中国物理学会高能物理分会第十一届全国会员代表大会暨学术年会

振幅计算和有效场论研究进展

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大连 2022年8月8-11日

Outline

- Modern amplitude methods
- Effective field theory and form factors
- Some recent progress

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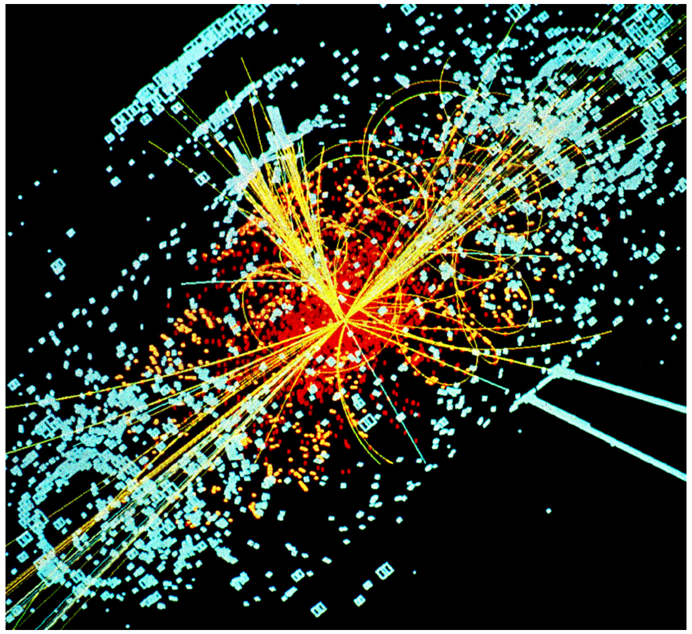
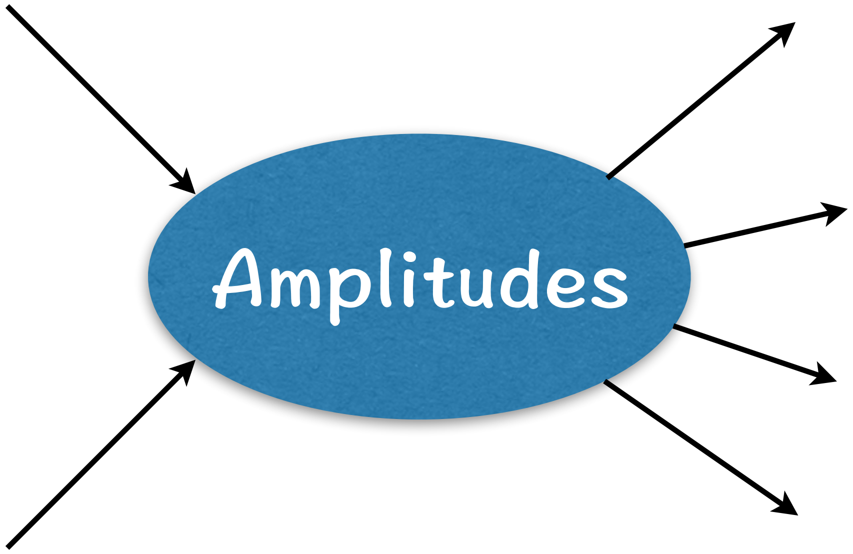
- Modern amplitude methods
- EFT and form factors
- Some recent progress

Scattering amplitudes



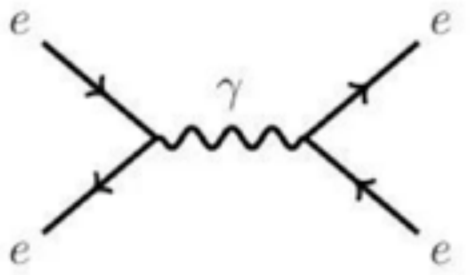


In past 30 years, significant progress has been made in the studies of scattering amplitudes.

Feynman diagram

Standard textbook method:

- universal
- simple rules
- intuitive picture



Feynman diagram

“Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses.”

— Schwinger

Feynman diagram

"Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses. Yes, one can analyze experience into individual pieces of topology. But eventually one has to put it all together again. And then the piecemeal approach loses some of its attraction."

— Schwinger

Feynman diagram

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

Surprising simplicity

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

n-gluon MHV tree amplitudes: [Parke, Taylor, 1986]

$$A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

Written in spinor helicity formalism (Chinese Magic)
by Xu, Zhang, Chang 1984

Lessons from modern amplitudes

Such simplicity is totally unexpected using traditional Feynman diagrams.

Conceptually:

New structures and new formulations

Methodologically:

New powerful computational methods

Modern amplitudes methods

"A Renaissance of the S-Matrix Program"

S-matrix program

Wheeler 1937
Heisenberg 1943

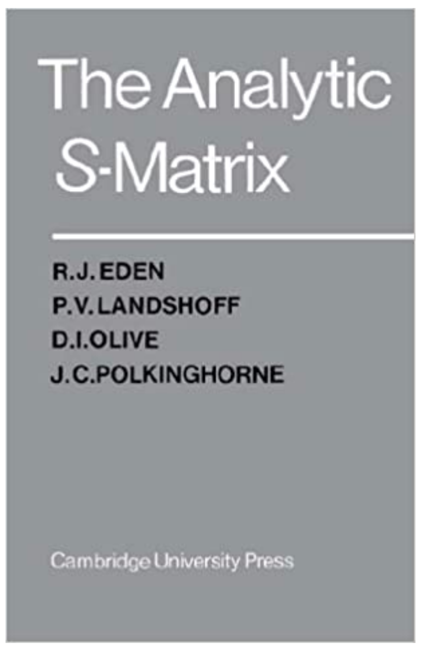
→

S-matrix bootstrap by
Chew, Mandelstam, etc
1950s-1960s

→

Modern amplitudes
On-shell methods

S-matrix program



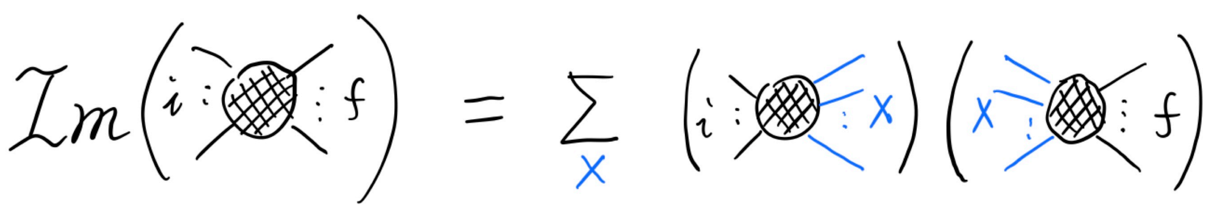
“The S-matrix is a Lorentz-invariant analytic function of all momentum variables with only those singularities required by unitarity.”

“One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, ….”

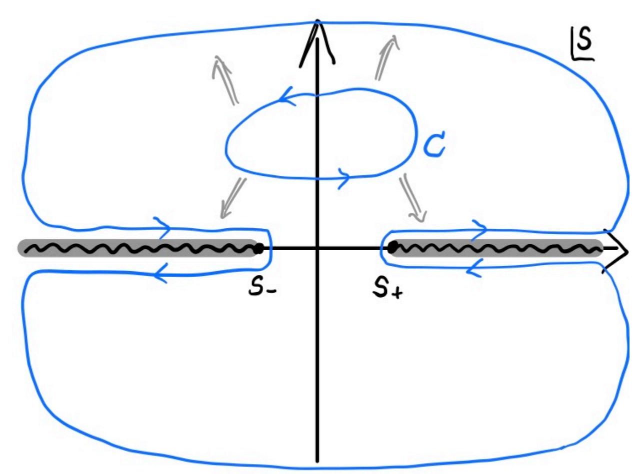
— Eden et.al, “The Analytic S-matrix”, 1966

S-matrix bootstrap

Unitarity: $S^\dagger S = 1 = S S^\dagger \longrightarrow -\mathbb{i}(\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle) = \sum_X \langle f|T^\dagger|X\rangle\langle X|T|i\rangle$



Dispersion relation:



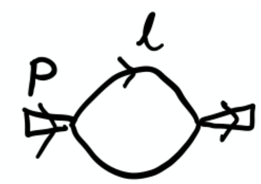
$$\mathcal{I}m[A] \Rightarrow A(s) \sim \int \frac{\mathcal{I}m[A]}{s'-s}\, ds'$$

(plus possible poles and asymptotic contributions)

A bubble-integral example

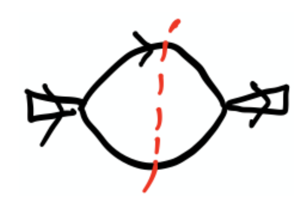
Let us compute this integral via S-matrix bootstrap:

$$I_2(P^2) = \int \frac{d^D l_1}{(2\pi)^D} \frac{1}{l^2(l-P)^2}$$



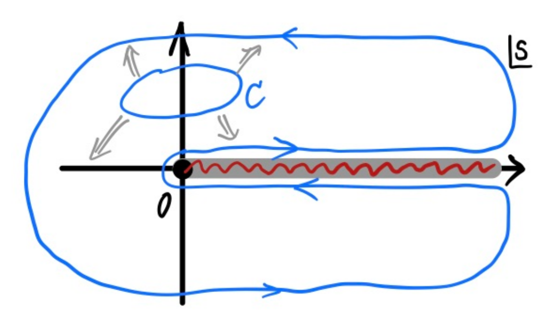
Step 1: compute discontinuity

$$\text{Disc}[I_2(P^2)] = \int \frac{d^D l_1}{(2\pi)^D}(-2\pi \mathbb{i})\delta(l^2)(-2\pi \mathbb{i})\delta((l-P)^2) = -\frac{(P^2)^{-\epsilon}}{(4\pi)^{2-2\epsilon}}\frac{\pi^{\frac{3}{2}-\epsilon}}{\Gamma(\frac{3}{2}-\epsilon)}$$

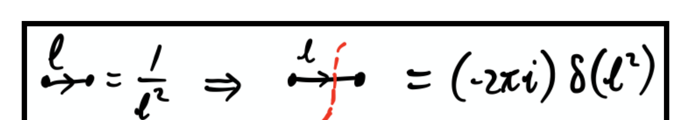


Step 2: apply dispersion relation $s = P^2 < 0,$

$$I_2(s) = \frac{1}{2\pi \mathbb{i}}\int_0^\infty \frac{dt}{t-s}\text{Disc}[I_2(t)] = \frac{\mathbb{i}}{(4\pi)^{\frac{D}{2}}}(-s)^{-\epsilon}\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$



Cutkosky cutting rule:



Modern amplitudes methods

S-matrix program is replaced by the Standard Model since late1960s.

New ingredients in the modern on-shell methods:

- Working at perturbative level
- Generalized unitarity cuts
- Use of good variables, e.g. spinor helicity
- New mathematical functional structures (e.g. symbol)
- Using simple toy models (N=4 SYM) as testing ground

e.g. tree-level BCFW recursion relations, unitarity-cut methods

Bern, Dixon, Durban, Kosower 1994; Britto, Cachazo, Feng, Witten, 2004

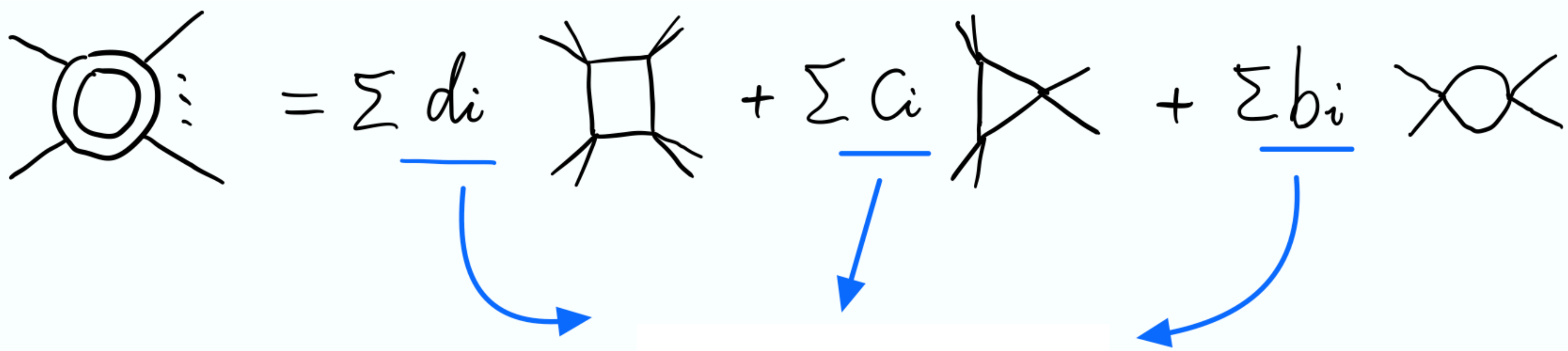
Modern amplitudes methods

A question:

In the optical theorem, unitarity can be used to compute only the imaginary part. How can the modern on-shell methods compute the full amplitudes via unitarity cuts?

One-loop structure

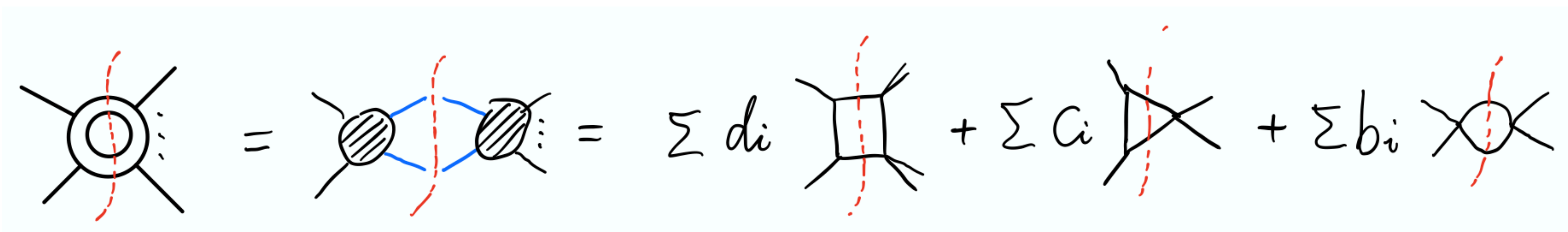
Consider one-loop amplitudes:



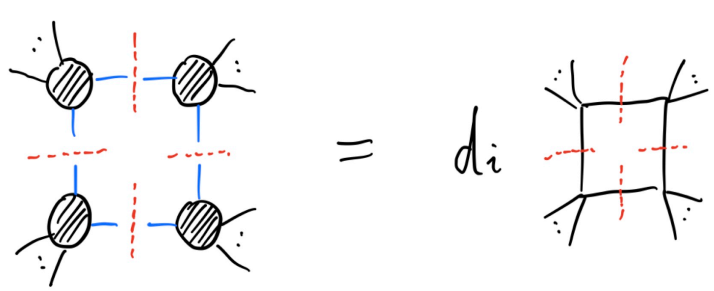
What we really want

Unitarity cuts

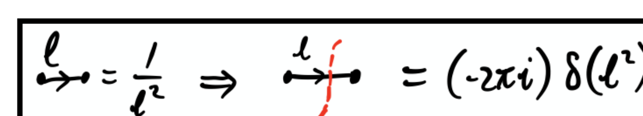
Using simpler tree-level blocks, one can derive the coefficients more efficiently:



[Bern, Dixon, Dunbar, Kosower 1994]

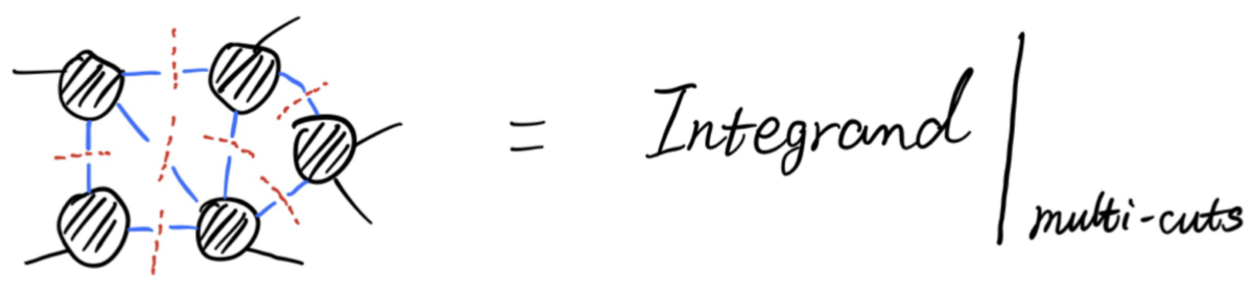


generalized multiple cuts [Britto, Cachazo, Feng 2004]

Cutkosky cutting rule: 

Loop integrands

Both the basis coefficients and integrand are rational functions, once they are obtained, one has the information for the full amplitudes.



Comments on the integral reduction and evaluation:

work by Bo Feng, Song He, Zhao Li, Li-Lin Yang, Yang Zhang, etc.. (an incomplete list)

Notable new efficient numerical method by Xiao Liu and Yan-Qing Ma: AMFlow package

See the talk by Hua-Xing Zhu

Outline

- Modern amplitude methods
- Effective field theory and form factors
- Some recent progress

Effective field theory

Standard Model effective field theory:

$$\mathscr{L}_{\text{SMEFT}} = \mathscr{L}_{SM} + \sum_{n\geq 1} \sum_{i} \frac{C_i^{(n)}}{M^n} \mathscr{O}_i^{(n)}$$

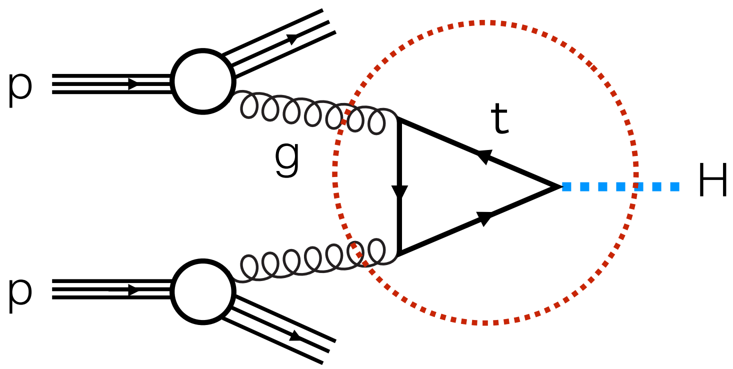
Contribution from higher dimensional operators are suppressed by powers of $\left(\frac{E}{M}\right)^n$

Fermi theory of weak interactions is such an example.

Effective field theories, mostly for QCD:

- HQET, SCET, χPT, NRQCD (NRQED, NRGR)

Higgs EFT



$m_t \to \infty$

Effective gluon-Higgs vertex:

$$\mathcal{L}_{\text{eff}} = C_0 O_0 + \frac{1}{m_t^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_t^4}\right)$$

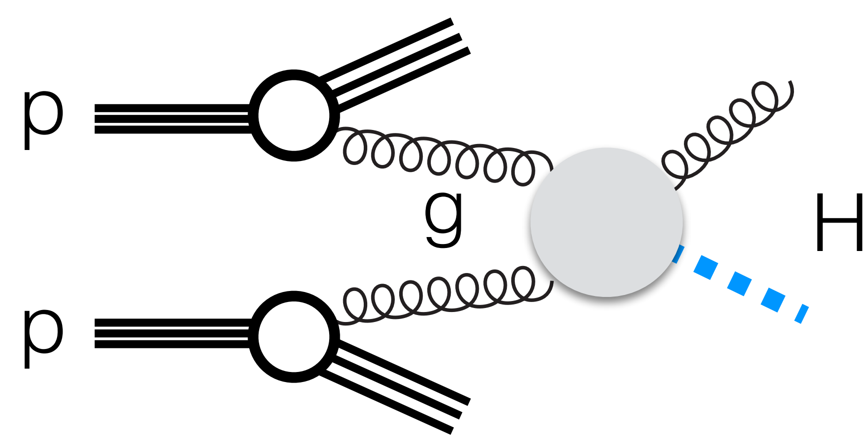
Dimension-5 operator

$O_0 = H\text{tr}(F_{\mu\nu}F^{\mu\nu})$

Dimension-7 operators

$$O_1 = H\text{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu),$$
$$O_2 = H\text{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu}),$$
$$O_3 = H\text{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu}),$$
$$O_4 = H\text{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu}).$$

Higgs plus jet production



Effective field theory

A effective field theory:

$$\mathscr{L}_{\text{EFT}} = \mathscr{L}_0 + \sum_{n\geq 1} \sum_i \frac{C_i^{(n)}}{M^n} \mathscr{O}_i^{(n)}$$

Two central ingredients:

$\mathscr{O}_i^{(n)}(\mu)$ Local operators

$C_i^{(n)}(\mu)$ Wilson coefficients

Effective field theory

Problems in EFT studies:

$$\mathscr{L}_{\mathrm{EFT}} = \mathscr{L}_0 + \sum_{n\geq 1}\sum_i \frac{C_i^{(n)}}{M^n}\mathscr{O}_i^{(n)}$$

- Classification of operators   work by Yi Liao, Jing Shu, Jiang-Hao Yu, etc..
- Constraining Wilson coefficients   work by Cen Zhang, Shuang-Yong Zhou, etc..
- Renormalization and RG
- Amplitudes in EFT

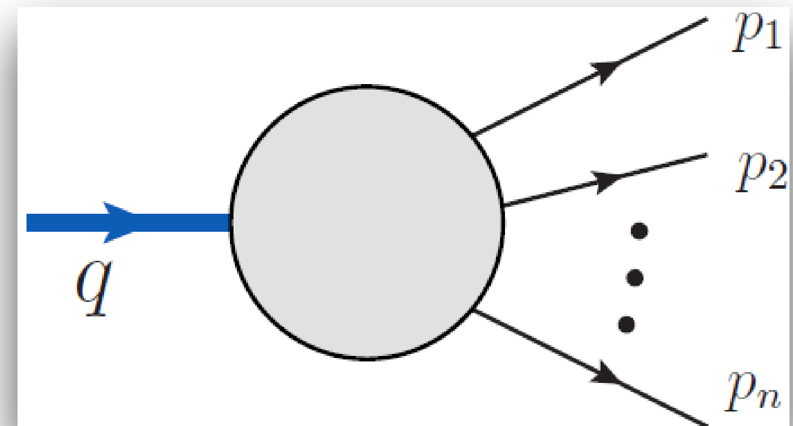
On-shell form factors

Hybrids of on-shell states and off-shell operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x} \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

$$= \delta^{(4)}(\sum_{i=1}^{n} p_i - q) \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$

(work in momentum space)



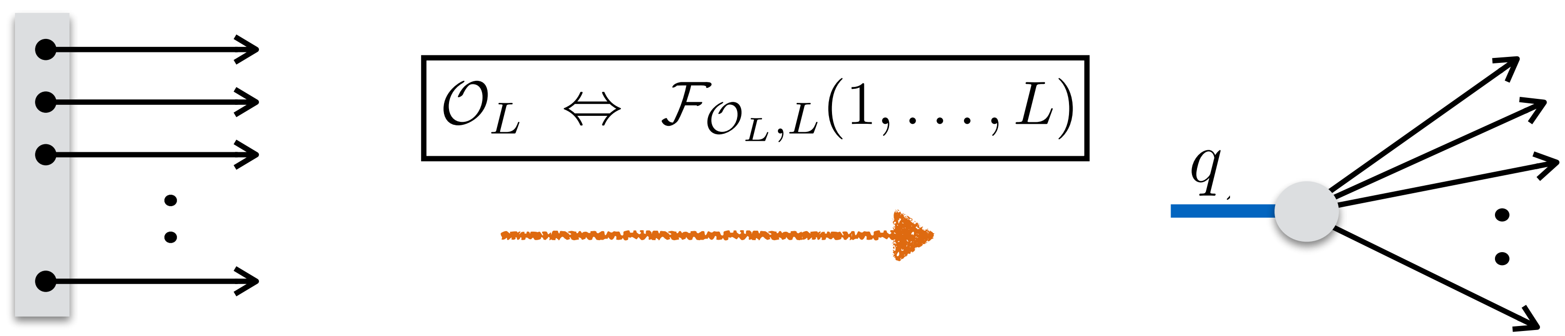
$$q = \sum_i p_i, \quad q^2 \neq 0$$

$$\langle p_1 p_2 \ldots p_n | 0 \rangle$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$$

Minimal tree form factors

One can translate any local operator into “on-shell” kinematics !



Examples:

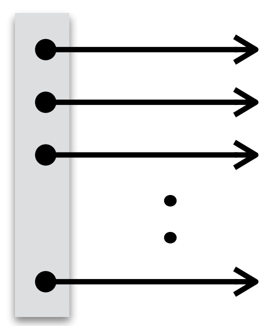
$$\mathcal{O}_3 = \phi\partial_\mu\phi\partial^\mu\phi \qquad \mathcal{F}_{\mathcal{O}_3,3}(1,2,3) = \delta^D(q - \sum_{i=1}^{3} p_i)\Big(s_{12} + s_{23} + s_{13}\Big)$$

$$\mathcal{O}_2 = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \qquad \mathcal{F}_{\mathcal{O}_2,2}(1,2) = \delta^D(q - \sum_{i=1}^{2} p_i)\Big[(\epsilon_1\cdot\epsilon_2)(p_1\cdot p_2) - (\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)\Big]$$

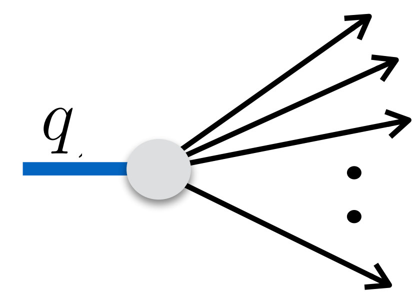
or $$\mathcal{F}_{\mathcal{O}_2,2}(1^-,2^-) = \delta^D(q - \sum_{i=1}^{2} p_i)\ \langle 12\rangle^2$$

Minimal tree form factors

One can translate any local operator into “on-shell” kinematics !



$$\mathcal{O}_L \;\Leftrightarrow\; \mathcal{F}_{\mathcal{O}_L,L}(1,\ldots,L)$$



Dictionary for YM operators:

operator	$D_{\dot\alpha\alpha}$	$f_{\alpha\beta}$	$\bar f_{\dot\alpha\dot\beta}$
spinor	$\tilde\lambda_{\dot\alpha}\lambda_\alpha$	$\lambda_\alpha\lambda_\beta$	$-\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}$

4-dim

$$F_{\mu\nu} \to F_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta}$$

Zwiebel 2011, Wilhelm 2014

$$\mathrm{tr}(\bar F_{\dot\alpha}{}^{\dot\beta}\bar F_{\dot\beta}{}^{\dot\gamma}\bar F_{\dot\gamma}{}^{\dot\alpha}) \to \tilde\lambda_1^{\dot\alpha}\tilde\lambda_{1\dot\beta}\tilde\lambda_2^{\dot\beta}\tilde\lambda_{2\dot\gamma}\tilde\lambda_3^{\dot\gamma}\tilde\lambda_{3\dot\alpha} = [1\,2][2\,3][3\,1]$$

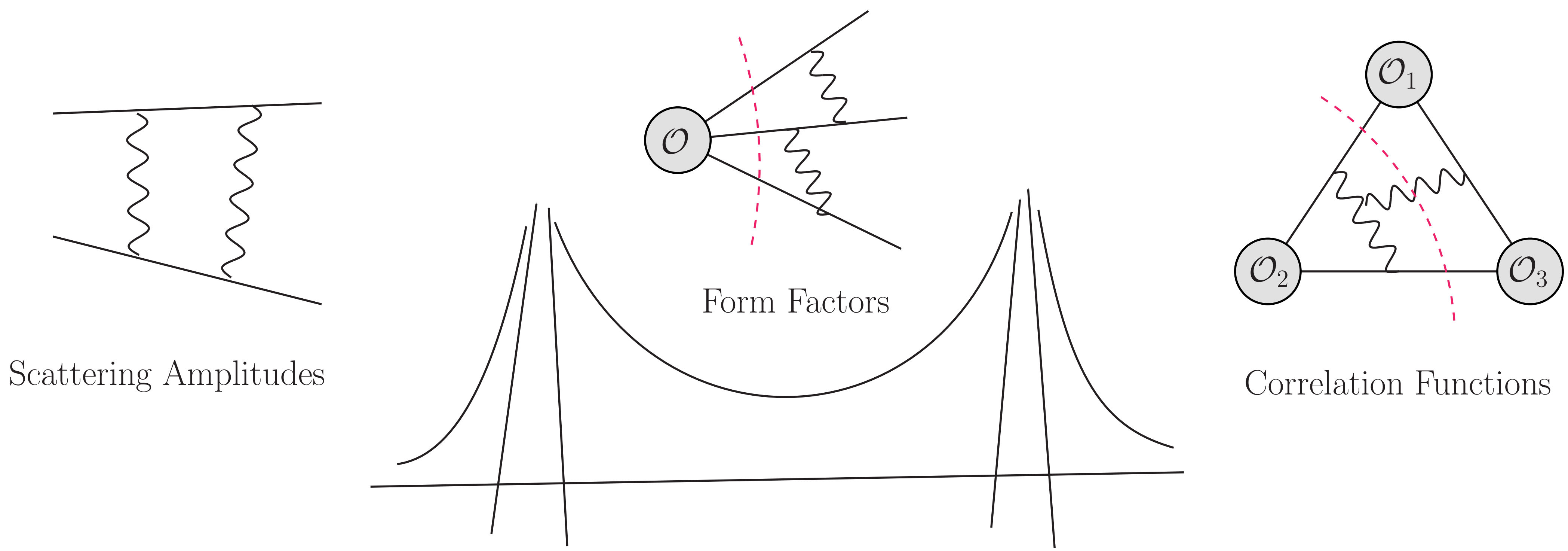
operator	D_μ	$F_{\mu\nu}$
kinematics	p_μ	$p_\mu\varepsilon_\nu - p_\nu\varepsilon_\mu$

D-dim

Important for capturing
“Evanescent operators”

Jin, Ren, GY, Yu, 2202.08285

On-shell methods



On-shell methods can be applied to operators and study EFT, for both the operator construction and high-loop renormalization.

Outline

- Modern amplitude methods
- EFT and form factors
- Some recent progress

YM Spectrum and Higgs Amplitudes

- 1804.04653 [Phys.Rev.Lett. 121 (2018) 10], 1904.07260, 1910.09384, with Qingjun Jin (靳庆军)
- 2011.02494 with Qingjun Jin, Ke Ren (任可);
- 2202.08285, 2208.xxxxx, with Qingjun Jin, Ke Ren; Rui Yu (余睿)

High-dimensional YM operators

We consider Lorentz scalar gauge invariant local operators:

$$\mathcal{O}(x) \sim c(a_1, ..., a_n) X(\eta^{\mu\nu}) \big(D_{\mu_{11}} ... D_{\mu_{1m_1}} F_{\nu_1\rho_1}\big)^{a_1} \cdots \big(D_{\mu_{n1}} ... D_{\mu_{nm_n}} F_{\nu_n\rho_n}\big)^{a_n}(x)$$

Classically, operators are generally not independent:

Equation of motion:	Bianchi identities:
$D_\mu F^{\mu\nu} = 0$	$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$

At quantum level, different operators can mixing with each other via renormalization:

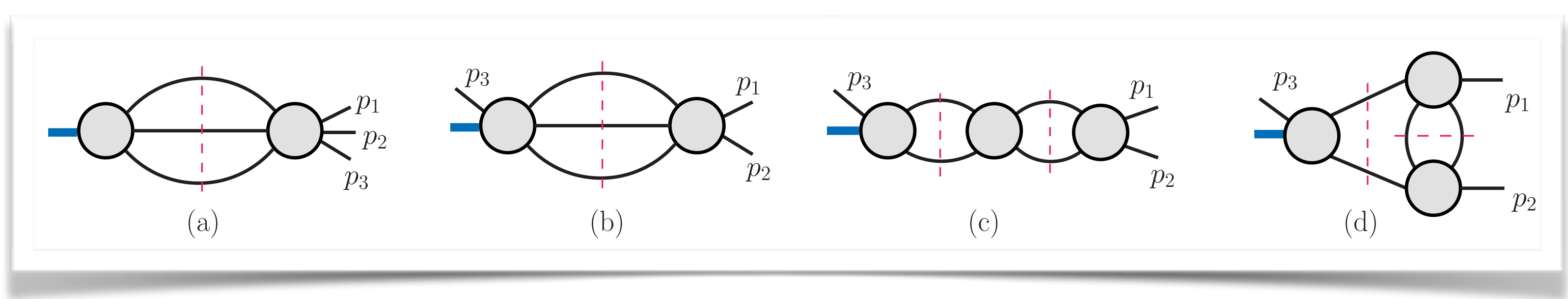
$$\mathcal{O}_{R,i} = Z_i^{\ j} \mathcal{O}_{B,j} \longrightarrow \mathcal{D} = -\frac{d \log Z}{d \log \mu} \longrightarrow \mathcal{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}$$

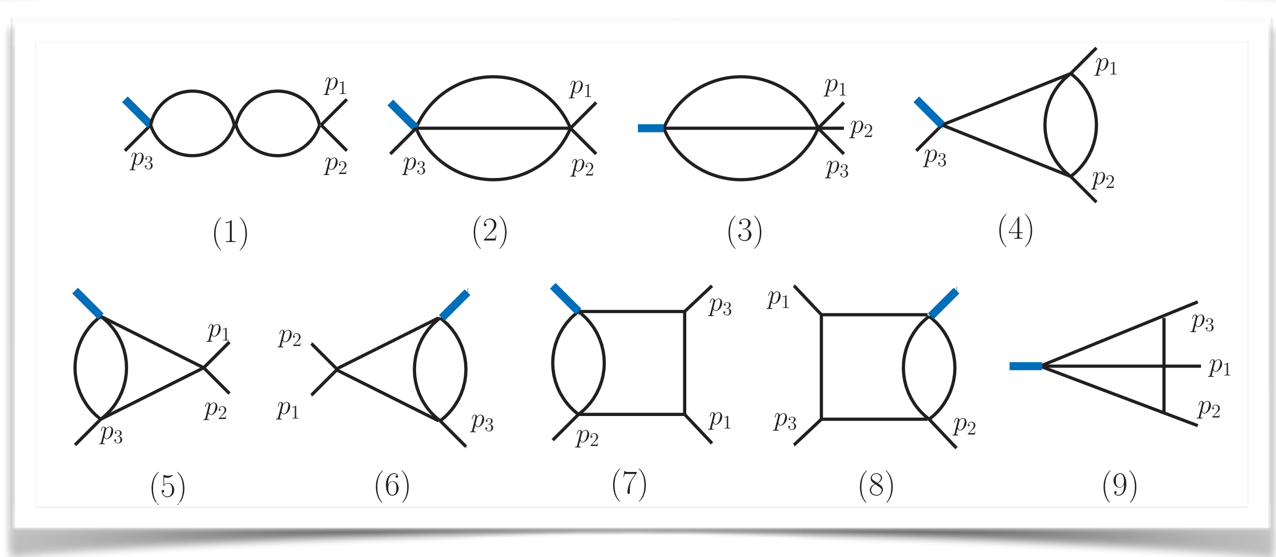
Form factors and on-shell methods

Previous results were known mostly at one-loop up to dimension-8.

Gracey 2002; Dawson, Lewis, Zeng 2014; …

Form factors can help tackle these problems to high dimensions and to high loop orders.





Mixing matrices and spectrum

Two-loop anomalous dimensions for length-3 operators up to dimension 16:

Jin, Ren, GY 2020

dim	4	6	8	10	12	14	16
$\gamma^{(1)}_{f,\alpha}$	$-\frac{22}{3}$	/	$\frac{7}{3}$	$\frac{71}{15}$	$\frac{241}{30}, \frac{101}{15}$	$\frac{61}{6}, \frac{172}{21}$	$\frac{331}{35}, \frac{1212\pm\sqrt{3865}}{105}$
$\gamma^{(2)}_{f,\alpha}$	$-\frac{136}{3}$	/	$\frac{269}{18}$	$\frac{2848}{125}$	$\frac{49901119}{1404000}, \frac{8585281}{234000}$	$\frac{4392073141}{87847200}, \frac{685262197}{15373260}$	$\frac{231568398949}{4253886000}, \frac{355106171452034\pm95588158951\sqrt{3865}}{6576507756000}$
$\gamma^{(1)}_{f,\beta}$	$-\frac{22}{3}$	1	/	$\frac{17}{3}$	9	$\frac{43}{5}$	$\frac{67}{6}$
$\gamma^{(2)}_{f,\beta}$	$-\frac{136}{3}$	$\frac{25}{3}$	/	$\frac{2195}{72}$	$\frac{79313}{1800}$	$\frac{443801}{9000}$	$\frac{63879443}{1058400}$
$\gamma^{(1)}_{d,\alpha}$	/	/	/	$\frac{13}{3}$	$\frac{41}{6}$	$\frac{551\pm3\sqrt{609}}{60}$	$\frac{321\pm\sqrt{1561}}{30}$
$\gamma^{(2)}_{d,\alpha}$	/	/	/	$\frac{575}{36}$	$\frac{46517}{1440}$	$\frac{5809305897\pm19635401\sqrt{609}}{131544000}$	$\frac{229162584707\pm225658792\sqrt{1561}}{4130406000}$
$\gamma^{(1)}_{d,\beta}$	/	/	/	/	9	/	$\frac{67}{6}$
$\gamma^{(2)}_{d,\beta}$	/	/	/	/	$\frac{150391}{3600}$	/	$\frac{174229}{3150}$

Two-loop renormalization for higher length operators. Jin, Ren, GY, Yu to appear

(Evanescent operators are important for computing 2-loop AD.)

Master-bootstrap method

- 2106.01374 [Phys.Rev.Lett. 127 (2021) 15, with Yuanhong Guo (郭圆宏), Lei Wang (王磊)
- 2205.12969, with Yuanhong Guo, Qingjun Jin, Lei Wang

Master bootstrap method

Guo, Wang, GY PRL 2021

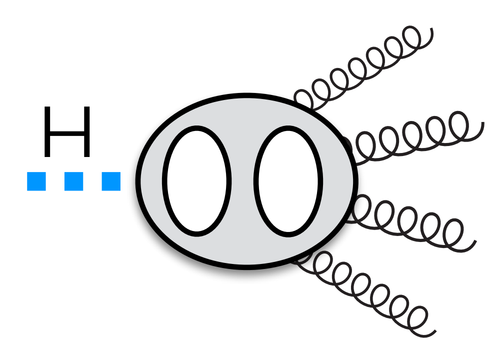
Ansatz in master integral expansion → **Physical constraints** → **Solution of coefficients**

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

- IR divergences
- Collinear factorization
- Spurious-pole cancellation
- Unitarity cut

$$\mathcal{F}^{(l),\text{ansatz}} = \sum_i C_i I_i^{(l)}$$

We apply this strategy to the frontier two-loop five-point scattering (Higgs plus four partons):



Master bootstrap method

Ansatz in master integral expansion → **Physical constraints** → Solution of coefficients

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- IR divergences
- Collinear factorization
- Spurious-pole cancellation
- Unitarity cut

$$\mathscr{F}^{(l),\text{ansatz}} = \sum_i {\color{red} C_i} \, I_i^{(l)}$$

The strategy does not rely on special symmetries of the theory, thus can be applied to general theories.

Maximal Transcendentality Principle

Maximal transcendentality principle

Kotikov, Lipatov, Onishchenko, Velizhanin 2004

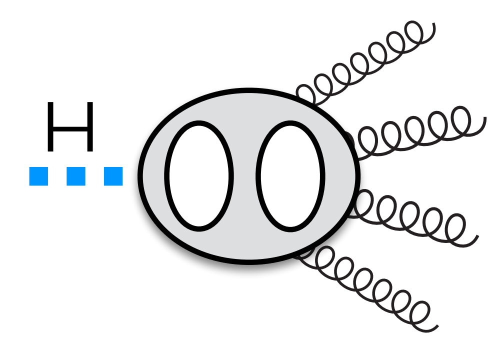
N=4 SYM ⟷ **QCD**

N=4 result is equal to the maximally transcendental part in QCD

Conjecture for certain quantities

We are able to prove the previously observed maximally transcendental correspondence for Higgs amplitudes (form factors) and also find new non-trivial example.

Guo, Jin, Wang, GY 2205.12969



Color-kinematics duality and double-copy of form factors

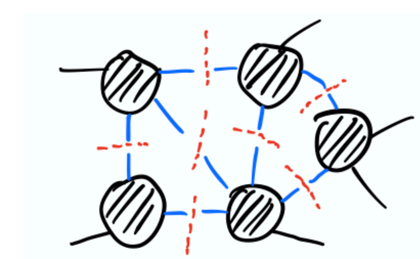
- 2106.01374 [Phys. Rev. Lett. 127 (2021) 17], 2111.03021, 2112.09123, with Guanda Lin (林冠达), Siyuan Zhang (张思源)
- 2111.12719, 220x.xxxxx, with Guanda Lin

Strategy of loop computation

CK-duality

Conjecture

Ansatz of the loop integrand



Unitarity cuts

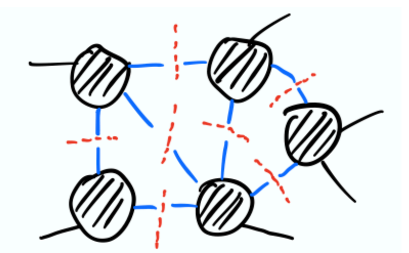
Solving linear equations

Strategy of loop computation

CK-duality

Conjecture

Ansatz of the loop integrand



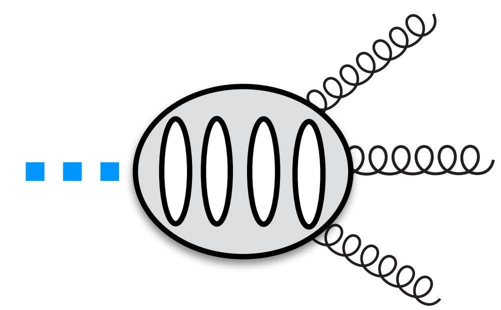
Unitarity cuts

Solving linear equations

Main challenge: it is a prior not known whether the solution exists

Results up to four loops

$$\mathcal{F}_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1, p_2, p_3 \,|\, \mathrm{tr}(F^2)(x) \,|\, 0 \rangle$$



L loops	$L{=}1$	$L{=}2$	$L{=}3$	$L{=}4$
# of cubic graphs	2	6	29	229
# of planar masters	1	2	2	4
# of free parameters	1	4	24	133

It is promising to go to higher loops.

Lin, GY, Zhang PRL 2021, 2112.09123

In the large-N limit, the remainder function was computed recently to **8 loops** via symbol bootstrap and the (non-perturbative) OPE input.

Dixon, Gurdogan, McLeod, Wilhelm 2204.11901
Sever, Tumanov, Wilhelm, 2009.11297 (FFOPE)

Form factor double-copy

Gauge x Gauge —Double-copy→ Gravity

- An surprising new mechanism for form factors: Lin, GY 2111.12719

Spurious poles (Gauge theory) —Double-copy→ Real propagators (Gravity theory)

- Hidden factorization relations of gauge form factors

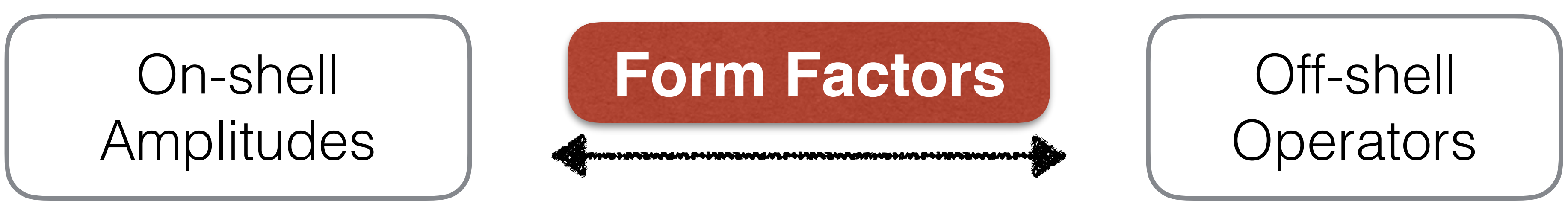
$$\vec{v} \cdot \vec{\mathcal{F}}_n \Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

Summary and outlook

Summary

- We review the S-matrix bootstrap and modern on-shell methods

- EFT and operators can be studied using on-shell techniques.



- We briefly mention some recent progress on form factors and their applications.

Outlook

Expectation:

All quantities that can be calculated using Feynman diagrams can be computed more efficiently with on-shell methods

- Consider more generic operators in general EFT, such as operators with fermion or massive fields, non-local operators, etc. A goal is to provide a two-loop framework for general EFT renormalization and EFT amplitudes.

- Explore hidden structure of renormalization and EFT amplitudes.

- Bootstrap beyond perturbation

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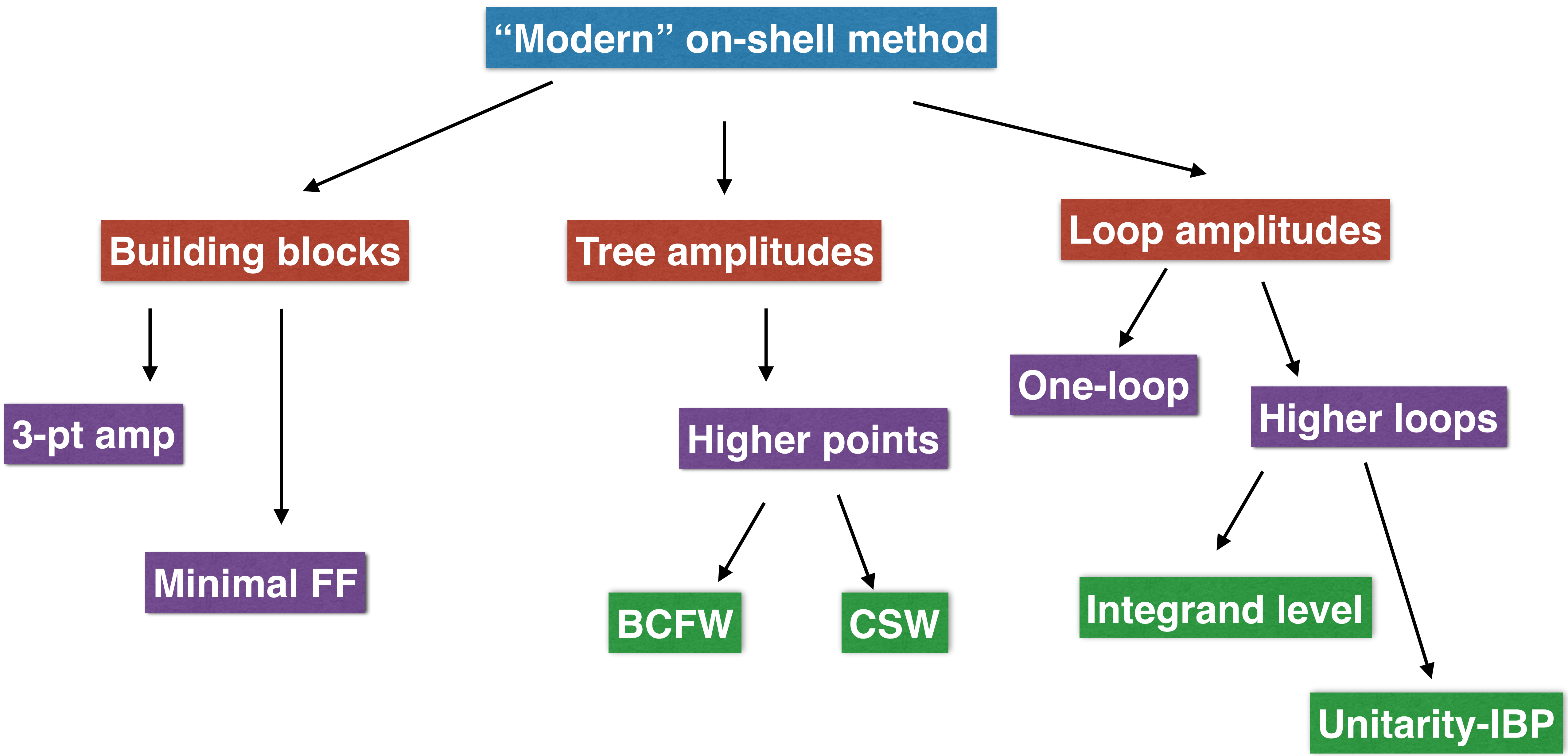
- Bootstrap beyond perturbation

Thank you for your attention!

Extra slides

A bird's eye view



MHV tree form factors

MHV structure of form factors: Brandhuber, Spence, Travaglini, GY 2010

$$F_n^{\rm MHV}(1^+,..,i_\phi,..,j_\phi,..,n^+;{\rm tr}(\phi^2)) = \delta^4(\sum_{i=1}^n p_i - q)\frac{\langle ij\rangle^2}{\langle 12\rangle\cdots\langle n1\rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

Compare with Parke-Taylor formula for amplitudes:

$$A_n^{\rm MHV}(1^+,..,i^-,..,j^-,..,n^+) = \delta^4(\sum_{i=1}^n p_i)\frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

$$0 = \sum_i p_i, \quad p_i^2 = 0$$

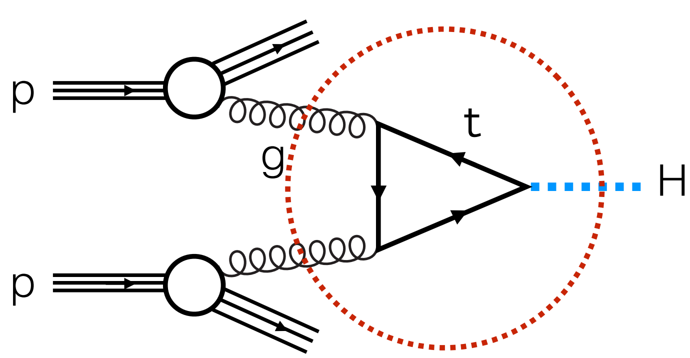
High-dimensional YM operators

We consider Lorentz scalar gauge invariant local operators:

$$\mathcal{O}(x) \sim c(a_1,\dots,a_n)X(\eta^{\mu\nu})\big(D_{\mu_{11}}\dots D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1}\cdots\big(D_{\mu_{n1}}\dots D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}(x)$$

$$D_\mu \star = \partial_\mu + ig[A_\mu,\star]\,, \qquad [D_\mu, D_\nu]\star = ig[F_{\mu\nu},\star] \qquad\qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a,T^b] = if^{abc}T^c$$

They are color-singlet gluon states and also appear as Higgs-gluon effective interaction vertices in Higgs EFT:



$$\xrightarrow{m_t \to \infty}$$

$$\mathcal{L}_{\text{eff}} = \hat{C}_0 H\mathcal{O}_{4;0} + \sum_{k=1}^{\infty}\frac{1}{m_{\text{t}}^{2k}}\sum_i \hat{C}_i H\mathcal{O}_{4+2k;i}$$

Evanescent YM operators

Parity-even gluonic evanescent operators start to appear at mass dimension 10:

$$\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

$$\mathcal{O}_{\rm e} = \frac{1}{16}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{\nu_1\nu_2\nu_3\nu_4\nu_5}\mathrm{tr}(D_{\nu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}D_{\mu_5}F_{\nu_1\nu_2}F_{\nu_3\nu_4})$$

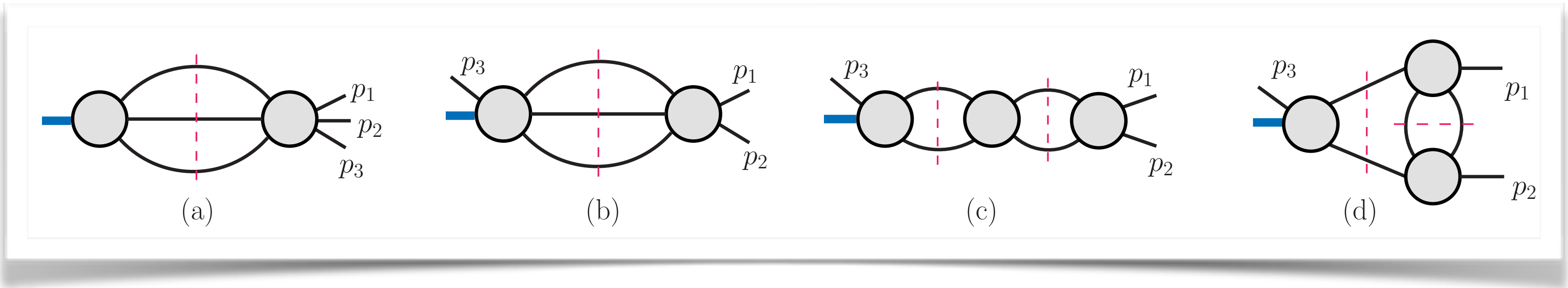
$$\delta^{\mu_1..\mu_n}_{\nu_1...\nu_n} = \det(\delta^{\mu}_{\nu}) = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \dots & \delta^{\mu_1}_{\nu_n} \\ \vdots & & \vdots \\ \delta^{\mu_n}_{\nu_1} & \dots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}$$

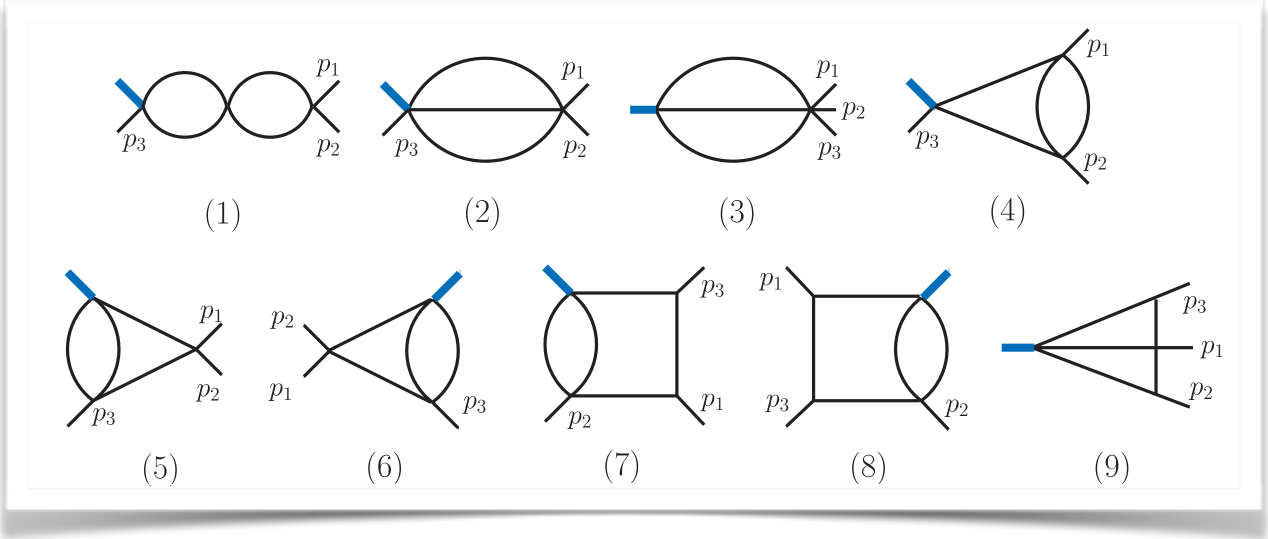
$$\mathcal{F}^{(0)}_{\mathcal{O}_{\rm e}}(1,2,3,4) = 2\delta^{e_1e_2p_1p_2p_3}_{e_3e_4p_3p_4p_1} + 2\delta^{e_1e_4p_1p_4p_2}_{e_2e_3p_2p_3p_4}$$

High-dimensional YM operators

On-shell unitarity-IBP method:

$$\mathcal{F}^{(l)}\Big|_{\text{cut}} = \prod(\text{Tree blocks}) = \text{Cut integrand} \xrightarrow{\text{IBP with cuts}} \sum_i c_i \left(I_i\big|_{\text{cut}}\right)$$





Mixing matrices and spectrum

$$\boxed{\text{Loop form factor} = (\text{Universal IR div.}) + (\text{UV div.}) + (\text{Finite part})}$$

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

Dimension-8: (up to length-3)

$$\mathcal{O}_{8;1} = \partial^4 \mathrm{Tr}(F^2), \quad \mathcal{O}_{8;2} = \partial^2 \mathrm{tr}(F^3), \quad \mathcal{O}_{8;3} = \mathrm{tr}(D_1 F_{23} D_4 F_{23} F_{14}),$$

$$\mathbb{D}_{\mathcal{O}_8} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix} \qquad \hat{\gamma}^{(1)}_{\mathcal{O}_8} = \left\{-\frac{22}{3}; 1; \frac{7}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_8} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}\right\}$$

Finite remainder

The finite remainders -> Higgs amplitudes with high-order top mass corrections in Higgs EFT.

There are "universal building blocks" that are independent of the operators:

The full transcendentality degree-4 part is universal:

$$\mathcal{R}_{\mathcal{O}}^{(2),\pm}\Big|_{\text{deg-4}} = -\frac{3}{2}\text{Li}_4(u) + \frac{3}{4}\text{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{4}\log(w)\left[\text{Li}_3\left(-\frac{u}{v}\right) + \text{Li}_3\left(-\frac{v}{u}\right)\right] + \frac{\log^2(u)}{32}\left[\log^2(u) + \log^2(v) + \log^2(w) - 4\log(v)\log(w)\right] + \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 + \text{perms}(u,v,w)\,,$$

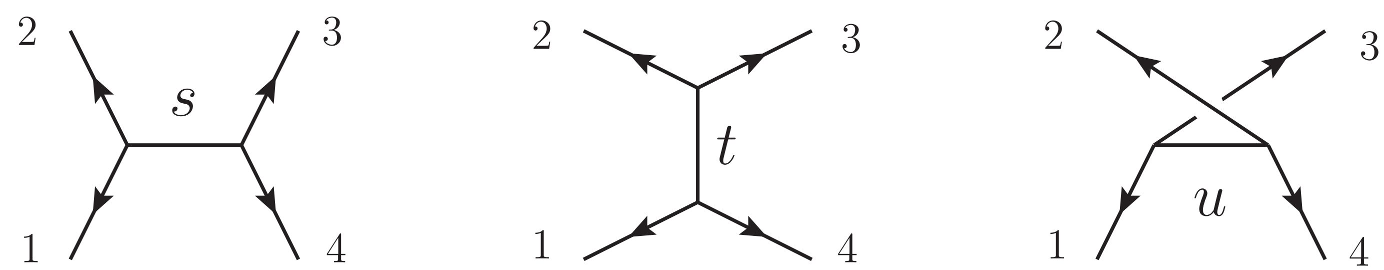
"maximal transcendentality principle"

Color-kinematics duality

In 2008 Bern, Carrasco and Johansson proposed an intriguing duality between color and kinematics factors:

Color factor	Duality ⟷ (conjecture)	Kinematic factor
$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c)$		$s_{ij} = (p_i + p_j)^2$
Gauge symmetry		**Spacetime symmetry**

Example: 4-pt amplitude



$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}$$

$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity

dual Jacobi relation

Color-kinematics duality

Proved at tree-level:

- String Monodromy relation Bjerrum-Bohr et.al 2009; Stieberger 2009
- BCFW recursion Feng, Huang, Jia 2010

Still a conjecture at loop level, relying on explicit constructions:

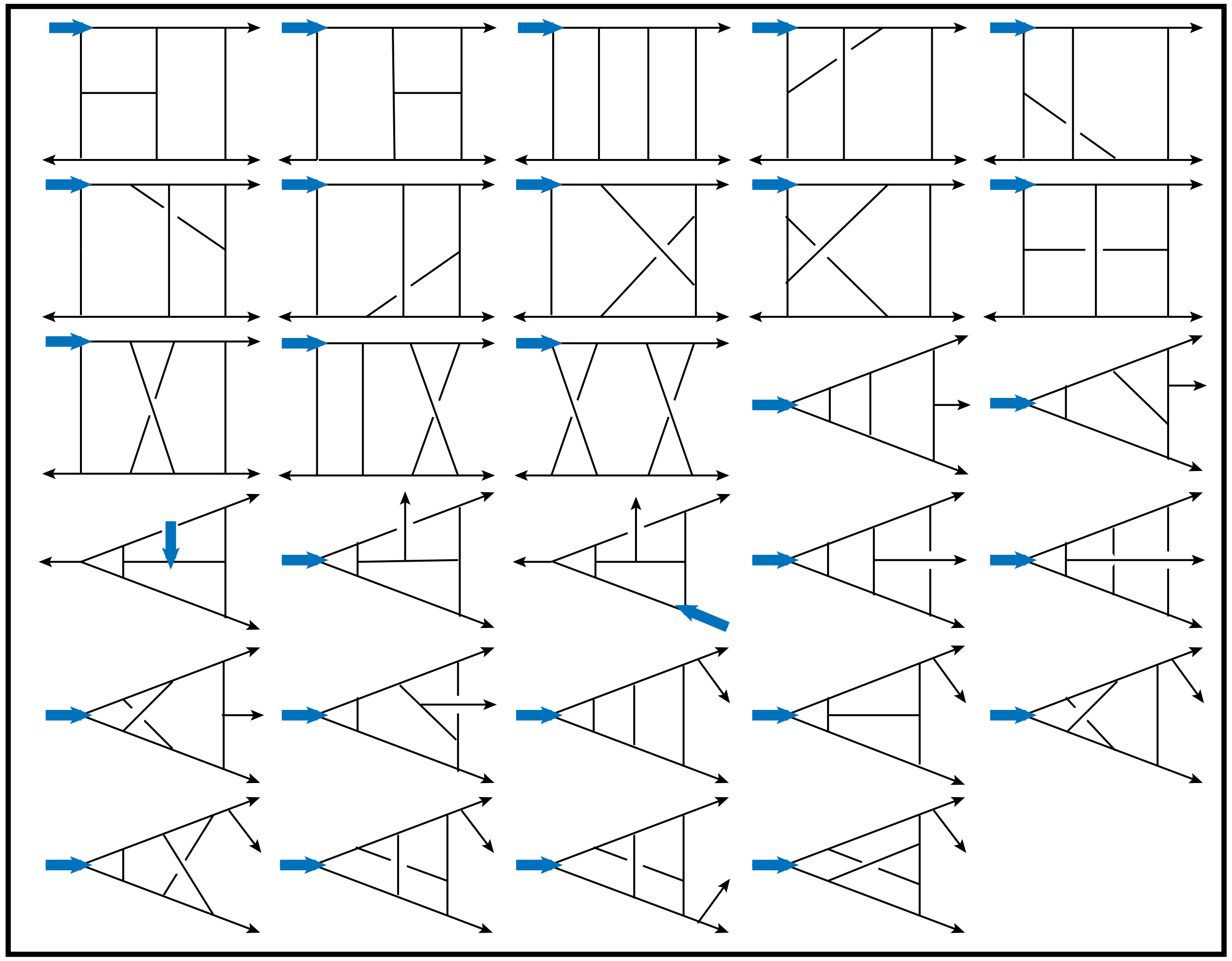
- 4-loop 4-point amplitudes in N=4 Bern, et.al, 2012
- 5-loop Sudakov form factor in N=4 G. Yang, 2016
- 2-loop 5-point amplitudes in pure YM O'Connell and Mogull 2015

It is usually non-trivial to find CK dual solution at high loops.

3-loop solution

3-loop integral topologies:

Lin, GY, Zhang PRL 2021



$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} \sum_{\sigma_3} \sum_i \int \prod_{j=1}^{3} d^D \ell_j \frac{1}{S_i} \frac{C_i N_i}{\prod_{\alpha_i} P^2_{\alpha_i}}$$

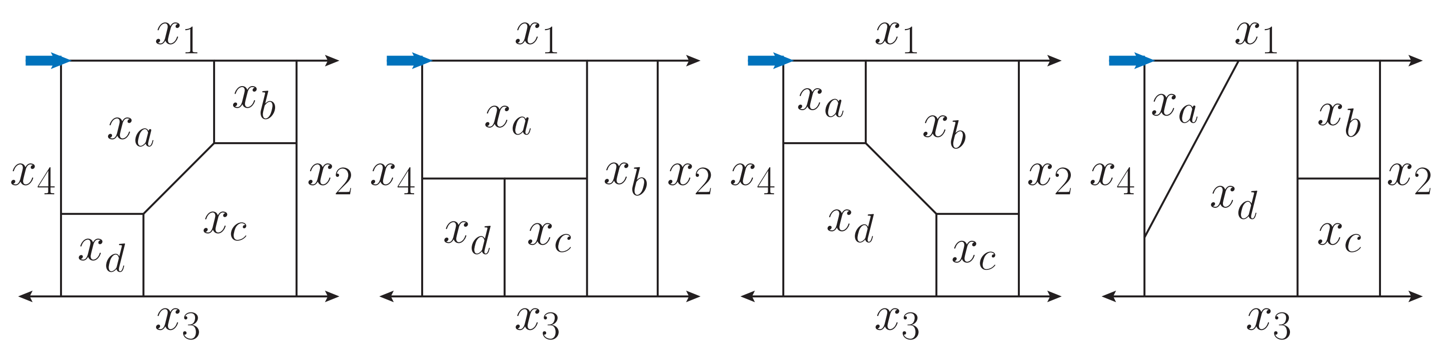
Finally physical solutions still contains 24 free parameters !

We also perform (numerical) integration and obtain the integrated results, including 3-loop non-planar corrections.

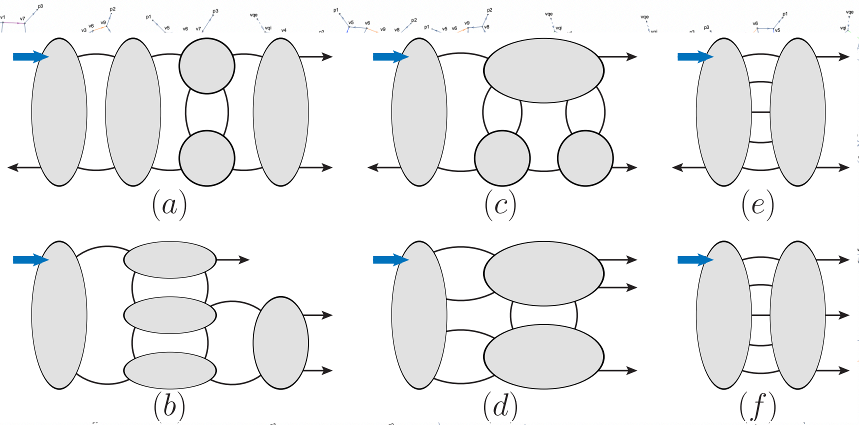
(~10 million CPU core-hours)

Four loops

Master graphs



Unitarity cuts



Final solution with 133 free parameters!

$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^{4} d^D \ell_j \frac{1}{S_i} \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

G. Lin, GY, S. Zhang, 2112.09123

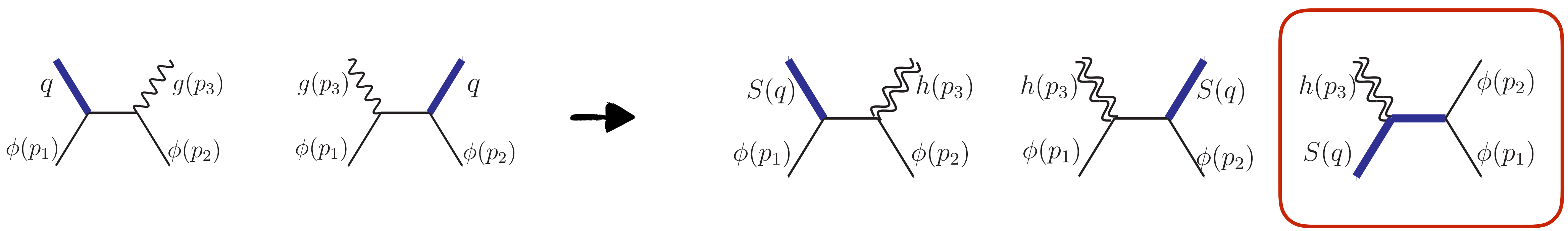
Example: 3-point form factor

$$\mathcal{G}_3 = \frac{(N_1^{\rm CK})^2}{s_{23}} + \frac{(N_2^{\rm CK})^2}{s_{13}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}}\Big(\mathcal{F}_3(1^\phi, 3^g, 2^\phi)\Big)^2$$

There is a nice factorization behavior at the new pole:

$$s_{13} + s_{23} = q^2 - s_{12} = 0$$

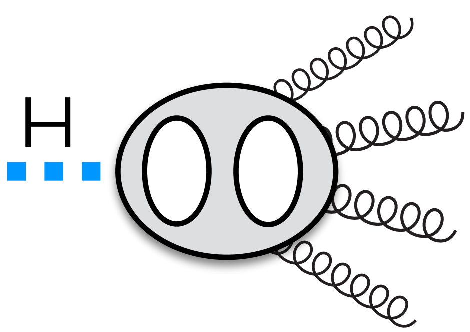
$$\text{Res}\,[\mathcal{G}_3]_{s_{12}=q^2} = (\epsilon_3 \cdot q)^2 = \Big(\mathcal{F}_2(1^\phi, 2^\phi)\Big)^2 \times \left(\mathcal{A}_3(\mathbf{q}_2^S, 3^g, -q^S)\right)^2$$



A new graph in gravity

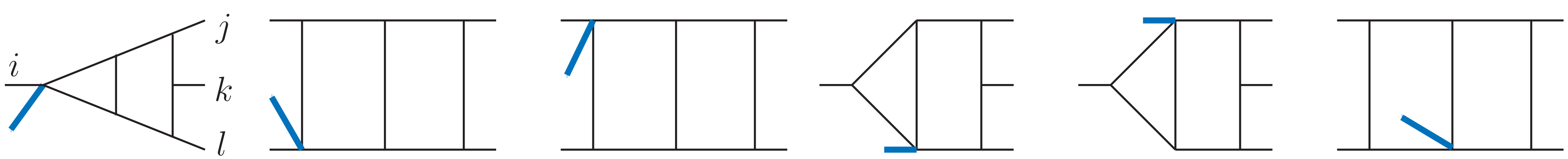
Ansatz of the form factors

Our result provides a first two-loop five-point example with a color-singlet off-shell leg.



$$\mathscr{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x}\langle 1,2,3,4\,|\,\mathcal{O}(x)\,|\,0\rangle$$

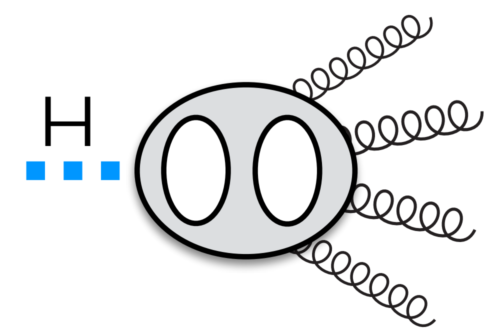
$\{s_{12}, s_{23}, s_{34}, s_{14}, s_{13}, s_{24}, \mathrm{tr}_5\}$; $\qquad \mathrm{tr}_5 = 4i\varepsilon_{p_1p_2p_3p_4}$



The form factor we consider

A two-loop four-point form factor in N=4 SYM:

$$\mathcal{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x} \langle 1_\phi, 2_\phi, 3_\phi 4^+ \,|\, \mathrm{tr}(\phi^3)(x) \,|\, 0 \rangle$$



As an N=4 version of Higgs+4-parton amplitudes in QCD.

Five-point two-loop amplitudes are at frontier and under intense study:

There have been many massless five-point two-loop amplitudes obtained in analytic form. See e.g. Abreu, Dormans, Cordero, Ita, Page 2019 and many others….

For five-point two-loop amplitudes with one massive leg, very limited results are available: $u\bar{d} \to W^+ b\bar{b}$ Badger, Hartanto, Zoia 2021

Outline of two-loop computation

Ansatz in master integral expansion

$$\mathcal{F}^{(2),\text{ansatz}}_{\mathcal{O},4} = \sum_i C_i I_i^{(l)}$$

Guo, Wang, GY PRL 2021

Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	22
IR (Function)	17
Collinear limit (Funcion)	10
If keeping only to ϵ^0 order	6
Simple unitarity cuts	0

Solution of coefficients

$$\mathcal{F}^{(2)}_{\mathcal{O},4} = \sum_i C_i I_i^{(l)}$$