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$\mathcal{N}=4$ supersymmetric Yang-Mills thermodynamics to order λ^2

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Outline:

- Background and Motivations
- Free energy to λ^2 of SYM[a]$_{4,4}$ theory
- Large-N_c generalized Padé approximant
- Comparison for scaled entropy density
- Summary

[a]The notation SYM$_{x,y}$ indicates $\mathcal{N}$ = x supersymmetric Yang-Mills theory in y spacetime dimensions.

Background and Motivations

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM$_4$)

Definition

A super Yang-Mills theory with the maximum number $\mathcal{N} = 4$ of supersymmetries, where **$\mathcal{N} = 4$ refers to the number of supersymmetric charges**.

Prominent feature

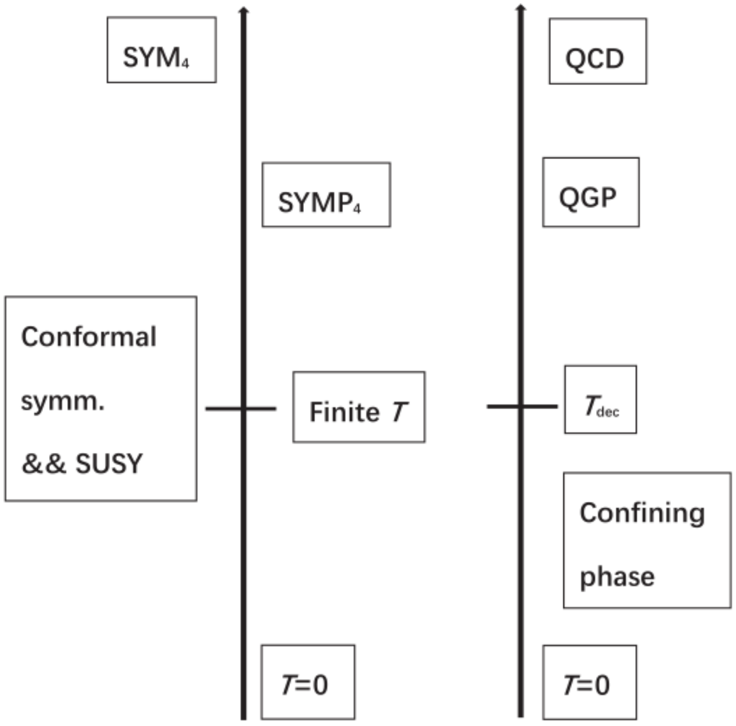
The theory is finite. Proof: The one-loop β function equals to 0.
This means that the coupling does not run.

Application

- The **most famous example** of a conformal field theory (CFT) in 4 dimension.
- Be often **taken as a model for hot QCD** in the large N_c and strong 't Hooft coupling λ limits, $\lambda = g^2 N_c$, N_c is number of colors, g is the gauge coupling constant.

Background and Motivations

✓ Relation between SYM_4 theory and QCD



QGP and **$SYMP_4$** are surprisingly similar to each other **in the weak coupling regime** (**high temperature T**).

Key difference

(energy densities, Debye masses, shear viscosity, energy loss)

The number and types of degrees of freedom.
(Four Majorana fermions and six scalars)

Background and Motivations

The **perturbative expansion** of the free energy of the $SYM_{4,4}$ at high T is

$$F(\lambda \to 0) \sim T^4 \left[a_0\lambda^0 + a_2\lambda^1 + a_3\lambda^{3/2} + (a_4 + a_4'\log\lambda)\lambda^2 + \mathcal{O}\left(\lambda^{5/2}\right)\right], \quad (1)$$

$\sim a_0\lambda^0$ ⟹ Free energy of the ideal $SYM_{4,4}$ plasma ⟹ non-analytic, e.g. $\mathcal{O}(\lambda^{3/2})$

⇓

$\sim a_2\lambda^1$ ⟹ The 2-loop Feynman diagrams ⟹ non-analytic, e.g. $\mathcal{O}(\lambda^2 \log\lambda)$

A. D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys.53 (1981) 43.

⇓

$\sim\mathcal{O}(\lambda^{n>1})$ ⟹ Reorganizing the perturbation theory[2] to account for the thermal mass of the gauge bosons and scalar fields

A. P.B. Arnold and C.-X. Zhai, hep-ph/9408276; hep-ph/9410360.

⇑

Like QCD[3], there are uncanceled infrared divergences at the three-loop level

Full $\mathcal{O}(\lambda^2)$ ⟹ Require 3-loop calculation ⟹ Generate $\mathcal{O}(\lambda^{5/2})$

⇑

Consider the dressed propagators

Background and Motivations

In the **weak-couping limit** the $SYM_{4,4}$ free energy has been calculated through order $\lambda^{3/2}$ giving[4]

A. Fotopoulos and T. R. Taylor. Phys. Rev. D 59:061701,1999.
B. C.-j. Kim and S.-J. Rey. Nucl. Phys. B 564:430–440, 2000.
C. M. A. Vazquez-Mozo. Phys. Rev. D 60:106010, 1999.

$$\frac{\mathcal{F}}{\mathcal{F}_{\mathrm{ideal}}} = \frac{\mathcal{S}}{\mathcal{S}_{\mathrm{ideal}}} = 1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{\frac{3}{2}} + \mathcal{O}(\lambda^2)\,, \qquad (2)$$

The **strong coupling** behavior of the free energy has been computed using the anti-de Sitter space/CFT (AdS/CFT) correspondence[5]

A. S. S. Gubser, I. R. Klebanov. Nucl. Phys. B 534:202–222, 1998.

$$\frac{\mathcal{F}}{\mathcal{F}_{\mathrm{ideal}}} = \frac{\mathcal{S}}{\mathcal{S}_{\mathrm{ideal}}} = \frac{3}{4}\left[1 + \frac{15}{8}\zeta(3)\lambda^{-\frac{3}{2}} + \mathcal{O}(\lambda^{-2})\right], \qquad (3)$$

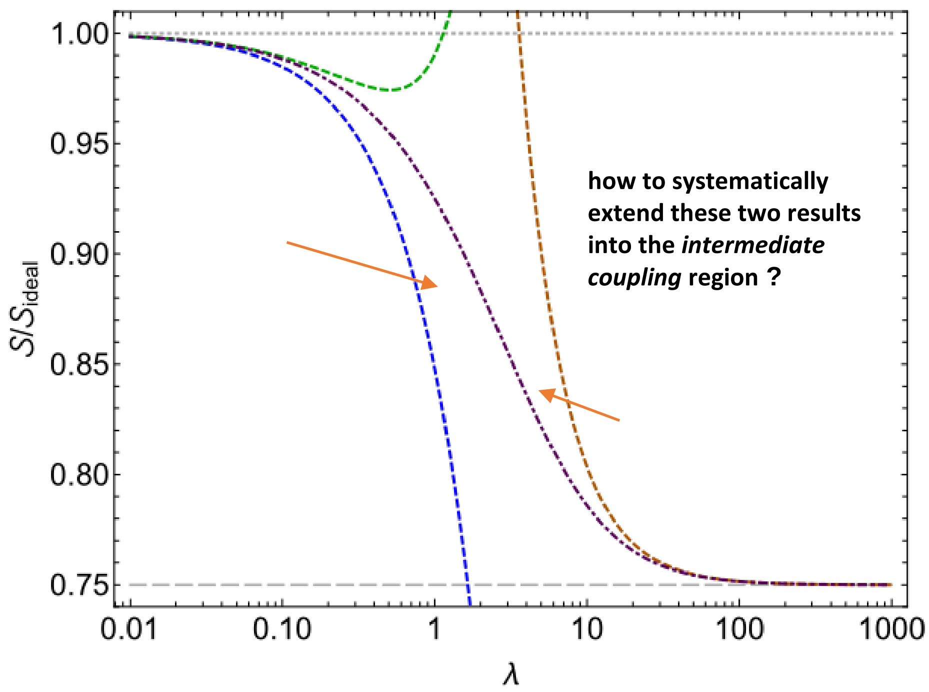
$\mathcal{F}_{\mathrm{ideal}} = -d_A\pi^2T^4/6$,$\mathcal{S}_{\mathrm{ideal}} = 2d_A\pi^2T^3/3$: the ideal or Stefan-Boltzmann limit of the **free energy** and **entropy density**.

$\boldsymbol{d_A} = N_c^2 - 1$: the **dimension** of the adjoint representation .

Background and Motivations

✓ Comparison of the scaled entropy density（ $\mathcal{S}/\mathcal{S}_{\mathbf{ideal}}$ ）between the weak and strong coupling results in SYM$_4$ and $R_{[4,4]}$ *Padè* approximation

Padè approximant is constructed by interpolating between the weak- and strong-coupling limits.



A. J. P. Blaizot, E. Iancu, U. Kraemmer. JHEP, 06:035, 2007.

B. C.-j. Kim and S.-J. Rey. Nucl. Phys. B 564:430–440, 2000.

Motivation of our work ⟹ **Systematically extend the weak coupling result into the *intermediate coupling* region.**

The aim of this work was to get the 4$^{\text{th}}$ term $\sim(a_4 + a_4'\log\lambda)\lambda^2$

Background and Motivations

- **Under the scheme called**
  regularization by dimensional reduction **(RDR)**[7]

A. **W.Siegel**, Phys. Lett. B 84(1979) 193.
B. D.Capper, D.Jones and P.Van Nieuwenhuizen, Nuclear Physics B 167(1980) 479.
C. P.Howe, A.Parkes and P.West, Physics Letters B 147 (1984) 409.

RDR ⇨ A modified version of the dimensional regularization based on dimensional reduction which manifestly preserves gauge invariance, unitarity, and supersymmetry

⇩

Applied to pure Yang-Mills theory; supersymmetric QED and $\boldsymbol{\mathcal{N} = 1, 2,}$ **and 4 SYM** theory[8]

A. L.V. Avdeev and A.A. Vladimirov, Nucl. Phys. B 219 (1983) 262.

Dimensional reduction(DR)[9] ⇨ $SYM_{4,4}$ can be obtained from $SYM_{1,10}$ **without thermal mass contributions.**

A. L. Brink, J. H. Schwarz, and J. Scherk. Nucl. Phys. B 121:77–92, 1977.

- Lagrangian density for $SYM_{1,10}$ and $SYM_{4,4}$

$\mathcal{L}$ for $SYM_{1,10}$:

$$\mathcal{L}_{SYM_{1,10}} = \mathrm{Tr}[-\frac{1}{2}G_{MN}^2 + 2i\overline{\psi}\Gamma^M D_M\psi],\ M,N = 0,\cdots,9, \quad (4)$$

the field strength tensor: $G_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N]$, the covariant derivative in the adjoint representation of $SU(N_c)$: $D_M = \partial_M - ig[A_M,\cdot]$.

$\mathcal{L}$ for $SYM_{4,4}$ can be obtained by DR from $SYM_{1,10}$:

$$\mathcal{L}_{SYM_{4,4}} = \mathrm{Tr}[-\frac{1}{2}G_{\mu\nu}^2 + (D_\mu\Phi_A)^2 + i\overline{\psi}_i \not{D}\psi_i - \frac{1}{2}g^2(i[\Phi_A,\Phi_B])^2 - ig\overline{\psi}_i[\alpha_{ij}^p X_P + i\beta_{ij}^q \gamma_5 Y_q, \psi_j]],\ \mu,\nu = 0,\cdots,3, \quad (5)$$

$\psi_i\ (i = 1,\cdots,4)$: four Majorana fermions, $\Phi \equiv (X_1,Y_1,X_2,Y_2,X_3,Y_3)$: six independent real scalar fields.

Add the gauge-fixing and ghost terms to quantize $SYM_{1,10}$ and $SYM_{4,4}$.

All fields belong to the adjoint representation of the SU(N_c) gauge group.

Background and Motivations

Feynman diagrams calculations **require**

In RDR scheme

Keep all **fields** to be **integer-dimensional tensors or spinors** in the case of extended supersymmetry

Preserve supersymmetry

Keep all **momenta** to be **such as 4−2ϵ dimensional vectors**, where ϵ is infinitesimal

Ensure ultraviolet finiteness

Variations of RDR

- Results from them must be the same as RDR
- **Disadvantage:** computations are more complex

Be Discarded!

Free energy to λ^2 of SYM$_{4,4}$ theory

➢ The resummed Lagrangian density

Following Arnold and Zhai, the resummed Lagrangian density can be written as

$$\mathcal{L}^{\text{resum}}_{\text{SYM}_{4,4}} = \{\mathcal{L}_{\text{SYM}_{4,4}} + \text{Tr}[m_D^2 A_0^2 \delta_{p_0} - M_D^2 \Phi_A^2 \delta_{p_0}]\} - \text{Tr}[m_D^2 A_0^2 \delta_{p_0} - M_D^2 \Phi_A^2 \delta_{p_0}] \quad (6)$$

Contribute to the gluon and scalar counterterms

Contribute to the gluon and scalar propagators

where

1) m_D and M_D only contribute to the zero Matsubara modes of the two fields, and

$$m_D^2 = \lambda(d-2)[(D+4)b_1 - 8f_1] = 2\lambda T^2 + \mathcal{O}(\epsilon),$$
$$M_D^2 = \lambda[(D+4)b_1 - 8f_1] = \lambda T^2 + \mathcal{O}(\epsilon). \quad (7)$$

2) δ_{p_0} is shorthand for the Kronecker delta function $\delta_{p_0,0}$.

Free energy to λ^2 of SYM$_{4,4}$ theory

The calculation of the thermodynamics up to $\mathcal{O}(\lambda^2)$

Feynman diagrams up to 2-loop order

⇓

Compute directly in SYM$_{4,4}$ under the RDR scheme , since we must dress the gluon and scalar propagators differently

⇓

Feynman diagrams at the 3-loop order

3-loop gluonic and scalar counterterm with bare propagators

⇓

Compute directly in SYM$_{4,4}$

⇓

$g_M^M = 4$, Tr$I_4 = 4$, the dimension for the gluon field $D = 4$ and for all momentum $d = 4 - 2\epsilon$

Massless 3-loop diagrams

⇓

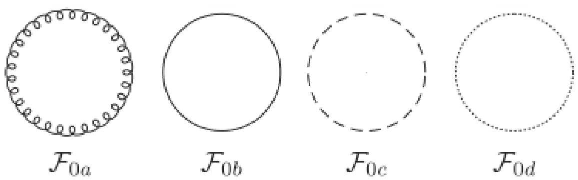
Compute in SYM$_{1,10}$ under the RDR scheme

⇓

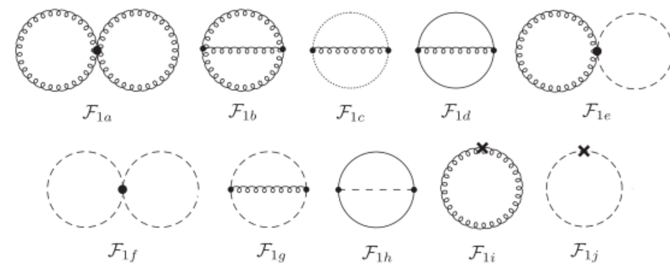
$g_M^M = 10$, Tr$I_{32} = 8$, the dimension for the gluon field $\mathcal{D} = 10$ and for all momentum $d = 4 - 2\epsilon$

Free energy to λ^2 of $SYM_{4,4}$ theory

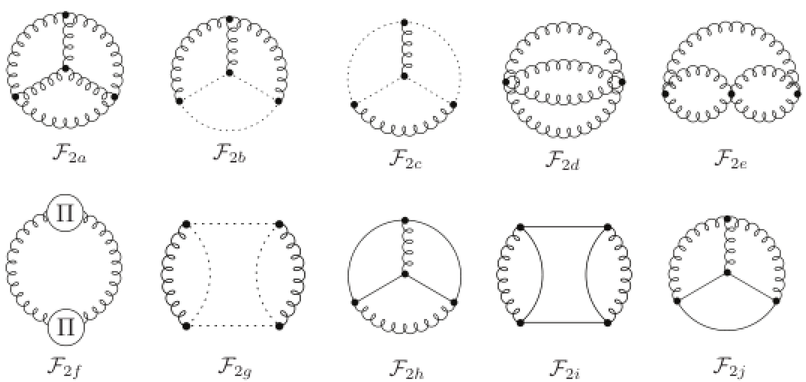
✓ Feynman diagrams up to the three-loop level



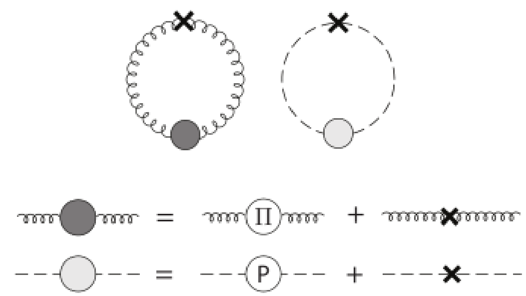
The 1-loop diagrams for $SYM_{4,4}$



The 2-loop diagrams for $SYM_{4,4}$



The 3-loop diagrams with bare propagators for $SYM_{1,10}$



The 3-loop gluonic and scalar counterterm diagrams with bare propagators for $SYM_{4,4}$

Dashed lines indicate a scalar field and dotted lines indicate a ghost field. The crosses are the thermal counterterms.

Free energy to λ^2 of SYM$_{4,4}$ theory

- **The resummed two-loop free energy**

$$F_{1-\mathrm{loop}}^{\mathrm{resum}} = \mathcal{F}_{\mathrm{ideal}}\left[1 + \frac{3+\sqrt{2}}{\pi^3}\lambda^{\frac{3}{2}}\right]. \tag{8}$$

- **The resummed one-loop free energy**

$$F_{2-\mathrm{loop}}^{\mathrm{resum}} = \mathcal{F}_{\mathrm{ideal}}\left[-\frac{3}{2\pi^2}\lambda - \frac{3}{2\pi^4}\left(\frac{23}{8} + \frac{3\sqrt{2}}{4} + \frac{15\log 2}{4} - \log\lambda\right)\lambda^2\right]. \tag{9}$$

- **The resummed three-loop free energy**

$$F_{\text{3-loop}}^{\mathrm{resum}} = \mathcal{F}_{\text{3-loop}} + \mathcal{F}_{\text{3-loop}}^{\mathrm{sct}} + \mathcal{F}_{\text{3-loop}}^{\mathrm{bct}}, \tag{10}$$

where

$$\begin{aligned} F_{\text{3-loop}} &= d_A\lambda^2[\mathcal{F}_{2a} + \mathcal{F}_{2b} + \mathcal{F}_{2c} + \mathcal{F}_{2d} + \mathcal{F}_{2e} + \mathcal{F}_{2f} + \mathcal{F}_{2g} \\ &\quad + \mathcal{F}_{2h} + \mathcal{F}_{2i} + \mathcal{F}_{2j}]|_{d=4-2\epsilon}^{\mathcal{D}=10}, \end{aligned} \tag{11}$$

which will generated infrared divergences which will be canceled by $\mathcal{F}_{3-\mathrm{loop}}^{\mathrm{sct}}$ and $\mathcal{F}_{3-\mathrm{loop}}^{\mathrm{bct}}$.

$$F_{\text{3-loop}}^{\mathrm{resum}} = \mathcal{F}_{\mathrm{ideal}}\frac{\lambda^2}{2\pi^4}\left[\frac{27}{8} + 3\gamma + 3\frac{\zeta'(-1)}{\zeta(-1)} + 5\log 2 - 6\log\pi\right]. \tag{12}$$

Free energy to λ^2 of $\text{SYM}_{4,4}$ theory

- **$\text{SYM}_{4,4}$ thermodynamic functions to $\mathcal{O}(\lambda^2)$**

The final result for the **resummed free energy up to order λ^2** for $\text{SYM}_{4,4}$ **in the RDR scheme** is

$$\mathcal{F} = \mathcal{F}_{\text{ideal}}\left\{1 - \frac{3}{2}\frac{\lambda}{\pi^2} + \left(3+\sqrt{2}\right)\left(\frac{\lambda}{\pi^2}\right)^{3/2} + \left[-\frac{21}{8} - \frac{9\sqrt{2}}{8} + \frac{3}{2}\gamma_E + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{25}{8}\log 2 + \frac{3}{2}\log\frac{\lambda}{\pi^2}\right]\left(\frac{\lambda}{\pi^2}\right)^2\right\}. \quad (13)$$

1) It **holds for all N_c**, and is **independent of** the momentum scale μ
2) **Infrared divergences** are **canceled** by considering m_D^2 and M_D^2
3) **No coupling constant renormalization counterterm** is required, since λ does not run in $\text{SYM}_{4,4}$

Large-N_c generalized Padé approximant

Based on the large-N_c structure of the strong-coupling expansion, we find the following form can reconstruct all known coefficients in both the weak- and strong-coupling limits

$$\frac{S}{S_{\text{ideal}}} = \frac{1 + a\lambda^{1/2} + b\lambda + c\lambda^{3/2} + d\lambda^2 + e\lambda^{5/2}}{1 + a\lambda^{1/2} + \bar{b}\lambda + \boxed{\frac{4}{3}}c\lambda^{3/2} + \boxed{\frac{4}{3}}d\lambda^2 + \boxed{\frac{4}{3}}e\lambda^{5/2}} \,. \quad (14)$$

To ensure that in the strong-coupling limit
1) one obtains the correct asymptotic limit of 3/4
2) terms of the form $\lambda^{-1/2}$, $\lambda^{-1/2}\log\lambda$, λ^{-1}, and $\lambda^{-1}\log\lambda$ do not appear in the strong-coupling expansion

Large-N_c generalized Padé approximant

To fix the remaining coefficients

- Make sure it **reproduces the weak-coupling** result through $\mathcal{O}(\lambda^2, \lambda^2 \log \lambda)$
- Make sure it **reproduces the strong-coupling result** through $\mathcal{O}(\lambda^{-3/2})$

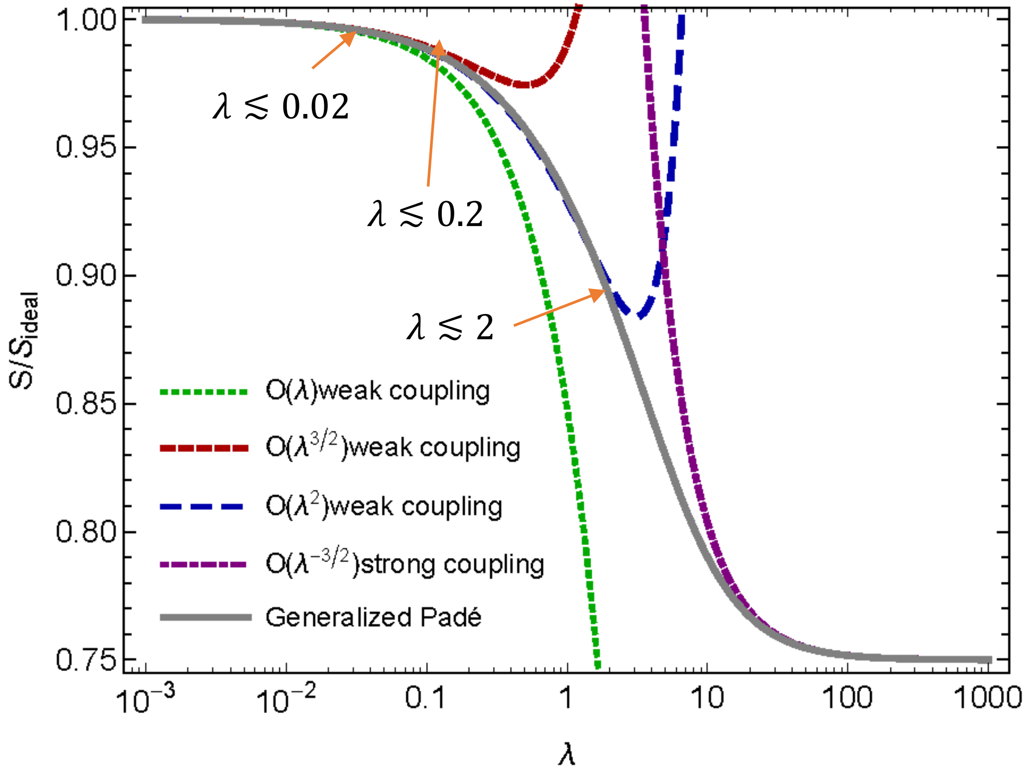
$$a = \frac{4\pi^2}{135\zeta(3)} + \frac{2(3+\sqrt{2})}{3\pi}, \quad \bar{b} = b + \frac{3}{2\pi^2}, \quad c = \frac{2}{15\zeta(3)},$$

$$d = \frac{180(3+\sqrt{2})\zeta(3)+8\pi^3}{2025\pi\zeta^2(3)}, \quad e = \frac{2b}{15\zeta(3)} - \frac{3}{5\pi^2\zeta(3)}, \tag{15}$$

$$b = \frac{1}{\pi^2}\log\left(\frac{\lambda}{\pi^2}\right) + \frac{16\pi\left[45(3+\sqrt{2})\zeta(3)+\pi^3\right]}{18225\zeta^2(3)} + \frac{36\left(\gamma_E + \frac{\zeta'(-1)}{\zeta(-1)}\right) + 69\sqrt{2} + 59 - 75\log(2)}{36\pi^2}$$

Comparison for scaled entropy density

✓ Comparison of $\mathcal{S}/\mathcal{S}_{\mathbf{ideal}}$ between the update Padè approximation and the perturbative results up to $\mathcal{O}(\lambda^2)$



At $\mathcal{O}(\lambda^2)$, $\mathcal{P}/\mathcal{P}_{\text{ideal}} < 1 \implies$ SYM$_{4,4}$: $\lambda \lesssim 10$; QCD : $\lambda \lesssim 3.5$ at $\hat{\mu} = 1$

The perturbative expansion of the SYM$_{4,4}$ free energy might have better convergence than QCD

Summary

- Finish the calculation resummed free energy to order λ^2 for $SYM_{4,4}$ under the RDR scheme.
- Construct a new large-N_c Padè approximant based on our result.
- Compare our final result for the scaled entropy density to the updated Padè approximants.
- Currently we compute the coefficient of $\lambda^{5/2}$ in the $SYM_{4,4}$ free energy.

Thanks for your attention!