Projected Three-Point Energy-Correlator to NNLL in Collinear Limit

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in preparation

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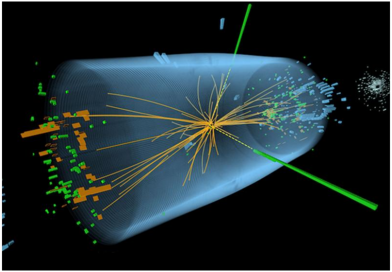
Precision Collider Physics and Event Shapes



Event shape variables for testing QCD and for improving our understanding of its dynamics:

Thrust :

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

Energy-energy correlator :

$$\mathrm{EEC}(\theta) = \frac{\sum_{i,j} E_i E_j \delta(\cos\theta - \cos\theta_{ij})}{\left(\sum_i E_i\right)^2}$$

Other important examples include jet-broadening, spherocity and C-parameter, etc.

Resummation of Energy Correlators in Collinear Limit

The energy-energy correlator in terms of $z = (1 - \cos\theta)/2$,

$$\text{EEC}(z) \equiv \frac{1}{\sigma_{\text{tot}}}\frac{d\sigma}{dz} = \sum_{i,j}\int d\sigma\,\frac{E_i E_j}{Q^2}\,\delta\left(z - \frac{1-\cos\theta_{ij}}{2}\right)$$

In the collinear limit, interesting for jet-substructure, dominated by large logarithms $\ln z$.

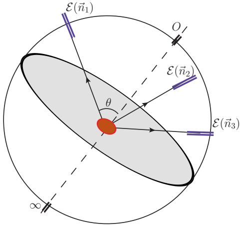
$$\frac{1}{\sigma_{\text{tot}}}\frac{d\sigma^{e^+e^-}(z)}{dz} = \frac{2a_s}{z} + \frac{a_s^2}{z}\left(-\frac{173}{15}\ln z + \cdots\right)$$
$$+\frac{a_s^3}{z}\left[\frac{20317}{450}\ln^2 z + \ln z\left(\frac{3704}{81}\zeta_3 - \frac{343252}{1215}\zeta_2 - \frac{686702711}{1093500}\right) + \cdots\right]$$
$$= \frac{2a_s}{z} + \frac{a_s^2}{z}(-11.5333\ln z + 81.4809) + \frac{a_s^3}{z}\left(45.1489\ln^2 z - 1037.73\ln z + 2871.36\right)$$

The logarithmic series to all-orders,

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\sigma}{dz} = \sum_{L=1}^{\infty}\sum_{j=-1}^{L-1} a_s^L c_{L,j}\,\mathcal{L}^j(z) + \ldots \qquad \text{where} \quad \mathcal{L}^j(z) = \left[\ln^j z/z\right]_+$$
$$= \underbrace{\sum_{L=1}^{\infty} a_s^L c_L \left[\ln^{L-1} z/z\right]_+}_{\text{LL}} + \underbrace{\sum_{L=2}^{\infty} a_s^L d_L \left[\ln^{L-2} z/z\right]_+}_{\text{NLL}} + \underbrace{\sum_{L=3}^{\infty} a_s^L f_L \left[\ln^{L-3} z/z\right]_+}_{\text{NNLL}} + \cdots$$

All-order resummation achieved through to NNLL via RG evolution.
[L. Dixon, I. Moult, H.X. Zhu, arXiv:1905.01310]

Generalization to Higher Point Energy Correlators



Three-point energy correlator (EEEC)

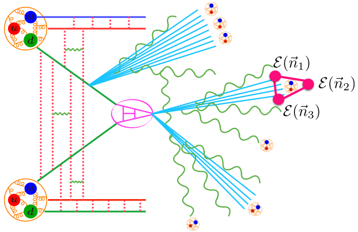
$$\frac{1}{\sigma_{\text{tot}}} \frac{d^3\sigma}{dx_1 dx_2 dx_3} = \sum_{ijk} \int d\sigma \frac{E_i E_j E_k}{Q^3} \times \delta\left(x_1 - \frac{1-\cos\theta_{jk}}{2}\right) \delta\left(x_2 - \frac{1-\cos\theta_{ik}}{2}\right) \delta\left(x_3 - \frac{1-\cos\theta_{ij}}{2}\right)$$

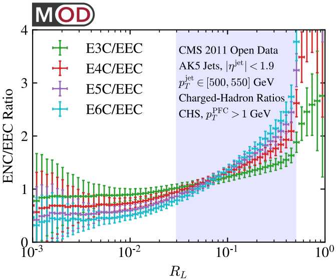
captures non-trivial shape dependence of the events or jets.

- defined and calculated in the collinear limit in [H. Chen, M.X. Luo, I. Moult, T.Z. Yang, X.Y. Zhang, H.X. Zhu, arXiv:1912.11050];
- full analytic calculation at LO in $\mathcal{N} = 4$ SYM [K. Yan, X.Y. Zhang, 2203.04349] , and in QCD [T.Z. Yang, X.Y. Zhang, 2208.01051].

Show interesting mathematical structure and benefit the understanding of trijet events as well as jet-substructure. See also HuaXing's talk.

Generalization to Higher Point Energy Correlators





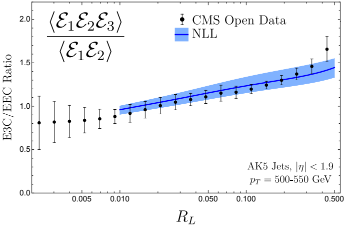
P.T. Komiske, I. Moult, J. Thaler, H.X. Zhu, 2201.07800

Projected N-point energy correlator (ENC)

$$\frac{d\sigma^{[N]}}{dz} = \sum_n \sum_{1\le i_1,\cdots i_N \le n} \int d\sigma \frac{\prod_{a=1}^N E_{i_a}}{Q^N} \delta(z - \max\{x_{i_1,i_2}, x_{i_1,i_3}, \cdots x_{i_{N-1},i_N}\})$$

- Enjoy simple factorization property.
- Ratio between ENC and EEC more robust to non-perturbative effects.
- Nice representation with light ray operators.

See also Hao Chen's talk.



K. Lee, B. Mecaj, I. Moult, 2205.03414

Resummation Strategy and Ingredients Needed for NNLL

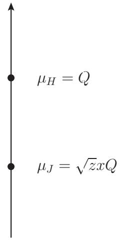
The factorization of projected N-point correlators in the collinear limit $z \to 0$,

$$\Sigma^{[N]}\left(z, \ln\frac{Q^2}{\mu^2}, \mu\right) = \int_0^z dz' \, \frac{d\sigma^{[N]}}{dz}\left(z', \ln\frac{Q^2}{\mu^2}, \mu\right) \overset{z\to 0}{=} \int_0^1 dx \, x^N \vec{J}^{[N]}\left(\ln\frac{zx^2Q^2}{\mu^2}, \mu\right) \cdot \vec{H}\left(x, \frac{Q^2}{\mu^2}, \mu\right)$$

Evolve the Jet function and the Hard function to a common scale with the RG equations.

$$\frac{d\vec{H}(x, \ln\frac{Q^2}{\mu^2})}{d\ln\mu^2} = -\int_x^1 \frac{dy}{y}\,\widehat{P}(y)\cdot\vec{H}\left(\frac{x}{y}, \ln\frac{Q^2}{\mu^2}\right)$$

$$\boxed{\frac{d\vec{J}^{[N]}(\ln\frac{zQ^2}{\mu^2})}{d\ln\mu^2} = \int_0^1 dy\, y^N \vec{J}^{[N]}(\ln\frac{zy^2Q^2}{\mu^2})\cdot\widehat{P}(y)}$$



Towards NNLL for ENC in Collinear Limit:

Jet function and hard function through to 2-loop needed as boundary conditions.

Time-like splitting function, or corresponding moments thereof, and β-function to 3-loop.

The last piece for E3C, the 2-loop jet function, has now been calculated.

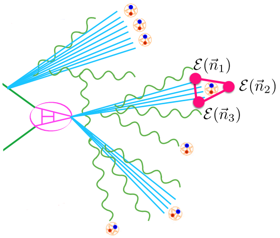
Calculation of the Two-loop Jet Function

The gluon jet function for EEEC, then project to the side with largest angle,

$$J_g(x_1, x_2, x_3, Q, \mu^2) = \int \frac{dl^+}{2\pi} \frac{1}{2(N_C^2-1)} \mathrm{Tr} \int d^4x e^{il\cdot x} \langle 0| \mathcal{B}^{a,\mu}_{n,\perp}(x) \boxed{\widehat{\mathcal{M}}_{\mathrm{EEEC}}} \delta(Q + \bar{n}\cdot\mathcal{P}) \delta^2(\mathcal{P}_\perp) \mathcal{B}^{a,\mu}_{n,\perp}(0)|0\rangle$$

$$\underbrace{\qquad}_{\sum_{i,j,k} \frac{E_i E_j E_k}{Q^3} \delta\left(x_1 - \frac{\theta_{ij}^2}{4}\right) \delta\left(x_2 - \frac{\theta_{jk}^2}{4}\right) \delta\left(x_3 - \frac{\theta_{ki}^2}{4}\right)}$$

Similarly for quark jet function with collinear quark field $\chi_n \equiv W_n^\dagger \xi_n$.



- Contact contributions (i, j, k not all non-identical) effectively depend on only one angle.

$$\delta\left(x - \frac{1-\cos\theta_{ij}}{2}\right) = \frac{(p_i\cdot p_j)}{x} \delta\left[2x(p_i\cdot Q)(p_j\cdot Q) - p_i\cdot p_j\right]$$

$$= \frac{1}{2\pi \mathrm{i}} \frac{(p_i\cdot p_j)}{x} \left\{ \frac{1}{\left[2x(p_i\cdot Q)(p_j\cdot Q) - p_i\cdot p_j\right] - \mathrm{i}0} - \frac{1}{\left[2x(p_i\cdot Q)(p_j\cdot Q) - p_i\cdot p_j\right] + \mathrm{i}0} \right\}$$

IBP reduction to linear combination of same master integrals as in NLO EEC calculation.

[L. Dixon, M.X. Luo, V. Shtabovenko, T.Z. Yang, H.X. Zhu, 1801.03219]

Calculation of the Two-loop Jet Function

The gluon jet function for EEEC, then project to the side with largest angle,

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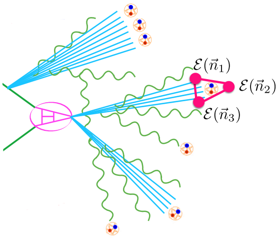
$$\widehat{\mathcal{M}}_{\text{EEEC}} = \sum_{i,j,k} \frac{E_i E_j E_k}{Q^3} \delta\left(x_1 - \frac{\theta_{ij}^2}{4}\right) \delta\left(x_2 - \frac{\theta_{jk}^2}{4}\right) \delta\left(x_3 - \frac{\theta_{ki}^2}{4}\right)$$

Similarly for quark jet function with collinear quark field $\chi_n \equiv W_n^\dagger \xi_n$.



- Non-identical contributions (i, j, k all non-identical) contain Θ-functions to keep the largest angle.

$$\int_S d\mathrm{Re}(z) d\mathrm{Im}(z) J_{\widehat{ijk}}$$

$$= A \cdot \int_S d\mathrm{Re}(z) d\mathrm{Im}(z) \int \mathrm{d}\Phi_c^{(3)} \frac{4g^4}{s_{123}^2} \sum_{i,j,k} P_{ijk} \widehat{\mathcal{M}}_{\text{EEEC}}$$

IBP reduction in the parametric representation. [W. Chen, arXiv:1902.10387; 1912.08606; 2007.00507]

Performed also direct integration with the proper parametrization.

Calculation of the Two-loop Jet Function

Combining the contact contributions and the non-identical contributions,

$$\frac{dJ_q^{\text{2-loop}}}{dz} \overset{\mu=Q}{=} \left(\frac{\alpha_s}{4\pi}\right)^2 \Big\{\delta(z)\Big[3.656 C_F T_F n_f - 0.6627 C_F^2 - 7.664 C_F C_A\Big] \rightarrow \text{jet function constant}$$

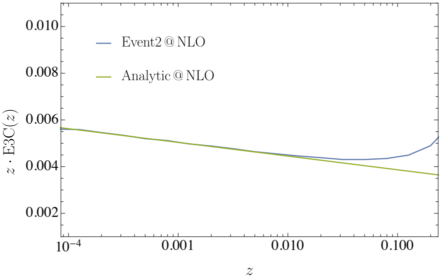
$$+ \underbrace{\frac{1}{z}\Big[C_F T_F n_f\Big(0.695 \ln z - 4.144\Big) + C_F^2\Big(2.944 \ln z - 1.050\Big) + C_F C_A\Big(-2.448 \ln z + 11.13\Big)\Big]}_{\rightarrow \text{ leading power distribution}}\Big\}$$

$$\frac{dJ_g^{\text{2-loop}}}{dz} \overset{\mu=Q}{=} \left(\frac{\alpha_s}{4\pi}\right)^2 \Big\{\delta(z)\Big[-3.343 C_F T_F n_f - 43.38 C_A T_F n_f + 33.13 C_A^2 + 8.201 T_F^2 n_f^2\Big] \rightarrow \text{jet function constant}$$

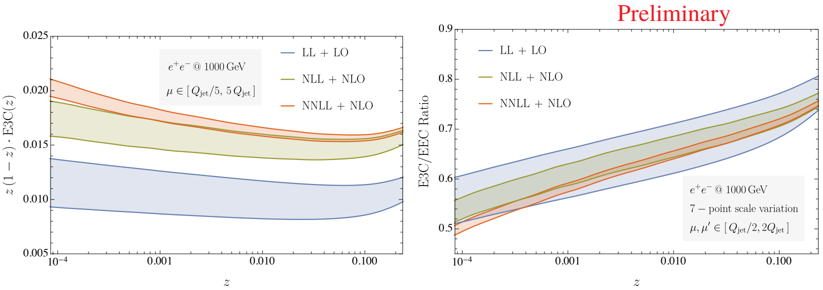
$$+ \frac{1}{z}\Big[C_F T_F n_f\Big(-0.4125 \ln z + 0.0875\Big) + C_A T_F n_f\Big(-0.4125 \ln z - 7.064\Big)$$

$$+ C_A^2\Big(0.28 \ln z + 20.69\Big) + T_F^2 n_f^2\Big(0.2 \ln z - 1.092\Big)\Big]\Big\}$$

checking the leading power distribution with `Event2`.



E3C Distribution and its Ratio to EEC at NNLL+NLO



Evident convergence from LL to NLL to NNLL, with decreasing scale uncertainty.

Uncertainty bands for E3C distribution barely overlap

$\Longrightarrow$ Higher order perturbative contributions sizable.

α_s-dependence of the slope of the ratio

$\Longrightarrow$ extraction of the coupling α_s from jet data.

So far, parton level only. $\mu_J = \sqrt{z}xQ$

$\Longrightarrow$ sensitive to non-perturbative QCD effects in the collinear limit $z \to 0$.

Non-Perturbative Corrections to the ENC

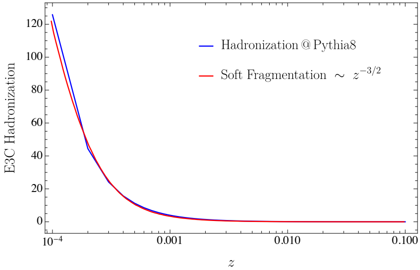
In the collinear limit, the ENC is sensible to the low-energy non-perturbative QCD effects.

$$\frac{d\sigma^{\rm PT}}{dz} \sim \frac{\alpha_s(\mu)}{z} \cdot \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^0 \cdot \left[\sum_{L=1}^{\infty} a_s^L\, c_L \left[\ln^L z\right] + \cdots\right] \quad \text{(RG Improved PT)}$$

$$\frac{d\sigma^{\rm NP-\mathit{soft}}}{dz} \sim \frac{\alpha_s(\mu)}{z^{3/2}} \cdot \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^1 + \cdots \quad \text{(Dispersive analysis at LL)}$$

$$\frac{d\sigma^{\rm NP-\mathit{collinear}}}{dz} \sim \frac{\alpha_s^2(\mu)}{z^2} \cdot \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^2 + \cdots \quad \text{(Renormalon analysis at NLL)}$$

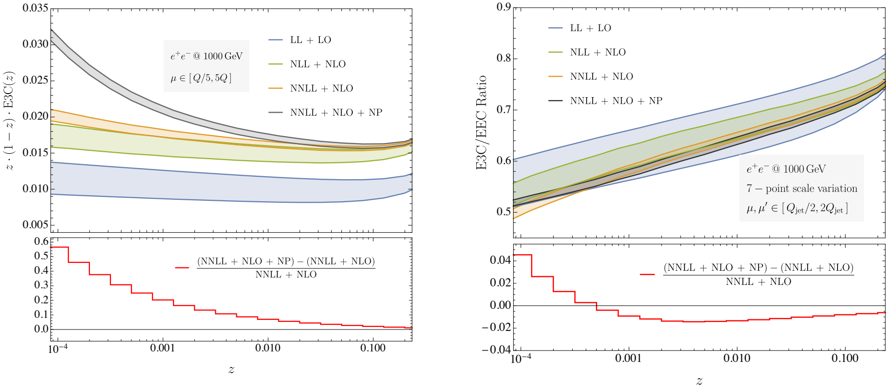
Soft contribution gives leading non-perturbative power correction in $\Lambda_{\rm QCD}$.
Collinear contribution at non-perturbative level relevant only below $z \sim (\Lambda_{\rm QCD}/Q)^2$.



Extract a single non-perturbative parameter from Pythia data to include $\mathcal{O}(\Lambda_{\rm QCD})$ correction.

Including the Non-Perturbative Corrections

Preliminary



In the collinear limit, hadronization correction to E3C is enhanced compared with the perturbative contribution.

Hadronization correction to E3C/EEC ratio are in fact largely reduced.

Summary

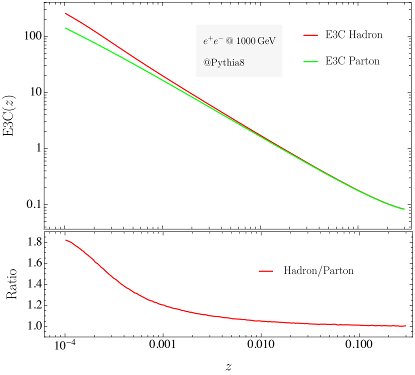
- Energy correlators are powerful event-shape variables in precision collider physics.
- E3C and its ratio with EEC obtained at NNLL+NLO accuracy.
- Non-perturbative corrections to ENC are sizable, and are largely cancelled in ratios.

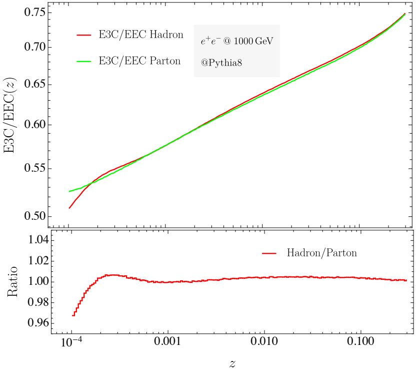
Outlook

- Higher point ($N > 3$) energy correlators at NNLL
  - two-loop jet function constants
- Resummation for LHC processes
  - more complicated hard function
- Combining small-z and small-x resummation for energy correlators
  - joint resummation with the small-x contributions and NP corrections

Extra Slides

Parton-Level and Hadron-Level Pythia Data





Hadronization correction is sizable in the collinear region.

Iterative Solution to the Jet Function RGE

Logarithmic structure of the jet function

$$\frac{d\vec{J}(\ln \frac{zQ^2}{\mu^2})}{d\ln\mu^2} = \int_0^1 dy\, y^\nu \vec{J}\,(\ln \frac{z\,y^2 Q^2}{\mu^2}) \cdot \widehat{P}(y)$$

Determine the n-th order coefficient of the logarithms in the jet function,

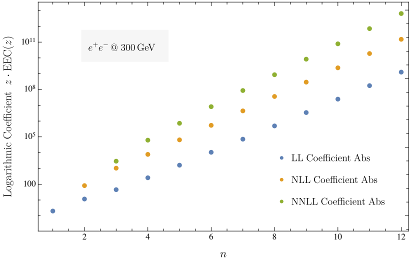
$$J_q = \sum_{n=0} a_s^n(\mu) q_{\rm L}^{(n)} \left(\ln \frac{\mu^2}{zQ^2}\right)^n + \sum_{n=1} a_s^n(\mu) q_{\rm NL}^{(n)} \left(\ln \frac{\mu^2}{zQ^2}\right)^{n-1} + \cdots$$

and similarly for the gluon jet. Recursive relations for coefficients from the RG equation

$$q_{\rm L}^{(n)} \cdot n = (n-1)\beta_0 q_{\rm L}^{(n-1)} - (q_{\rm L}^{(n-1)} \gamma_{T,qq}^{(0)} + g_{\rm L}^{(n-1)} \gamma_{T,gq}^{(0)})$$
$$g_{\rm L}^{(n)} \cdot n = (n-1)\beta_0 g_{\rm L}^{(n-1)} - (q_{\rm L}^{(n-1)} \gamma_{T,qg}^{(0)} + g_{\rm L}^{(n-1)} \gamma_{T,gg}^{(0)})$$

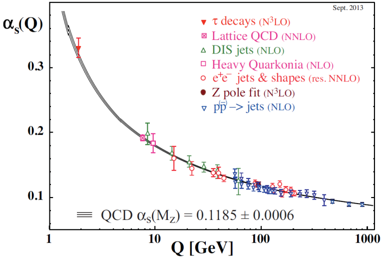
Coefficient of logarithmic correction can be obtained iteratively to high orders.

- Asymptotically divergent.
- Truncation at finite order.
- Residual compensated by non-perturbative corrections.



Strong QCD Coupling and Collider Processes

▹ Strong coupling at low-energy QCD



▹ Proton-proton collision at colliders

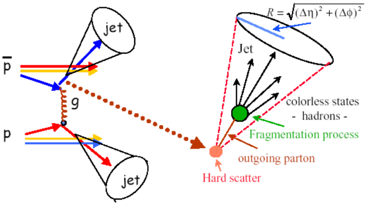


Image from: Brenna Flaugher CTEQ School 2002

▹ QCD factorization of hadronic cross sections

$$\frac{d\sigma}{dQ^2} \simeq \sum_{i,j=q,g} \int_0^1 dx_1 dx_2\, \mathcal{H}_{ij}(Q^2,\mu^2) f_{i\leftarrow h}(x_1,\mu^2) f_{j\leftarrow h}(x_2,\mu^2)$$

perturbative expansion of the partonic hard scattering cross section

$$\mathcal{H}_{ij} = \sum_{n=0}^{\infty} \alpha_s^n\, \mathcal{H}_{ij}^{(n)}$$

Need to take into account also the non-perturbative corrections to the observables.

Perturbation Theory at Large Orders and Renormalons

▹ Perturbative expansion as an *asymptotic* series:

$$f(\alpha) = \sum f_n \alpha^n , \qquad f_n \overset{n\to\infty}{\sim} a^n n^b n!$$

Analytic continuation with Borel transformation, ambiguous for fixed-sign factorials,

$$B[f](t) = \sum \frac{f_n}{n!} t^n \quad \Longrightarrow \quad f^B(\alpha) = \int_0^\infty e^{-t/\alpha} B[f](t)$$

e.g. $\sum \frac{|a|^n n!}{n!} t^n = \frac{1}{1 - |a|t}$ stronger divergence $\Longrightarrow$ larger *ambiguity* suppressed as $e^{-1/\alpha|a|}$

IR renormalons arise from the logarithms from renormalization procedure:

$$\int_0^q \frac{dk^2}{k^{2-m}} \left[|\beta_0| \ln\left(\frac{q^2}{k^2}\right)\right]^n \sim \left(\frac{2|\beta_0|}{m}\right)^n n! \quad \overset{\text{QCD}}{\Longrightarrow} \quad \text{power corrections } \delta \sim \left(\Lambda_{\text{QCD}}/Q\right)^{a/2|\beta_0|}$$

[Mueller, CU-TP-573; Beneke, hep-ph/9807443]