of tensor polarization comes from $\mathcal{O}(\partial^2)$

$$(\partial\beta)(\partial\beta), (\partial\beta)\mu, \mu\mu, \partial\partial\beta, \partial\mu$$

Thermal current $\beta_\nu(x) \sim \mathcal{O}(1)$ Spin potential $\mu_{\rho\sigma}(x) \sim \mathcal{O}(\partial)$

LEDO

- Free Lagrangian for neutral vector bosons $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu$
- Local equilibrium density operator (LEDO)

$$\widehat{\rho}_{\text{LE}} = \frac{1}{Z_{\text{LE}}} \exp \left\{ - \int_{\Sigma(\tau)} d\Xi_\mu(y) [\widehat{T}^{\mu\nu}(y) \beta_\nu(y) - \frac{1}{2} \widehat{S}^{\mu\rho\sigma}(y) \mu_{\rho\sigma}(y)] \right\} \quad (2)$$

Thermal current $\beta_\nu(y) \sim \mathcal{O}(1)$ Spin potential $\mu_{\rho\sigma}(y) \sim \mathcal{O}(\partial)$

Canonical operator $T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L}$, $S^{\mu\rho\sigma} = -F^{\mu\rho}A^\sigma + F^{\mu\sigma}A^\rho$

- LEDO maximizes the entropy

$$S[\Sigma(\tau)] = -\text{Tr}(\widehat{\rho}_{\text{LE}} \ln \widehat{\rho}_{\text{LE}}), \quad (3)$$

under constraints $\widehat{t}_\mu T^{\mu\nu}(x) = \widehat{t}_\mu \left\langle \widehat{T}^{\mu\nu}(x) \right\rangle_{\text{LE}} [\beta, \mu]$, $\widehat{t}_\mu S^{\mu\rho\sigma}(x) = \widehat{t}_\mu \left\langle \widehat{S}^{\mu\rho\sigma}(x) \right\rangle_{\text{LE}} [\beta, \mu]$

D. Zubarev *et al.*, Theo. and Math. Phys..40.821 (1979)

C. van Weert, Ann. of Phys..140.133 (1982)

F. Becattini *et al.*, Particles.2.197 (2019)

MVSD and Cumulant Expansion

- LEDO $\hat{\rho}_{\text{LE}} = \frac{1}{Z_{\text{LE}}} \exp \left\{ - \int_{\Sigma(\tau)} d\Xi_\mu(y) [\hat{T}^{\mu\nu}(y)\beta_\nu(y) - \frac{1}{2}\hat{S}^{\mu\rho\sigma}(y)\mu_{\rho\sigma}(y)] \right\}$
- Matrix-valued spin-dependent distribution (MVSD)

$$\hat{f}_{rs}(x, \mathbf{k}) = \int \frac{d^3 q}{(2\pi)^3} e^{-iq \cdot x} \hat{a}_{\mathbf{k} - \frac{\mathbf{q}}{2}}^{s\dagger} \hat{a}_{\mathbf{k} + \frac{\mathbf{q}}{2}}^r \quad (4)$$

$$f_{rs}(x, \mathbf{k}) = \text{Tr} \left[\hat{\rho}_{\text{LE}} \hat{f}_{rs} \right] = \text{Tr} \left[e^{\hat{A} + \hat{B}} \hat{f}_{rs}(x, \mathbf{k}) \right] / \text{Tr} \left[e^{\hat{A} + \hat{B}} \right] \quad (5)$$

"Gaussian" term $\hat{A} = -\beta_\nu(x)\hat{T}^\nu = \beta_\nu(x) \int d\Xi_\mu(y) \hat{T}^{\mu\mu}(y)$

"Perturbative" terms $\hat{B} = - \int d\Xi_\mu(y) [\hat{T}^{\mu\nu}(y)(\beta_\nu(y) - \beta_\nu(x)) - \frac{1}{2}\hat{S}^{\mu\rho\sigma}(y)\mu_{\rho\sigma}(y)]$

- Cumulant expansion $e^{\hat{A} + \hat{B}} = e^{\hat{A}} \sum_{n=0}^{\infty} \hat{B}_n$, with $\hat{B}_0 = 1$, $\hat{B}_n \sim \mathcal{O}(\partial^n)$,

$$f_{rs}(x, \mathbf{k}) = \frac{\sum_{n=0}^{\infty} \langle \hat{B}_n \hat{f}_{rs}(x, \mathbf{k}) \rangle_0}{\sum_{n=0}^{\infty} \langle \hat{B}_n \rangle_0} \quad (6)$$

with $\langle \hat{O} \rangle_0 = \text{Tr} (e^{\hat{A}} \hat{O}) / \text{Tr} (e^{\hat{A}})$. And $\langle \hat{O} \rangle_0$ easy to derive.

Zeroth-Order Result

- Cumulant expansion: $f_{rs}^{(0)}(x, \mathbf{k}) = \langle \hat{f}_{rs}(x, \mathbf{k}) \rangle_0$
- “Free” distribution: $\langle \hat{a}_{\mathbf{k}}^{s\dagger} \hat{a}_{\mathbf{q}}^r \rangle_0 = (2\pi)^3 \delta^{rs} \delta^{(3)}(\mathbf{k} - \mathbf{q}) n_B(\beta(x) \cdot k)$

Bose-Einstein distribution: $n_B(x) = 1/(e^x - 1)$

- Zeroth-order: Nothing but Bose-Einstein distribution.

$$f_{rs}^{(0)}(x, \mathbf{k}) = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q} \cdot \mathbf{x}} \left\langle \hat{a}_{\mathbf{k} - \frac{\mathbf{q}}{2}}^{s\dagger} \hat{a}_{\mathbf{k} + \frac{\mathbf{q}}{2}}^r \right\rangle_0 = \boxed{\delta_{rs} n_B(\beta(x) \cdot k)} \quad (7)$$

1st-Order Diagrams

$$\text{Cumulant expansion: } f_{rs}^{(1)}(x, \mathbf{k}) = \left\langle \widehat{B}_1 \widehat{f}_{rs}(x, \mathbf{k}) \right\rangle_{0,c}$$

$$\begin{aligned} f_{rs}^{(1)}(x, \mathbf{k}) &= \partial_{\alpha_1} \beta_{\nu_1}(x) \int_0^1 d\lambda_1 \int d\Xi_{\mu_1}(y_1) (y_1 - x)^{\alpha_1} \left\langle -\widehat{T}^{\mu_1 \nu_1}(y_1 - i\lambda_1 \beta(x)) \widehat{f}_{rs}(x, \mathbf{k}) \right\rangle_{0,c} \\ &+ \mu_{\rho_1 \sigma_1}(x) \int_0^1 d\lambda_1 \int d\Xi_{\mu_1}(y_1) \left\langle \frac{1}{2} \widehat{S}^{\mu_1 \rho_1 \sigma_1}(y_1 - i\lambda_1 \beta(x)) \widehat{f}_{rs}(x, \mathbf{k}) \right\rangle_{0,c} \end{aligned} \quad (8)$$

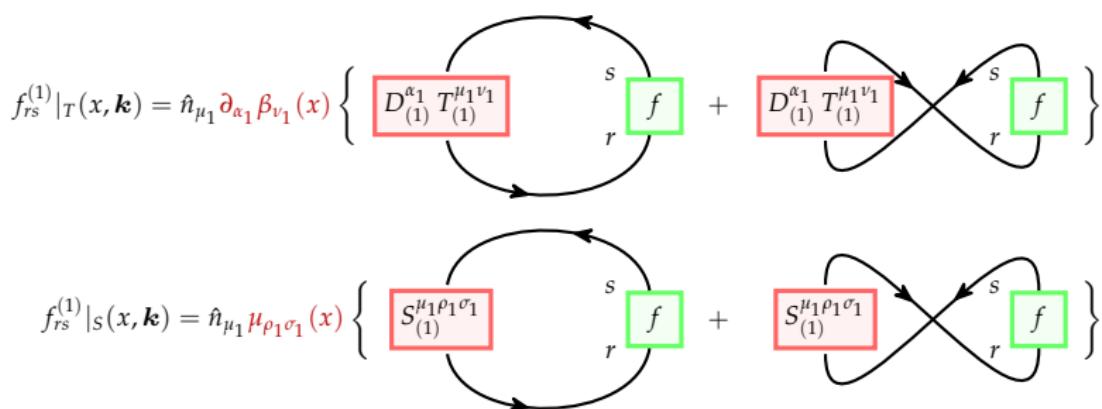


Diagram Rules

$$f(x, \mathbf{k}) = \int_0^1 d\lambda_1 D_{(1)}^{\alpha_1} t_{(1)}^{\mu_1 \nu_1} \gamma_1 \gamma_2 (p_1, -p_2) (n_B(\beta \cdot p_1) + 1) n_B(\beta \cdot p_2) \epsilon_s^{\gamma_1}(p_1) \epsilon_r^{\gamma_2*}(p_2)$$

Vertex **B-E distributions from lines** **$p_1 = k, p_2 = k$**

Polarization vectors from lines **Momentum conservation**

$$t_{(l)}^{\mu_l \nu_l} \gamma_i \gamma_j (p_i, p_j) = \frac{e^{-\lambda_l \hat{n} \cdot (p_i + p_j) \hat{n} \cdot \beta}}{\sqrt{2E_{p_i} 2E_{p_j}}} \left[-p_i^{\mu_l} p_j^{\nu_l} \eta_{\gamma_i \gamma_j} + p_i \gamma_j p_j^{\nu_l} \eta^{\nu_l \nu_l} \gamma_i + \frac{1}{2} (p_i \cdot p_j) \eta^{\mu_l \nu_l} \eta_{\gamma_i \gamma_j} - \frac{1}{2} p_i \gamma_j p_j \gamma_i \eta^{\mu_l \nu_l} + \frac{1}{2} m^2 \eta^{\mu_l \nu_l} \eta_{\gamma_i \gamma_j} \right]$$

$$D_{(l)}^{\alpha_m} = -i \left[\frac{1}{2} \sum_i \sigma_i^{(l)} \left(\eta^{\alpha_m \zeta_m} - \hat{p}_i^{\alpha_m} \hat{n}^{\zeta_m} \right) \frac{\partial}{\partial p_i^{\zeta_m}} - \lambda_l \Delta^{\alpha_m \zeta_m} \beta_{\zeta_m} \right], \sigma_i^{(l)} = \pm 1, \hat{p}_i^{\alpha_m} = p_i^{\alpha_m} / (\hat{n} \cdot p_i)$$

- Inner lines: $n_B(\beta \cdot p_i)(-\eta^{\gamma_i \gamma_j} + p_i^{\gamma_i} p_j^{\gamma_j} / m^2)$ or $[n_B(\beta \cdot p_i) + 1](-\eta^{\gamma_i \gamma_j} + p_i^{\gamma_i} p_j^{\gamma_j} / m^2)$
- Adaptability (higher-order results, additional operators in LEDO, fermions)

1st-Order Result

Using FeynCalc, V. Shtabovenko *et al.*, Comput.Phys.Commun..256.107478 (2020)

$$f_{rs}^{(1)}|_T(x, \mathbf{k}) = \frac{i}{2} n_B (1 + n_B) [\partial_{\alpha_1}^{\perp} \beta_{\nu_1}] (x) \left[\frac{\gamma_k (\gamma_k - 1)}{\gamma_k + 1} (e_s \cdot \hat{k}) e_r^{\alpha_1 *} \hat{k}^{\nu_1} - \text{h.c.}(r \leftrightarrow s) \right] \quad (9)$$

$$\begin{aligned} f_{rs}^{(1)}|_S(x, \mathbf{k}) &= \frac{i}{2} n_B (1 + n_B) \mu_{\rho_1 \sigma_1} (x) \left[e_s^{\rho_1} e_r^{\sigma_1 *} + (e_s \cdot \hat{k}) e_r^{\rho_1 *} \right. \\ &\quad \times \left. \left(\frac{2\gamma_k}{\gamma_k + 1} \hat{n}^{\sigma_1} + \frac{\gamma_k (\gamma_k - 1)}{\gamma_k + 1} \hat{k}^{\sigma_1} \right) - h.c.(r \leftrightarrow s) \right] \end{aligned} \quad (10)$$

with $\gamma_k = E_{\mathbf{k}}/m$ and $e_{\pm 1}^{\mu} = -(0, i, 0, \pm 1)/\sqrt{2}$, $e_0^{\mu} = (0, 0, 1, 0)$.

- Hermit: $f_{rs}^{(1)}(x, \mathbf{k}) = f_{sr}^{(1)*}(x, \mathbf{k})$
- Space-time reversal (PT) odd: $f_{rs}^{(1)}(x, \mathbf{k}) = -(-1)^{r+s} f_{-s, -r}^{(1)}(x, \mathbf{k})$

From MVSD to Tensor Polarization

- Space-time reversal property caused by power counting rules

$$f_{r,s}^{(n)} = (-1)^{r+s+n} f_{-s,-r}^{(n)} \quad (11)$$

with $f_{r,s}^{(n)}$ the n-order MVSD.

- Spin density matrix in phase space $\rho_{rs}(x, \mathbf{k})$

$$f_{rs}(x, \mathbf{k}) = \rho_{rs}(x, \mathbf{k}) f(x, \mathbf{k}) \quad (12)$$

with $f(x, \mathbf{k}) = \sum_r f_{rr}(x, \mathbf{k})$ the scalar distribution

- The leading order of polarization (y-axis as the spin axis)

$$\begin{aligned} \{P_x, P_y, P_z\} &= \frac{2}{3n_B} \left\{ \sqrt{2} \operatorname{Im} f_{01}^{(1)}, f_{11}^{(1)}, \sqrt{2} \operatorname{Re} f_{01}^{(1)} \right\} \\ \{T_{2,0}, T_{2,1}, T_{2,2}\} &= \frac{1}{3n_B} \left\{ \sqrt{\frac{2}{3}} (f_{11}^{(2)} - f_{00}^{(2)}), -\sqrt{2} f_{01}^{(2)}, f_{1-1}^{(2)} \right\} \end{aligned}$$

$$\boxed{\rho_{00} - \frac{1}{3} = \frac{2}{9n_B} (f_{00}^{(2)} - f_{11}^{(2)})} \quad (13)$$

$$f_{rs}^{(2)}|_{TT}(x, \mathbf{k}) = \hat{n}_{\mu_1} \hat{n}_{\mu_2} \partial_{\alpha_1} \beta_{\nu_1}(x) \partial_{\alpha_2} \beta_{\nu_2}(x) \left\{ \begin{array}{c} D_{(1)}^{\alpha_1} T_{(1)}^{\mu_1 \nu_1} \\ D_{(2)}^{\alpha_2} T_{(2)}^{\mu_2 \nu_2} \end{array} \right. \begin{array}{l} s \\ r \end{array} f + \text{other 15 diagrams} \right\}$$

$$f_{rs}^{(2)}|_{TS}(x, \mathbf{k}) = \hat{n}_{\mu_1} \hat{n}_{\mu_2} \partial_{\alpha_1} \beta_{\nu_1}(x) \mu_{\rho_2 \sigma_2}(x) \left\{ \begin{array}{c} D_{(1)}^{\alpha_1} T_{(1)}^{\mu_1 \nu_1} \\ S_{(2)}^{\mu_2 \rho \gamma \sigma_2} \end{array} \begin{array}{l} s \\ r \end{array} f + \text{other 15 diagrams} \right\}$$

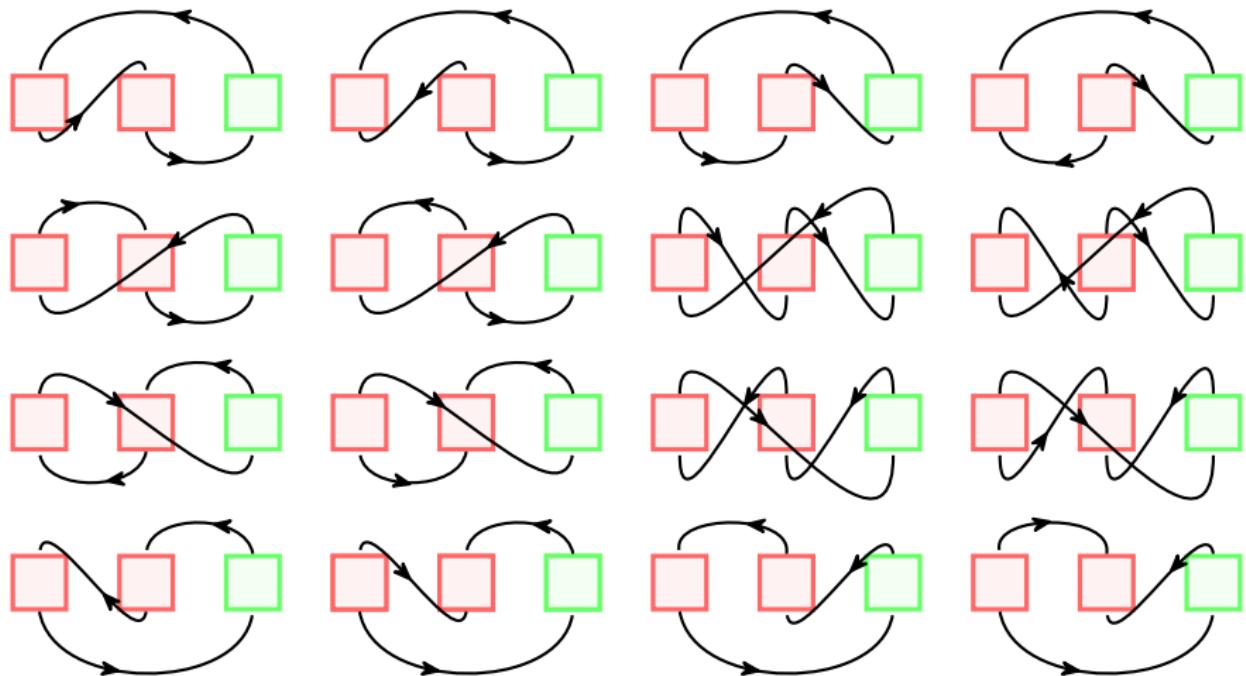
$$f_{rs}^{(2)}|_{ST}(x, \mathbf{k}) = \hat{n}_{\mu_1} \hat{n}_{\mu_2} \mu_{\rho_1 \sigma_1}(x) \partial_{\alpha_2} \beta_{\nu_2}(x) \left\{ \begin{array}{c} S_{(1)}^{\mu_1 \rho_1 \sigma_1} \\ D_{(2)}^{\alpha_2} T_{(2)}^{\mu_2 \nu_2} \end{array} \begin{array}{l} s \\ r \end{array} f + \text{other 15 diagrams} \right\}$$

$$f_{rs}^{(2)}|_{SS}(x, \mathbf{k}) = \hat{n}_{\mu_1} \hat{n}_{\mu_2} \mu_{\rho_1 \sigma_1}(x) \mu_{\rho_2 \sigma_2}(x) \left\{ \begin{array}{c} S_{(1)}^{\mu_1 \rho_1 \sigma_1} \\ S_{(2)}^{\mu_2 \rho_2 \gamma \sigma_2} \end{array} \begin{array}{l} s \\ r \end{array} f + \text{other 15 diagrams} \right\}$$

$$f_{rs}^{(2)}|_T(x, \mathbf{k}) = \hat{n}_{\mu_1} \partial_{\alpha_1} \partial_{\alpha_2} \beta_{\nu_1}(x) \left\{ \begin{array}{c} D_{(1)}^{\alpha_1} D_{(1)}^{\alpha_2} T_{(1)}^{\mu_1 \nu_1} \end{array} \begin{array}{l} s \\ r \end{array} f + \text{the other diagram} \right\}$$

$$f_{rs}^{(2)}|_S(x, \mathbf{k}) = \hat{n}_{\mu_1} \partial_{\alpha_1} \mu_{\rho_1 \sigma_1}(x) \left\{ \begin{array}{c} D_{(1)}^{\alpha_1} S_{(1)}^{\mu_1 \rho_1 \sigma_1} \end{array} \begin{array}{l} s \\ r \end{array} f + \text{the other diagram} \right\}$$

16 Diagrams



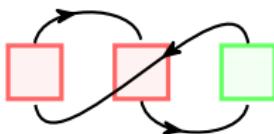
Take-Home Message

- Leading order of tensor polarization comes from $\mathcal{O}(\partial^2)$

$$(\partial\beta)(\partial\beta), (\partial\beta)\mu, \mu\mu, \partial\partial\beta, \partial\mu$$

$\partial_\mu\beta_\nu =$ Thermal vorticity + Thermal shear, $\mu_{\rho\sigma}$ – Spin potential

- The diagram scheme has been checked at global nonrelativistic equilibrium (with small vorticity) and vector mesons' $k = 0$.



Thank you!

Analogy to Feynman's Path Integral

Starting point of diagram scheme

$$f_{rs}(x, \mathbf{k}) = \frac{\sum_{n=0}^{\infty} \left\langle \hat{B}_n \hat{f}_{rs}(x, \mathbf{k}) \right\rangle_0}{\sum_{n=0}^{\infty} \left\langle \hat{B}_n \right\rangle_0}$$

~

$$\approx \frac{\langle \Omega | T \phi(x_1) \phi(x_2) | \Omega \rangle}{\langle 0 | T e^{i \int d^4 z \mathcal{L}_{int}(z)} \phi(x_1) \phi(x_2) | 0 \rangle}$$

Both denominators cancel out the vacuum fluctuation.

Cumulant Expansion

Cumulant expansion: $e^{\widehat{A}+x\widehat{B}} = e^{\widehat{A}} \sum_{n=0}^{\infty} x^n \widehat{B}_n$, as $x \rightarrow 0$

$$\widehat{B}_0 = 1 \quad (14)$$

$$\begin{aligned} \widehat{B}_1 &= \int_0^1 d\lambda_1 \widehat{B}(\lambda_1) \\ \dots \end{aligned} \quad (15)$$

$$\widehat{B}_n = \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_{n-1}} d\lambda_n \widehat{B}(\lambda_1) \widehat{B}(\lambda_2) \cdots \widehat{B}(\lambda_n) \quad (16)$$

with $\widehat{B}(\lambda_i) = e^{-\lambda_i \widehat{A}} \widehat{B} e^{\lambda_i \widehat{A}}$.