# Resummation of large logarithms in cross sections at subleading power

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# Introduction

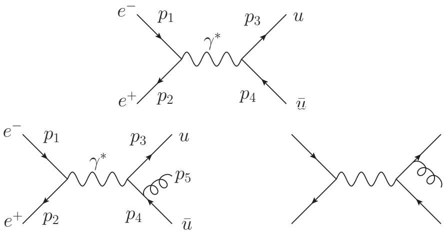
- Test of the Standard Model and New Physics searches require precise theoretical predictions.
- Perturbative QCD corrections are typically computed as an expansion in $\alpha_s$. State-of-the-art predictions are up to the third order for processes with a single scale, and up to NNLO for many processes with multiple scales.
- When the scales are widely separated, the presence of large logarithms requires an all-order resummation of the series $\alpha_s \ln Q_h/Q_s$. Threshold resummation and $p_T$ resummation.
- Analytical results of the amplitudes in the IR region have been applied in $N^3LO$ differential calculation.
- Resummation at LP is generally well understood. In contrast, NLP resummation is not fledged.

# Introduction

New features at NLP:

- Soft quark starts to contribute.
- The smaller light-cone components of a collinear momenta (field) can not be simply neglected.
- The convolution between H and J , or J and S, is ill-defined. Endpoint singularities appear.
- The RG equation is more non-local, and the solution becomes harder.
- The (off-diagonal) anomalous dimensions may contain double logarithms.

# An example




$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi}\left[\frac{2(3T^2 \overset{(1)}{-} 3T+2)}{T(1-T)} \ln\left(\frac{2T-1}{1-T}\right)^{(2)} - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$

with

$$T \equiv \max_{\vec{n}} T_{\vec{n}} = \max_{\vec{n}} \frac{\sum_i |\vec{n}\cdot\vec{p}_i|}{\sum_i |\vec{p}_i|}$$

No analytical NLO results though only one parameter appears. Numerical NNLO results have been obtained [Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, '07]. NNNLO soft functions [Baranowski, Delto, Melnikov, Wang, '22, Chen, Feng, Jia, Liu, '22].

# An example

In the limit $\tau \equiv 1 - T \to 0$,

$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi} \left[ \frac{4}{\tau} \ln\left(\frac{1}{\tau}\right) - 2 \ln\left(\frac{1}{\tau}\right) + \cdots \right]$$

Can we obtain these large logarithms without performing the complicated phase space integral? (Is there a simple way to derive these logarithms?)

Actually, since $\alpha_s \ln \tau \sim 1$ or even larger than 1, it is not valid any more to expand the cross section in $\alpha_s$. Infinite higher orders of such kind of logarithms matter.

# Leading power: Renormalization group view

Scale hierarchy: $E_{\rm cm} \gg \sqrt{\tau} E_{\rm cm} \gg \tau E_{\rm cm}$. The physics at different scales decouples from each other; no interference between waves of different length happens; the process factorizes into hard, jet, and soft functions.

$$\frac{d\sigma}{\sigma_B d\tau} = |C_H(\mu)|^2 \int d\tau_s d\tau_c d\tau_{\bar{c}} \delta(\tau - \tau_s - \tau_c - \tau_{\bar{c}}) \\ J(\tau_c, \mu) J(\tau_{\bar{c}}, \mu) S(\tau_s, \mu)$$

Laplace transform $\tilde{f}(N) = \int_0^\infty dx e^{-xN} f(x)$:

$$\frac{d\tilde{\sigma}(N)}{\sigma_B d\tau} = |C_H(\mu)|^2 \tilde{J}(N, \mu) \tilde{J}(N, \mu) \tilde{S}(N, \mu)$$

The RG equation is local:

$$\frac{d}{d \ln \mu^2} \tilde{J}(N, \mu) = \left[ \Gamma_J \ln \frac{\mu^2}{E_{\rm cm}^2 / N} + \gamma_J \right] \tilde{J}(N, \mu) \qquad (1)$$

# Lessons learned

The logarithms (at leading power) can be derived by

1. Factorization of the cross section
2. Calculation of the anomalous dimension of each ingredient

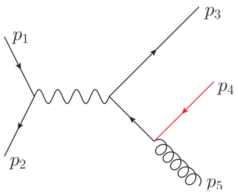
# Next-to-leading power contribution: "Gluon" thrust

At next-to-leading power, there are contributions from both soft/collinear gluons and soft/collinear quarks. [Boughezal, Liu, Petriello, '16, Moult, Rothen, Stewart, Tackman, Zhu, '16,'17]

NLP effects: a) NLP Lagrangian insertions, b) NLP operators, c) NLP measurement functions, d) NLP phase space, Only the first two exist for the quark contribution.

To focus on this situation, we consider the "gluon" thrust, where there is a $q\bar{q}$ pair recoiling against a gluon at the leading order.

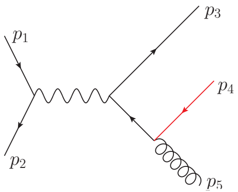
This talk is based on 2205.04479 [Beneke, Garny, Jaskiewicz, Strohm, Szafron, Vernazza, JW, '22.]




Firstly, we consider the soft limit of $p_4 \sim \mathcal{O}(\tau)$. The NLP x-sec:

$$\begin{aligned}\frac{1}{\sigma_B}\frac{d\sigma_s^{\rm NLP(1)}}{d\tau} =& \frac{g_s^2 C_F}{2(2\pi)^3}\int dn_+p_4 dn_-p_4 d^{d-2}p_{4\perp}\delta(p_4^2)\frac{2}{n_+p_4 E_{\rm cm}} \\ &\times\left[\delta(\tau - \frac{n_-p_4}{E_{\rm cm}})\theta(n_+p_4 - n_-p_4) + (n_- \leftrightarrow n_+)\right] \\ =& \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\tau^{-2\epsilon}E_{\rm cm}^{-2\epsilon} + \mathcal{O}(\epsilon^0) \qquad (2)\end{aligned}$$

with $p_3 \parallel n_-$, $p_5 \parallel n_+$.




Then, we consider the collinear limit of $p_4 \parallel p_3$. The NLP result

$$\frac{1}{\sigma_B}\frac{d\sigma_c^{\mathrm{NLP}(1)}}{d\tau} = \frac{g_s^2 C_F}{16\pi^2}\int ds_{34}\int_0^1 dz[z(1-z)]^{-\epsilon} s_{34}^{-\epsilon}\frac{2}{E_{\mathrm{cm}}^2}\frac{\bar{z}}{z}\delta(\tau - \frac{s_{34}}{E_{\mathrm{cm}}^2})$$
$$= \frac{\alpha_s C_F}{2\pi}\frac{-1}{\epsilon}\tau^{-\epsilon}E_{\mathrm{cm}}^{-2\epsilon} + \mathcal{O}(\epsilon^0) \quad (3)$$

The sum of the soft and collinear contribution is

$$\frac{1}{\sigma_{\mathrm{B}}}\frac{d(\sigma_s^{\mathrm{NLP}(1)} + \sigma_c^{\mathrm{NLP}(1)})}{d\tau} = \frac{\alpha_s C_F}{2\pi}E_{\mathrm{cm}}^{-2\epsilon}\left[\frac{1}{\epsilon}\tau^{-2\epsilon} - \frac{1}{\epsilon}\tau^{-\epsilon}\right] \quad (4)$$
$$= -\frac{\alpha_s C_F}{2\pi}\ln\tau \quad (5)$$

# Large logarithms

Summarize the results based on factorization of the cross section,

$$\sigma(\tau) = \sigma_B \sum_n \alpha_s^n \left[ c_n \delta(\tau) + \sum_{m=0}^{2n-1} \left( c_{nm} \frac{\ln^m \tau}{\tau} + \underbrace{d_{nm} \ln^m \tau}_{NLP} \right) + \cdots \right]$$

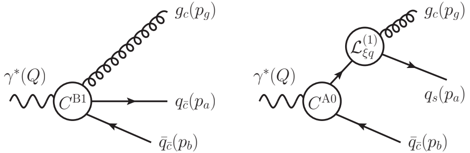
$c_{nm}$ are fully determined by the anomalous dimensions of the jet function and soft function. In this sense, they are universal.

$d_{nm}$ denote the power suppressed logarithms. The question is how to develop a factorization formula for resummation of this power suppressed contribution.

# Recent development

- Beyond leading logarithms (at $O(\alpha_s)$) [Boughezal, Isgro, Petriello, '18, Ebert, Moult, Stewart, Tackmann, Vita, Zhu, '18 ]
- Beyond $2 \to 1$ or $1 \to 2$ [Beekveld, Beenakker, Laenen, White '19,Boughezal, Isgro, Petriello, '19]
- Threshold/Thrust resummation at NLP [Moult, Stewart, Vita, Zhu, '18, Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, JW, '18-'21, Bahjat-Abbas, Bonocore, Damste, Laenen, Magnea, Vernazza, White '19, Ajjath, Mukherjee, Ravindran '20]
- Rapidity divergences in $q_T$ spectrum or energy-energy correlators [Ebert, Moult, Stewart, Tackmann, Vita, Zhu, '18, Moult, Vita, Yan, '19]
- Soft quark Sudakov [Liu, Penin, '17, '18, '21, Moult, Stewart, Vita, Zhu, '19, Liu, Mecaj, Neubert, Wang, Fleming, '20, JW, '20]
- Subleading power effects in B physics and heavy quarkonium production [Ma, Qiu, Sterman, Zhang '13, Lee, Sterman '20, Li, Lü, Sheng Wang, Wang, Wei, '17,'20]

# Endpoint divergences




$$\int_0^1 drdr'\, C^{B1}(r)C^{B1}(r')^* \otimes \mathcal{J}_{\bar{c}}^{q\bar{q}}(r,r') \times \mathcal{J}_c^{(g)} \times S^{(g)}$$

$$\int_0^\infty d\omega d\omega'\, \left|C^{A0}\right|^2 \times \mathcal{J}_{\bar{c}}^{(\bar{q})} \times \mathcal{J}_c(\omega,\omega') \otimes S_{\mathrm{NLP}}(\omega,\omega') \qquad (6)$$

Since $C^{B1}(r) \sim \frac{1}{r}$ and $\mathcal{J}_c(\omega,\omega') \sim \frac{1}{\omega\omega'}$, endpoint singularities appear as $r, r' \to 0$ or $\omega, \omega' \to \infty$.

Physical picture: collinear particles become soft, or soft particles become collinear. The overlapping regions appear already at leading power, or more generally in the calculation of loop integrals using expansion by regions .

Naive factorization and resummation are broken.

# Endpoint divergences

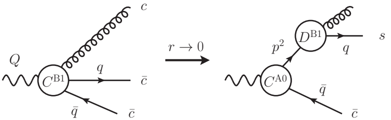
As seen in the leading order result, the crucial point is that the endpoint divergences from the A-type and B-type contributions cancel each other.
The reason is that the endpoint divergences come from the overlapping regions, where the integrals become scaleless in dimensional regularization after re-expansion the integrands.

$$\int_{-\infty}^{\infty} dr \theta(r)\theta(1-r) r^{-1-\epsilon} \to \int_0^{\infty} dr \; r^{-1-\epsilon} \tag{7}$$

$$\int_{-\infty}^{\infty} d\omega \theta(\omega-\tau)\omega^{-1-\epsilon} \to \int_0^{\infty} d\omega \; \omega^{-1-\epsilon} \tag{8}$$

For $r \to 0$ (soft quark), the matching coefficient of $g_c(q\bar{q})_{\bar{c}}$ B1 operator becomes a two-scale object, which can itself be factorized according to [Beneke et al, '20]

$$C_1^{\mathrm{B1}}(Q^2, r) = C^{\mathrm{A0}}(Q^2) \times \frac{D^{\mathrm{B1}}(rQ^2)}{r} + \mathcal{O}(r^0)\,, \qquad (9)$$




$D^{B1}$ is also called the radiative jet function, a universal structure in NLP factorization formula. [Beneke et al, '20, Liu, Neubert, Schnubel, Wang, '21] It exhibits all-order $(C_A - C_F)^n$ color coefficients.[Vogt, '10, Moult, Stewart, Vita, Zhu, '19] The Abelian analogue also appears in B-meson decay and $H \to \gamma\gamma$, and has been computed up to twoloop for the fixed order result [Liu, Neubert, '20] or up to oneloop for RG equation [Bodwin, Ee, Lee, Wang, '21].

# Endpoint factorization

$$D^{\rm B1}(p^2) = 1 + \frac{\alpha_s}{4\pi}\left(C_F - C_A\right)\left(\frac{2}{\epsilon^2} - 1 - \frac{\pi^2}{6}\right)\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right)^{\epsilon}. \qquad (10)$$

Twoloop results have been obtained. [Liu,Neubert, Schnubel, Wang, '21] The anomalous dimension can be derived from that for general B1 operators [Beneke, Garny, Szafron, JW, '17-'18] or from RG consistency condition in $H \to gg$ [Liu, Neubert, Schnubel, Wang, '21]

$$\frac{d}{d\ln\mu} D^{\rm B1}(p^2) = \int_0^\infty d\hat{p}^2\, \gamma_D(\hat{p}^2, p^2) D^{\rm B1}(\hat{p}^2)\,, \qquad (11)$$

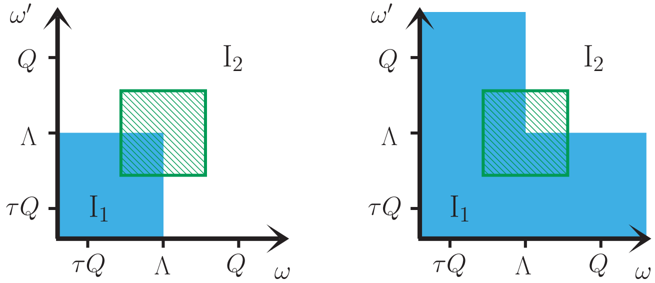
$$\gamma_D(\hat{p}^2, p^2) = \frac{\alpha_s(C_F - C_A)}{\pi}\,\delta(\hat{p}^2 - p^2)\ln\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right) + \frac{\alpha_s}{\pi}\left(\frac{C_A}{2} - C_F\right) p^2 \left[\frac{\theta(\hat{p}^2 - p^2)}{\hat{p}^2(\hat{p}^2 - p^2)} + \frac{\theta(p^2 - \hat{p}^2)}{p^2(p^2 - \hat{p}^2)}\right]_+ ,$$

The nonlocal term is the Lange-Neubert kernel that has appeared often in B-Meson Light-Cone Distribution Amplitude. Solutions of the RG equations containing such terms have been studied up to twoloop.[Bell, Feldmann, Wang, Yip, '13,Braun, Ji, Monashov, '14,'19, Galda, Neubert, Wang, '22]

Construct a scaleless integral

$$0 = |C^{\rm A0}(Q^2)|^2 \widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R)\widetilde{\mathcal{J}}_{c}^{(g)}(s_L) \times \int_0^\infty d\omega d\omega' \, \frac{D^{\rm B1}(\omega Q)}{\omega} \frac{D^{\rm B1*}(\omega' Q)}{\omega'} \left[\!\left[ \widetilde{S}_{\rm NLP}(s_R, s_L, \omega, \omega') \right]\!\right] \quad (12)$$

And split it to two parts to subtract the end-point divergences in the A-type ($I_2$) and B-type ($I_1$) contribution, respectively.



# Endpoint divergences subtracted A-type contribution

The Laplace transformed differential cross section:
$(\Lambda \gg 1/s_R, 1/s_L)$

$$
\begin{aligned}
\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}|_{\mathrm{A-type}} &= |C^{\mathrm{A0}}(Q^2)|^2\, \widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R) \int_0^\infty d\omega d\omega' \\
&\times \Big\{ \Big[1-\theta(\omega-\Lambda)\theta(\omega'-\Lambda)\Big] \widetilde{\mathcal{J}}_c(s_L,\omega,\omega')\, \widetilde{S}_{\mathrm{NLP}}(s_R,s_L,\omega,\omega') \Big\}
\end{aligned}
$$

# Endpoint divergences subtracted B-type contribution

The Laplace transformed differential cross section:

$$
\begin{aligned}
\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}\Big|_{\mathrm{B-type}} &= \widetilde{\mathcal{J}}_c^{(g)}(s_L)\,\widetilde{S}^{(g)}(s_R,s_L)\int_0^\infty dr dr' \\
&\times\Big[\theta(1-r)\theta(1-r')\,C_1^{\mathrm{B1}*}(Q^2,r')C_1^{\mathrm{B1}}(Q^2,r)\,\widetilde{\mathcal{J}}_{\bar{c}}^{q\bar{q}(8)}(s_R,r,r') \\
&\quad -\big[1-\theta(r-\Lambda/Q)\theta(r'-\Lambda/Q)\big] \\
&\quad \times [\![C_1^{\mathrm{B1}*}(Q^2,r')]\!]_0\,[\![C_1^{\mathrm{B1}}(Q^2,r)]\!]_0\,[\![\widetilde{\mathcal{J}}_{\bar{c}}^{q\bar{q}(8)}(s_R,r,r')]\!]_0\Big] \qquad (13)
\end{aligned}
$$

- The $\Lambda$ dependence cancels between the A- and B-type contributions.
- There is no endpoint divergence any more in each type. They are also scale independent.
- RG evolution can be derived as usual for the integrand.

The Wilson coefficient of the hard operator satisfies

$$\frac{d}{d\ln\mu} C^{\mathrm{A0}}\left(Q^2,\mu^2\right) = \left[C_F\gamma_{\mathrm{cusp}}\left(\alpha_s\right)\ln\frac{-Q^2}{\mu^2} + \gamma_{\mathrm{A0}}\left(\alpha_s\right)\right] C^{\mathrm{A0}}\left(Q^2,\mu^2\right) \tag{14}$$

with the solution at leading log

$$C_{\mathrm{LL}}^{\mathrm{A0}}\left(Q^2,\mu^2\right) = \exp\left[2C_F S\left(\mu_h,\mu\right)\right]\left(\frac{-Q^2}{\mu_h^2}\right)^{-C_F A_{\gamma_{\mathrm{cusp}}}\left(\mu_h,\mu\right)} C^{\mathrm{A0}}\left(Q^2,\mu_h^2\right) \tag{15}$$

Functions:

$$A\left(\nu,\mu\right) \approx \frac{2}{\beta_0}\ln\frac{\alpha_s\left(\mu\right)}{\alpha_s\left(\nu\right)} \approx -\frac{\alpha_s}{2\pi}\ln\frac{\mu^2}{\nu^2} \tag{16}$$

$$S\left(\nu,\mu\right) \approx \frac{4\pi}{\beta_0^2\alpha_s\left(\nu\right)}\left[1-\frac{\alpha_s\left(\nu\right)}{\alpha_s\left(\mu\right)}-\ln\frac{\alpha_s\left(\mu\right)}{\alpha_s\left(\nu\right)}\right] \approx -\frac{\alpha_s}{8\pi}\ln^2\frac{\mu^2}{\nu^2}$$

Similar equations for the LP collinear jet functions.

We focus on the asymptotic region.

$$\frac{d}{d\ln\mu}\left[\!\left[C_1^{\mathrm{B1}}\left(Q^2,r,\mu^2\right)\right]\!\right]_0 = \Bigg[C_F\gamma_{\mathrm{cusp}}\left(\alpha_s\right)\ln\frac{-Q^2}{\mu^2} - \left(C_F-C_A\right)\gamma_{\mathrm{cusp}}\left(\alpha_s\right)\ln\frac{-rQ^2}{\mu^2}+\gamma_{\mathrm{A0}}\left(\alpha_s\right)\Bigg]\left[\!\left[C_1^{\mathrm{B1}}\left(Q^2,r,\mu^2\right)\right]\!\right]_0 + Q\int_0^\infty d\hat{r}\,\frac{\hat{r}}{r}\,\hat{\gamma}_D\left(\hat{r}Q,rQ\right)\left[\!\left[C_1^{\mathrm{B1}}\left(Q^2,\hat{r},\mu^2\right)\right]\!\right]_0 . \tag{17}$$

The LL solution reads

$$\left[\!\left[C_1^{\mathrm{B1}}\left(Q^2,r,\mu^2\right)\right]\!\right]_0 = \exp\left[2C_F S\left(\mu_h,\mu\right)-2\left(C_F-C_A\right)S\left(\mu_{h\Lambda},\mu\right)\right] \times\left(\frac{-Q^2}{\mu_h^2}\right)^{-C_F A_{\gamma_{\mathrm{cusp}}}\left(\mu_h,\mu\right)}\left(\frac{-rQ^2}{\mu_{h\Lambda}^2}\right)^{\left(C_F-C_A\right)A_{\gamma_{\mathrm{cusp}}}\left(\mu_{h\Lambda},\mu\right)} \times\left[\!\left[C_1^{\mathrm{B1}}\left(Q^2,r,\mu_h^2,\mu_{h\Lambda}^2\right)\right]\!\right]_0 , \tag{18}$$

The NLP collinear function (operators of two collinear fields) has a similar structure.

Collecting every piece together, we have the Leading log resummed cross section

$$
\begin{aligned}
\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}\Big|_{\mathrm{LL}} \quad = \quad & \frac{\alpha_s(Q/(s_L e^{\gamma_E}))C_F}{\pi}\frac{1}{Q s_R} \\
& \times \exp\left[4C_F S(Q^2, \frac{Q}{s_R e^{\gamma_E}}) + 4C_A S(\frac{1}{s_L s_R e^{2\gamma_E}}, \frac{Q}{s_L e^{\gamma_E}})\right] \\
& \times \int_\sigma^Q \frac{d\omega}{\omega} \exp\left[-4\left(C_F - C_A\right) S(\omega Q, \frac{\omega}{s_R e^{\gamma_E}})\right] \\
& \quad \times \left(s_R e^{\gamma_E} Q\right)^{2C_F A(\omega/s_R e^{\gamma_E}, Q/s_R e^{\gamma_E}) + 2C_A A(Q/s_L e^{\gamma_E}, \omega/s_R e^{\gamma_E})}
\end{aligned}
$$

The integral over $\omega$ and the inverse Laplace transformation can be performed only numerically.

$$
\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{Q}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{s\tau Q}\, \frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}\Big|_{s_R=s_L=s}
$$

The double log limit:

$$\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L} = \frac{\alpha_s C_F}{\pi}\frac{1}{Qs_R}\, e^{-\frac{\alpha_s C_A}{\pi}\ln(s_L e^{\gamma_E}Q)\ln(s_R e^{\gamma_E}Q)} \int_\sigma^Q \frac{d\omega}{\omega}\left(\frac{\omega}{Q}\right)^{\frac{\alpha_s}{\pi}(C_F-C_A)\ln(s_R e^{\gamma_E}Q)}$$
$$= \frac{C_F}{C_F - C_A}\frac{1}{Qs_R\ln(s_R e^{\gamma_E}Q)}\exp\left[-\frac{\alpha_s C_A}{\pi}\ln\left(s_L e^{\gamma_E}Q\right)\ln\left(s_R e^{\gamma_E}Q\right)\right]$$
$$\times\left\{1-\left(\frac{\sigma}{Q}\right)^{\frac{\alpha_s}{\pi}(C_F-C_A)\ln(s_R e^{\gamma_E}Q)}\right\}$$

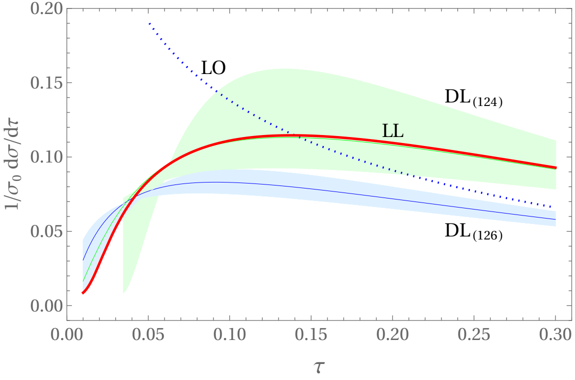
To obtain the thrust distribution we set $s_R = s_L = s$, and perform the inverse Laplace transformation,

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}\Big|_{\mathrm{DL}} = \frac{C_F}{C_F - C_A}\frac{1}{\ln(1/\tau)}\, e^{-\frac{\alpha_s C_A}{\pi}\ln^2\tau}\left\{1 - e^{-\frac{\alpha_s}{\pi}(C_F-C_A)\ln^2\tau}\right\}$$

This agrees with the result derived from different methods. [Moult, Stewart, Vita, Zhu, '19, Beneke, Garny, Jaskiewicz, Szafron, Vernazza, JW, '20]

# Resummation

Choose $Q = 91.1876$ GeV. The default scale of $\alpha_s$ is $\sqrt{\tau}Q$, and is changed from $\tau Q$ to $Q$. $\mathrm{DL}_{(124)}$ denotes the result after performing numerically the inverse Laplace transformation, while $\mathrm{DL}_{(126)}$ is obtained analytically.



# Summary

- The end-point divergences appearing in NLP factorization formula prevent a straight forward application of the traditional resummation techniques.
- We develop a novel end-point factorization relation for the "gluon" thrust, which is an off-diagonal process in a more general sense. The central property is that the soft quark contribution in the large limit coincides with the collinear quark contribution in the small limit.
- As such, a scaleless integral can be constructed to subtract the end-point divergences. Then the standard RG equations can be used. More nonlocal evolution kernels occur. The solution of these equations are still unknown.
- We reproduce the double log resummation formula, and obtain the LL one. More hard work is required to go beyond LL.
- Our framework may also be applied to the diagonal channel at NLP.

# Thank you !