New developments on Feynman integrals calculation

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Based on: 2201.11636(PRD), 2201.11637

高能物理学术年会

Date: 2022/8/11

Outline

I. Introduction

II. Auxiliary mass flow(AMF) method

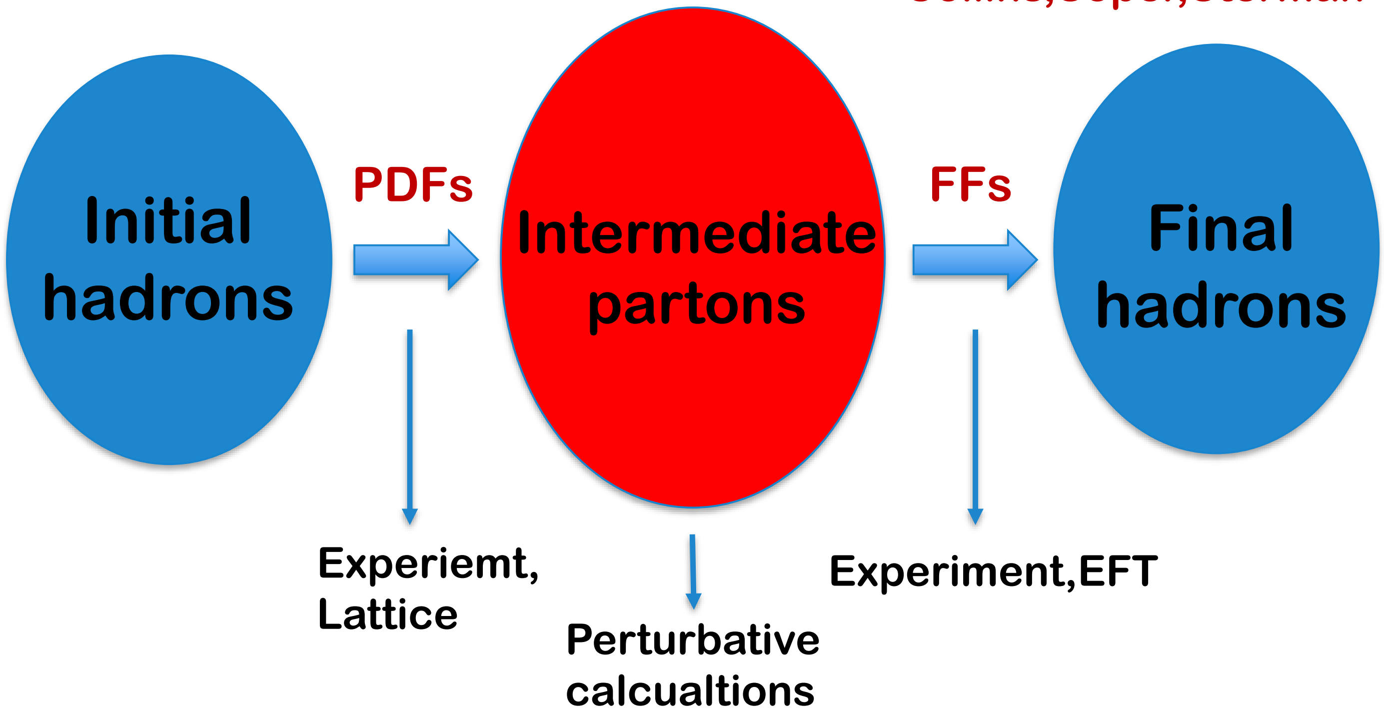
III. Extend AMF to any loops and New viewpoint

IV. New method to handle linear propagators and application

V. Summary and outlook

Hadrons scattering process

Collins,Soper,Sterman 1989



$$\sigma(\text{hadrons})=\text{PDFs}\otimes\sigma(\text{partons})\otimes\text{FFs}$$

- σ (hadrons) measured by experiment
- σ (partons) calculated by perturbative theory
- Requirement: uncertainty of perturbative calculations<=uncertainty of experiments
- Test Standard Model or find new physics

Perturbative calculation process

- Generate scattering amplitude expressed by Feynman integrals(FIs) Relatively easier

  Feynman diagram method

- Reduce the FIs to master integrals (MIs) by integration-by-part(IBP) identities Much harder

$$F\left(\mathcal{O}(10^4)\right) \xrightarrow{\text{IBP}} \sum_{i=1}^{\mathcal{O}(10^2)} c_i \times I_i$$

Laporta's algorithm
[Laporta, hep-ph/0102033]

Syzygy equations
[Gluza, et al, 1009.0472]

Block trianglular
[Guan,Liu,Ma, 1912.09294]

- Calculate master integrals Hard

Sector decomposition
[Binoth,Heinrich, hep-ph/0004013]

Canonical form
[Henn, 1304.1806]

Auxiliary mass flow
[Liu,Ma,Wang, 1711.09572]

- One-loop order: solved problem
- Higher-loop order?

['t Hooft,Veltman 1979]
[Oldenborgh,Vermaseren 1990]

Difficulties of perturbative calculations

Baikov, et al,1606.08659

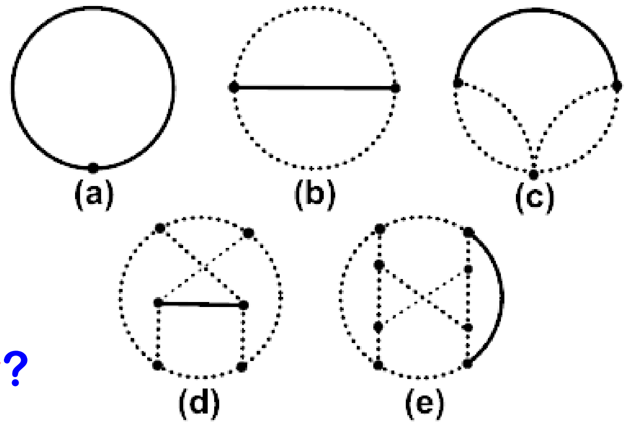
$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu)$$

# of loops	$\alpha_s^{(3)}(M_\tau)$	$\alpha_s^{(5)}(M_Z)$	$\alpha_s^{(5)}(M_H)$
3	0.33 ± 0.014	0.1195 ± 0.0015	0.1140 ± 0.0015
4	0.33 ± 0.014	0.1197 ± 0.0015	0.1142 ± 0.0015
5	0.33 ± 0.014	0.1198 ± 0.0015	0.1143 ± 0.0015

3 loop(1980)
4 loop(1997)
5 loop(2016)

Five-loop running of QCD coupling constant

- Involving Feynman integrals(FIs) up to 5 loops
- 7 months of running time by computer with about 160 cores
- How to calculate these FIs efficiently?
- How about 6-loop FIs?



Auxiliary mass flow method

- **Introduce an auxiliary mass to some propagator**

Liu,Ma,Wang, 1711.09572
Liu,Ma, 2107.01864

$$I_{\text{mod}}(\eta) = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \eta)^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}$$

- Physical integral

$$I_{\text{phy}} = \lim_{\eta \to \mathrm{i}0^-} I_{\text{mod}}(\eta)$$

- The limit is taken through following three steps:

(1) Set up η −DEs of modified integral

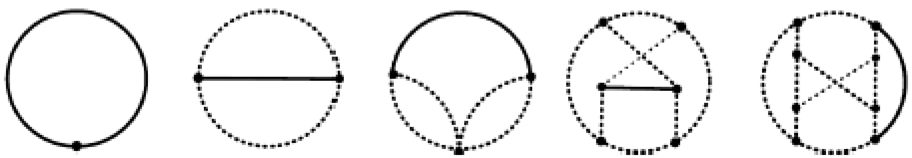
(2) Determination of boundary conditions(BCs) at $\eta = \infty$

(3) Evolution of η from ∞ to $\mathrm{i}0-$

Merits and demerits

➢ Advantages

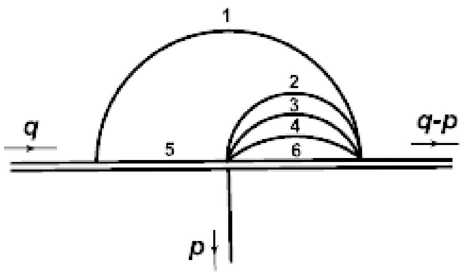
- Systematic and efficient for any loop FIs(given BCs)
- Simple BCs input: single mass vacuum Feynman integrals
- Arbitrary accuracy



Some single mass vacuum FIs up to 5 loops
(known analytic expressions up to 3 loops)

➢ Deficiencies

- No arbitrary accuracy at more than three loops
- Can't handle FIs containing linear propagators (often appear in many physical problems)

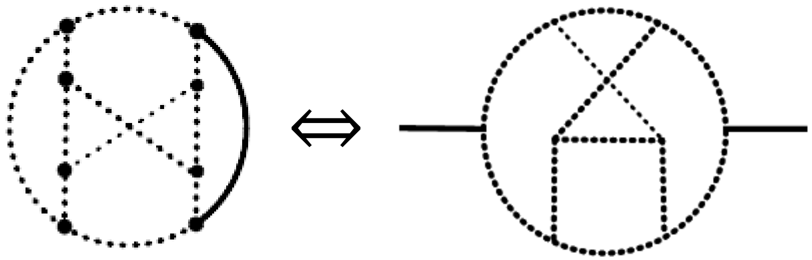


Arbitrary accuracy for any loops

ZFL,Ma, 2201.11637

- Loop-drop technique

A L-loop single mass vacuum FI is equivalent to
a (L-1)-loop massless propagator integral(p-integral).



- Use AMF and Loop-drop by turns

(L-1)-loop p-integrals
⇓ AMF
(L-1)-loop single mass tadpoles
⇓ Loop-drop
(L-2)-loop p-integrals
⇓ $(\text{AMF+Loop-drop})^{L-2}$
0-loop p-integrals(=1)

⇒ Loop integrations totally bypassed!

New viewpoint

ZFL,Ma, 2201.11637

➢ **Feynman integral is Linear algebra!**

$$\text{FIs} \triangleq \text{Linear algebra} \oplus \text{Vacuum integrals}$$

(2017-2021)

$$\Downarrow$$

$$\text{FIs} \triangleq \text{Linear algebra}$$

(2022)

- Only input is integration-by-part(IBP) identities(to set up η −DEs)
- No other input
- No kinematics
- No loops

For linear propagators

➢ **Change linear to quadratic propagators**

ZFL, Ma, 2201.11636

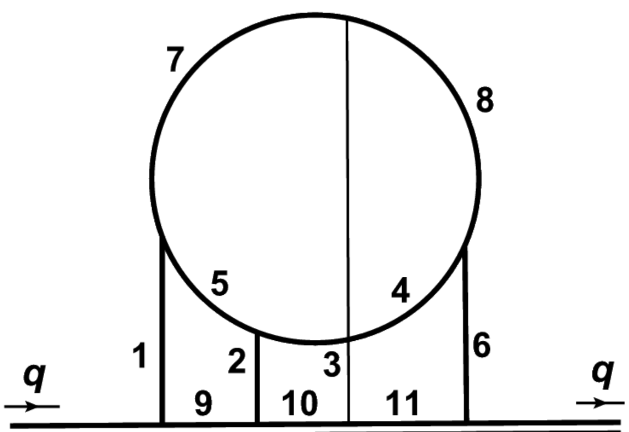
$$\frac{1}{p_a \cdot \ell_a + \Delta_a + \mathrm{i}0^+} \to \frac{1}{x\left(\ell_a^2 + \mathrm{i}0^+\right) + p_a \cdot \ell_a + \Delta_a}$$

- Quadratic integrals at finite x_0 calculated by AMF for any loops
- Original linear integrals obtained by taking $x \to 0^+$:
  1) Set up $x-\mathrm{DEs}$; 2) BCs at x_0 by AMF; 3) Solve $x-\mathrm{DEs}$
- A new method to calculate linear integrals systematically and efficiently
- Useful in effective field theory and region expansion

Application to four loops

ZFL, Ma, 2201.11636

- **Results of all four-loop MIs of vacuum integrals containing a gauge link for the first time**



The most complex diagram

$$
\begin{aligned}
I_{(1,\cdots,1,0,0,0)} &= 2.467401100272340\epsilon^{-4} \\
&- (2.107259492249774 - 31.00627668029982\mathrm{i})\epsilon^{-3} \\
&- (142.4061631115384 + 26.48060376335 31\mathrm{i})\epsilon^{-2} \\
&+ (258.9104506294486 - 157.4236379069742\mathrm{i})\epsilon^{-1} \\
&- (1638.226426723546 - 1859.681566826831\mathrm{i}) \\
&- (4750.006669884407 + 11690.896001211223\mathrm{i})\epsilon \\
&+ (22176.46421510774 + 23525.15655373790\mathrm{i})\epsilon^{2} \\
&- (120064.4744857791 + 165534.0450504650\mathrm{i})\epsilon^{3} \\
&+ (855055.4310121035 + 539390.2035592209\mathrm{i})\epsilon^{4} \\
&- (3796471.438494197 + 1678278.567906508\mathrm{i})\epsilon^{5}.
\end{aligned}
$$

- Higher precision and higher orders in ϵ achieved easily
- Useful for studying parton distribution functions

Summary and outlook

➢ Summary

- Use Loop-drop technique to make AMF calculate its BCs automatically and surprisingly find that FIs calculation can be reduced to Linear algebra
- Use quadratic integrals to calculate linear integrals systematically and efficiently
- Feynman integrals are just Linear algebra

➢ Outlook

- Deeper understanding of FIs calculation from Linear algebra
- A wide range of applications of AMF for all purposes(any loops, any dimension, linear propagators, ……)

Thank you!

References

[1] Zhi-Feng Liu and Yan-Qing Ma, "Automatic computation of Feynman integrals containing linear propagators via auxiliary mass flow", Phys.Rev.D 105 (2022) 7 (Editors' Suggestion), [arXiv:2201.11636 [hep-ph]].

[2] Zhi-Feng Liu and Yan-Qing Ma, "Feynman integrals are completely determined by linear algebra", [arXiv:2201.11637 [hep-ph]].

[3] Xiao Liu and Yan-Qing Ma, "AMFlow: a Mathematica package for Feynman integrals computation via Auxiliary Mass Flow", [arXiv:2201.11669 [hep-ph]].

Definition of FIs

➢ A family of Feynman integrals

$$I_{\vec{\nu}}(D,\vec{s}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_\alpha = A_{\alpha ij} \ell_i \cdot \ell_j + B_{\alpha ij} \ell_i \cdot p_j + C_\alpha$$

- $\ell_1, \ldots, \ell_L$: loop momenta; $p_1, \ldots, p_E$: external momenta;
- A, B: integers; C: linear combination of $\vec{s}$ (including masses)
- $\mathcal{D}_1, \ldots, \mathcal{D}_K$: inverse propagators; $\nu_1, \ldots, \nu_K$: integers
- $\mathcal{D}_{K+1}, \ldots, \mathcal{D}_N$: irreducible scalar products; $\nu_{K+1}, \ldots, \nu_N$: non-positive integers

➢ Difficulties of calculating FIs

- Analytical: known special functions are insufficient to express FIs
- Numerical: UV or IR divergence at $D \to 4$

Calculate p-integrals

Liu, YQM, 2201.11637

- **Apply AMF method on $(L-1)$-loop p-integral**

  1) IBP to setup η-DEs

  2) Single-mass vacuum integrals no more than $(L-1)$ loops as input

  Single-mass vacuum integrals with L loops are determined by that with no more than $(L-1)$ loops (besides IBP)

  - Boundary: 0-loop p-integrals equal 1

- **Only IBPs are needed to determine FIs!**

  - IBPs: linear algebra, exact (in $D, \vec{s}$) relations between FIs
  - Loop integrations are completely avoided!

New workflow

- The 'FICalc' to calculate FIs can be defined as (any given nonsingular D and $\vec{s}$):

LZF,YQM,2201.11637

① If it is a 0-loop p-integral, return 1;

② If it is a single-mass vacuum integral, express it by a p-integral, and call 'FICalc' to calculate the p-integral;

③ Otherwise:

a) Introduce η to one propagator (if the mass mode is not possible)

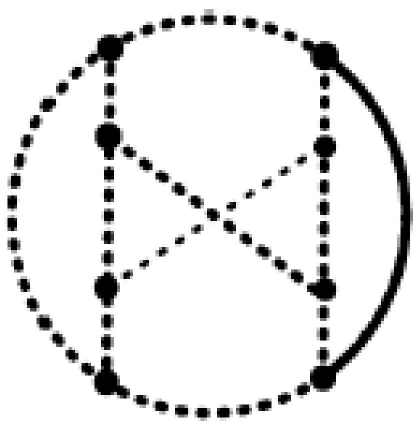
b) Setup η-DEs using IBP as input

c) Call 'FICalc' to calculate boundary FIs at $\eta \to \infty$

d) Numerically solve η-DEs with given BCs to obtain $\eta \to i0^-$

A five-loop example

LZF,YQM,2201.11637



$$= -2.07385551028674\epsilon^{-2} - 7.812755312590133\epsilon^{-1} - 17.25882864945875 + 717.6808845492140\epsilon + 8190.876448160049\epsilon^{2} + 78840.29598046500\epsilon^{3} + 566649.1116484678\epsilon^{4} + 3901713.802716081\epsilon^{5} + 23702384.71086095\epsilon^{6} + 142142936.8205112\epsilon^{7},$$

- IBP relations are the only input!
- Terms up to $O(\epsilon^4)$ agree with literature; Others are new ($D = 4 - 2\epsilon$)
  Lee, Smirnov, Smirnov, 1108.0732
- An arbitrary dimension D = 4/7, no other known method can do it

-9.793112097048649321808795980069111646428182547465428330614694726443l
516031830610056668242341877309401032293901004574319494017206091158244
708224654193885680661950372372090211196168499966402592016363 21*10^7

with about 130 significant digits

TABLE IX: Values of $\alpha_S(m_Z^2)$ determined at N^{3}LO accuracy from Γ_Z^{tot}, R_Z, and σ_Z^{had} individually, combined, as well as from a global SM fit, with propagated experimental, parametric, and theoretical uncertainties broken down. The last two rows list the expected values at the FCC-ee from all Z pseudoobservables combined and from the corresponding SM fit.

Observable	$\alpha_S(m_Z^2)$	uncertainties		
		exp.	param.	theor.
Γ_Z^{tot}	0.1192 ± 0.0047	± 0.0046	± 0.0005	± 0.0008
R_Z	0.1207 ± 0.0041	± 0.0041	± 0.0001	± 0.0009
σ_Z^{had}	0.1206 ± 0.0068	± 0.0067	± 0.0004	± 0.0012
All above combined	0.1203 ± 0.0029	± 0.0029	± 0.0002	± 0.0008
Global SM fit	0.1202 ± 0.0028	± 0.0028	± 0.0002	± 0.0008
All combined (FCC-ee)	0.12030 ± 0.00026	± 0.00013	± 0.00005	± 0.00022
Global SM fit (FCC-ee)	0.12020 ± 0.00026	± 0.00013	± 0.00005	± 0.00022

TABLE XII: PDG average of the categories of observables [March'22 update of the PDG'21 results]. These are the final input to the world average of $\alpha_S(m_Z^2)$.

category	$\alpha_S(m_Z^2)$	relative $\alpha_S(m_Z^2)$ uncertainty
τ decays and low Q^2	0.1178 ± 0.0019	1.6%
$Q\overline{Q}$ bound states	0.1181 ± 0.0037	3.1%
PDF fits	0.1162 ± 0.0020	1.7%
e^+e^- jets & shapes	0.1171 ± 0.0031	2.6%
electroweak	0.1208 ± 0.0028	2.3%
hadron colliders	0.1165 ± 0.0028	2.4%
lattice	0.1182 ± 0.0008	0.7%
world average (without lattice)	0.1176 ± 0.0010	0.9%
world average (with lattice)	0.1179 ± 0.0009	0.8%

Summary and outlook

➢ Summary

- Use Loop-drop technique to make AMF calculate its BCs automatically and surprisingly find that FIs calculation can be reduced to linear algebra
- Use quadratic integrals to calculate linear integrals systematically
- Automation calculation of all kinds of FIs has be implemented in the package AMFlow(2201.11669)

➢ Outlook

- Deeper understanding of FIs calculation from linear algebra
- A wide range of applications of AMF for all purposes(any loops, any dimension, linear propagators)