Inclusive and Semi-inclusive Production of Spin-3/2 Hadrons in e^+e^- Annihilation

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Reference: Jing Zhao, Zhe Zhang, Zuo-tang Liang, Tianbo Liu, and Ya-jin Zhou, arXiv: 2206.11742.

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Outline

- Introduction
- The description of spin-3/2 particles
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- Spin-3/2 hadrons produced in e^+e^- annihilation
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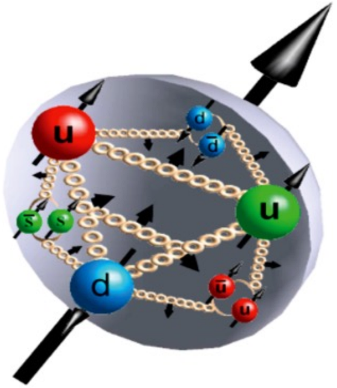
  - Semi-inclusive process $e^+e^- \to \Omega h X$
- Summary

Introduction

- Fragmentation functions (FFs): hadron momentum distribution in the final state, depend on z.

  Transverse momentum dependent (TMD) fragmentation functions (FFs): FFs that depend on z and P_{hT}.

- Studies of FFs provide important information for the study of the hadronization mechanism and the properties of strong interactions.



hadron bound states

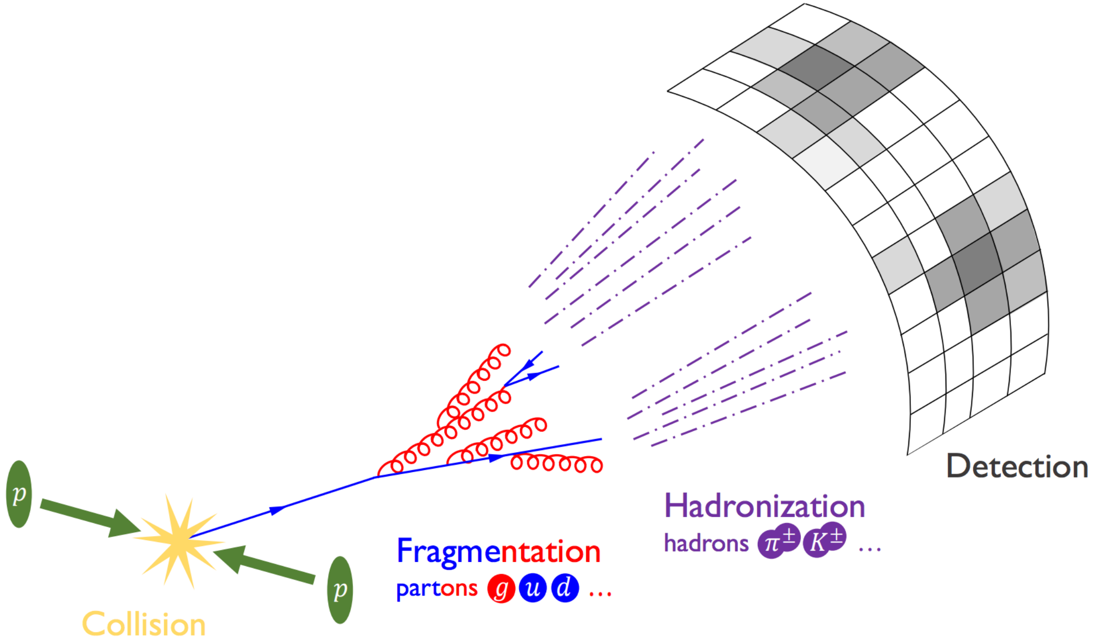


Figure from https://www.ericmetodiev.com/post/jetformation/.

Introduction

The leading-twist TMD fragmentation functions for spin-0, spin-1/2 and spin-1 particles:

		Quark Polarization			
		Unpolarized	Longitudinally Polarized	Transversely Polarized	
Hadron Polarization	U	$D_1\left(z,k_T^2\right)$		$H_1^{\perp}\left(z,k_T^2\right)$	2
	L		$G_{1L}\left(z,k_T^2\right)$	$H_{1L}^{\perp}\left(z,k_T^2\right)$	6
	T	$D_{1T}^{\perp}\left(z,k_T^2\right)$	$G_{1T}^{\perp}\left(z,k_T^2\right)$	$H_{1T}\left(z,k_T^2\right),H_{1T}^{\perp}\left(z,k_T^2\right)$	
	LL	$D_{1LL}\left(z,k_T^2\right)$		$H_{1LL}^{\perp}\left(z,k_T^2\right)$	
	LT	$D_{1LT}^{\perp}\left(z,k_T^2\right)$	$G_{1LT}^{\perp}\left(z,k_T^2\right)$	$H_{1LT}\left(z,k_T^2\right),H_{1LT}^{\perp}\left(z,k_T^2\right)$	10
	TT	$D_{1TT}^{\perp}\left(z,k_T^2\right)$	$G_{1TT}^{\perp}\left(z,k_T^2\right)$	$H_{1TT}'^{\perp}\left(z,k_T^2\right),H_{1TT}^{\perp}\left(z,k_T^2\right)$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	Spin>1 ?

A. Bacchetta and P. J. Mulders, Phys. Rev. D 62, 114004 (2000).

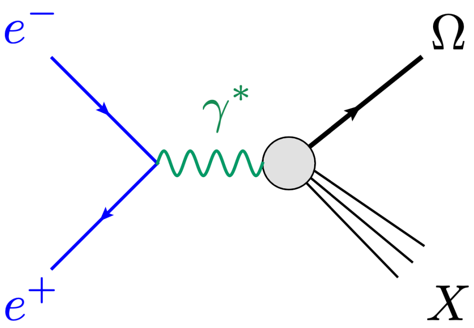
K. b. Chen, W. h. Yang, S. y. Wei and Z. t. Liang, Phys. Rev. D 94, 034003 (2016).

Introduction

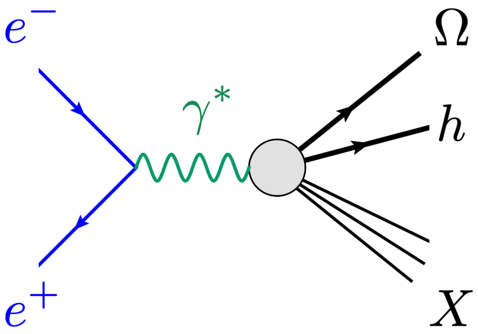
- Electron-positron annihilation is the cleanest progress to extract FFs.

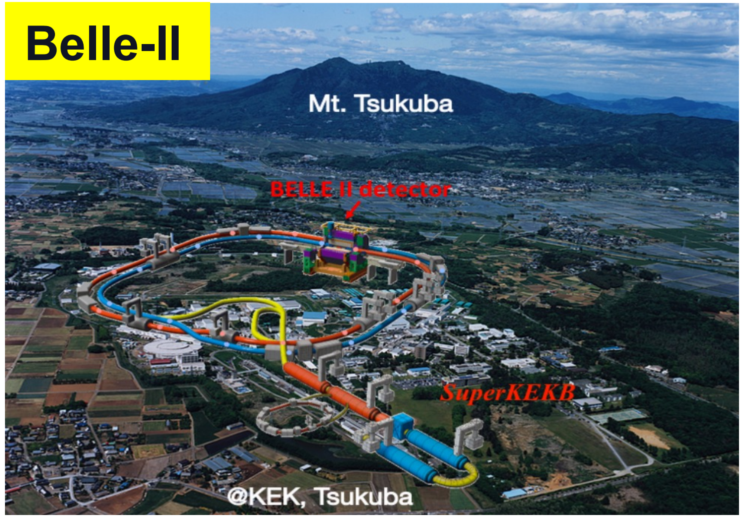
Spin-3/2 hadrons Ω produced in e^+e^- annihilation ⟹ Study rank-3 tensor polarized FFs

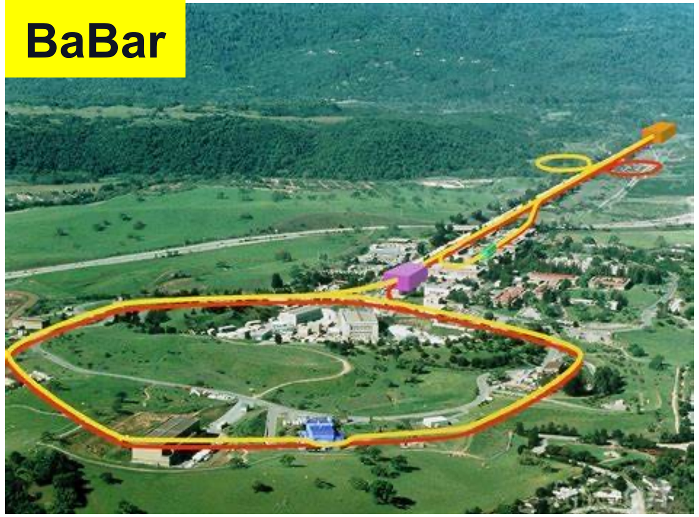
$e^+e^- \to \Omega X$ ⟹ collinear FFs



$e^+e^- \to \Omega h X$ ⟹ TMD FFs







The description of spin-3/2 particles

Spin-s: $m_s = \underbrace{-s, \cdots\cdots, s}_{(2s+1)}$ $\longrightarrow$ $\boldsymbol{\rho}$: $\mathbf{(2s+1)\times(2s+1)}$

The properties of spin density matrix: $\boldsymbol{\rho} = \boldsymbol{\rho}^\dagger, \quad \mathbf{Tr}\,\boldsymbol{\rho} = \mathbf{1}$

- **Spin-1/2:** $\boldsymbol{\rho} : \mathbf{2 \times 2}$ $\quad \boldsymbol{\rho} = \frac{1}{2}(\mathbf{1} + S^i \boldsymbol{\sigma}^i)$

  S^i : spin vector $\quad \boldsymbol{S} = (S_T^x, S_T^y, S_L)$ **—3 independent components**

- **Spin-1:** $\boldsymbol{\rho} : \mathbf{3 \times 3}$ $\quad \boldsymbol{\rho} = \frac{1}{3}\left(\mathbf{1} + \frac{3}{2} S^i \boldsymbol{\Sigma}^i + 3T^{ij}\boldsymbol{\Sigma}^{ij}\right)$

T^{ij} : symmetric traceless rank-2 spin tensor

$$T^{ij} = \frac{1}{2}\begin{pmatrix} S_{LL} + S_{TT}^{xx} & S_{TT}^{xy} & S_{LT}^{x} \\ S_{TT}^{xy} & S_{LL} - S_{TT}^{xx} & S_{LT}^{y} \\ S_{LT}^{x} & S_{LT}^{y} & -2S_{LL} \end{pmatrix}$$

$$\left.\begin{array}{ll} S_{LL}, S_{LT}^x, S_{LT}^y, S_{TT}^{xx}, S_{TT}^{xy} & \mathbf{5} \\ S_T^x, S_T^y, S_L & \mathbf{3} \end{array}\right\} \textbf{8 independent components}$$

A. Bacchetta and P. J. Mulders, Phys. Rev. D 62, 114004 (2000).

E. Leader, *Spin in particle physics, 2001*.

The description of spin-3/2 particles

- **Spin-3/2:** $\rho : 4 \times 4$ $\quad \rho = \frac{1}{4}\left(\mathbf{1} + \frac{4}{5}S^i\mathbf{\Sigma}^i + \frac{2}{3}T^{ij}\mathbf{\Sigma}^{ij} + \frac{8}{9}R^{ijk}\mathbf{\Sigma}^{ijk}\right)$

16 $\quad$ 1 $\quad$ 3 $\quad$ 5 $\quad$ 7

$$\mathbf{\Sigma}^{ij} = \frac{1}{2}\left(\mathbf{\Sigma}^i\mathbf{\Sigma}^j + \mathbf{\Sigma}^j\mathbf{\Sigma}^i\right) - \frac{5}{4}\mathbf{1}\delta^{ij}$$

$$\mathbf{\Sigma}^{ijk} = \frac{1}{6}\left(\mathbf{\Sigma}^{\{i}\mathbf{\Sigma}^j\mathbf{\Sigma}^{k\}}\right) - \frac{41}{60}\left(\mathbf{\Sigma}^i\delta^{jk} + \mathbf{\Sigma}^j\delta^{ik} + \mathbf{\Sigma}^k\delta^{ij}\right)$$

R^{ijk} : symmetric traceless rank-3 spin tensor $\quad R^{ijk} = \mathrm{Tr}[\mathbf{\Sigma}^{ijk}\rho] =< \mathbf{\Sigma}^{ijk} >$

$$\left.\begin{array}{ll} S^i : S_T^x, S_T^y, S_L & 3 \\ T^{ij} : S_{LL}, S_{LT}^x, S_{LT}^y, S_{TT}^{xx}, S_{TT}^{xy} & 5 \\ R^{ijk} : S_{LLL}, S_{LLT}^x, S_{LLT}^y, S_{LTT}^{xx}, S_{LTT}^{xy}, S_{TTT}^{xxx}, S_{TTT}^{yxx} & 7 \end{array}\right\}$$

15 independent components

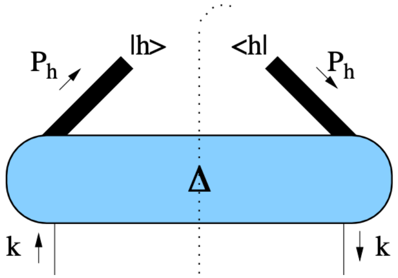
In the rest frame $\quad$ Lorentz covariant form

$S^i, T^{ij}, R^{ijk} \longrightarrow S^\mu, T^{\mu\nu}, R^{\mu\nu\rho}$

$$\boxed{P_\mu S^\mu = 0, \quad P_\mu T^{\mu\nu} = 0, \quad P_\mu R^{\mu\nu\rho} = 0}$$

TMD fragmentation functions

Parameterization of the quark-quark correlation function



$$\Delta_{\alpha\beta}\,(k,P,S,T,R) = \sum_X \int \frac{\mathrm{d}^4\xi}{(2\pi)^4} e^{ik\cdot\xi} \langle 0|\underline{\mathcal{L}(\infty,\xi)}\psi_\alpha(\xi)|P,S,T,R,X\rangle$$

$$\times \quad \langle P,S,T,R,X|\bar{\psi}_\beta(0)\underline{\mathcal{L}^\dagger(\infty,0)}|0\rangle$$

Gauge link

The correlation function can be decomposed by Dirac structures.

$$\Delta(k,P,S,T,R) \left\{ \begin{array}{l} \mathbf{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \mathrm{i}\sigma^{\mu\nu}\gamma_5 \\ \\ k^\mu, P^\mu, S^\mu, T^{\mu\nu}, R^{\mu\nu\rho} \end{array} \right.$$

$\mathbf{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \mathrm{i}\sigma^{\mu\nu}\gamma_5$ ⟶ basis

$k^\mu, P^\mu, S^\mu, T^{\mu\nu}, R^{\mu\nu\rho}$ ⟶ coefficient

⟹ The most general decomposition of correlation function.

Each term of the decomposition fulfills Hermiticity and parity invariance:

Hermiticity: $\Delta(k,P,S,T,R) = \gamma^0 \Delta^\dagger(k,P,S,T,R)\gamma^0$

Parity invariance: $\Delta(k,P,S,T,R) = \gamma^0 \Delta(\bar{k},\bar{P},-\bar{S},\bar{T},-\bar{R})\gamma^0$

TMD fragmentation functions

$$
\begin{aligned}
\Delta(k,P,S,T,R) = {} & \boxed{MB_1\mathbf{1} + B_2\not{P} + B_3\not{k} + \frac{B_4}{M}\sigma_{\mu\nu}P^\mu k^\nu}_{\text{Spin-0}} + \mathrm{i}B_5 k\cdot S\gamma_5 + MB_6\not{S}\gamma_5 + B_7\frac{k\cdot S}{M}\not{P}\gamma_5 + B_8\frac{k\cdot S}{M}\not{k}\gamma_5 \\
& + \mathrm{i}B_9\sigma_{\mu\nu}\gamma_5 S^\mu P^\nu + \mathrm{i}B_{10}\sigma_{\mu\nu}\gamma_5 S^\mu k^\nu + \mathrm{i}B_{11}\frac{k\cdot S}{M^2}\sigma_{\mu\nu}\gamma_5 P^\mu k^\nu + B_{12}\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k^\rho S^\sigma}{M} \quad \text{Spin-1/2} \\
& + \frac{B_{13}}{M}k_\mu k_\nu T^{\mu\nu}\mathbf{1} + \frac{B_{14}}{M^2}k_\mu k_\nu T^{\mu\nu}\not{P} + \frac{B_{15}}{M^2}k_\mu k_\nu T^{\mu\nu}\not{k} + \frac{B_{16}}{M^3}k_\mu k_\nu T^{\mu\nu}\sigma_{\rho\sigma}P^\rho k^\sigma + B_{17}k_\mu T^{\mu\nu}\gamma_\nu \\
& + \frac{B_{18}}{M}\sigma_{\nu\rho}P^\rho k_\mu T^{\mu\nu} + \frac{B_{19}}{M}\sigma_{\nu\rho}k^\rho k_\mu T^{\mu\nu} + \frac{B_{20}}{M^2}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma_5 P^\nu k^\rho k_\tau T^{\tau\sigma} \quad \text{Spin-1} \\
& + \mathrm{i}\frac{B_{21}}{M^2}k_\mu k_\nu k_\rho R^{\mu\nu\rho}\gamma_5 + \frac{B_{22}}{M^3}k_\mu k_\nu k_\rho R^{\mu\nu\rho}\not{k}\gamma_5 + \frac{B_{23}}{M^3}k_\mu k_\nu k_\rho R^{\mu\nu\rho}\not{P}\gamma_5 + \mathrm{i}\frac{B_{24}}{M^4}k_\mu k_\nu k_\rho R^{\mu\nu\rho}\sigma_{\tau\lambda}\gamma_5 k^\tau P^\lambda \\
& + \frac{B_{25}}{M}k_\mu k_\nu R^{\mu\nu\rho}\gamma_\rho\gamma_5 + \mathrm{i}\frac{B_{26}}{M^2}\sigma_{\rho\tau}\gamma_5 k^\tau k_\mu k_\nu R^{\mu\nu\rho} + \mathrm{i}\frac{B_{27}}{M^2}\sigma_{\rho\tau}\gamma_5 P^\tau k_\mu k_\nu R^{\mu\nu\rho} + \frac{B_{28}}{M^3}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu k^\nu P^\rho k_\tau k_\lambda R^{\tau\lambda\sigma} \quad \text{Spin-3/2}
\end{aligned}
$$

$B_{21} - B_{28}$: rank-3 tensor polarized terms, newly defined for spin-3/2 hadrons.

P^- as a large momentum component and the leading-twist TMD FFs can be projected out from the correlator by the Dirac matrices. $\gamma^-, \gamma^-\gamma_5, i\sigma^{i-}\gamma_5$

TMD fragmentation functions

32 leading-twist TMD fragmentation functions for spin-3/2 particles:

		Quark Polarization		
		Unpolarized	Longitudinally Polarized	Transversely Polarized
Hadron Polarization	U	$D_1(z, k_T^2)$		$H_1^\perp(z, k_T^2)$
	L		$G_{1L}(z, k_T^2)$	$H_{1L}^\perp(z, k_T^2)$
	T	$D_{1T}^\perp(z, k_T^2)$	$G_{1T}^\perp(z, k_T^2)$	$H_{1T}(z, k_T^2), H_{1T}^\perp(z, k_T^2)$
	LL	$D_{1LL}(z, k_T^2)$		$H_{1LL}^\perp(z, k_T^2)$
	LT	$D_{1LT}^\perp(z, k_T^2)$	$G_{1LT}^\perp(z, k_T^2)$	$H_{1LT}(z, k_T^2), H_{1LT}^\perp(z, k_T^2)$
	TT	$D_{1TT}^\perp(z, k_T^2)$	$G_{1TT}^\perp(z, k_T^2)$	$H_{1TT}^\perp(z, k_T^2), H_{1TT}^{\perp\perp}(z, k_T^2)$
	LLL		$G_{1LLL}(z, k_T^2)$	$H_{1LLL}^\perp(z, k_T^2)$
	LLT	$D_{1LLT}^\perp(z, k_T^2)$	$G_{1LLT}^\perp(z, k_T^2)$	$H_{1LLT}(z, k_T^2), H_{1LLT}^\perp(z, k_T^2)$
	LTT	$D_{1LTT}^\perp(z, k_T^2)$	$G_{1LTT}^\perp(z, k_T^2)$	$H_{1LTT}^\perp(z, k_T^2), H_{1LTT}^{\perp\perp}(z, k_T^2)$
	TTT	$D_{1TTT}^\perp(z, k_T^2)$	$G_{1TTT}^\perp(z, k_T^2)$	$H_{1TTT}^\perp(z, k_T^2), H_{1TTT}^{\perp\perp}(z, k_T^2)$

$L, T\ldots$: hadron polarized

$\perp$: the dependence of k_T

2

6

10

14

A. Bacchetta and P. J. Mulders, Phys. Rev. D 62, 114004 (2000).

K. b. Chen, W. h. Yang, S. y. Wei and Z. t. Liang, Phys. Rev. D94, 034003 (2016).

TMD fragmentation functions

The collinear FFs are defined from the k_T -integrated correlation function,

$$\Delta(z) = \frac{z}{4}\int \mathrm{d}^2\boldsymbol{k}_T\ \mathrm{d}k^+ \Delta\left(k, P, S, T, R\right)\Big|_{k^- = \frac{P_h^-}{z}}$$

Collinear FFs: **9**

$$\Delta_U(z) = \frac{1}{4} D_1\left(z\right) \not{n},$$

$$\Delta_L(z) = \frac{1}{4} G_{1L}\left(z\right) S_L \gamma_5 \not{n},$$

$$\Delta_T(z) = \frac{1}{4} H_{1T}\left(z\right) i\sigma_{\mu\nu}\gamma_5 n^\mu S_T^\nu,$$

$$\Delta_{LL}(z) = \frac{1}{4} D_{1LL}\left(z\right) S_{LL} \not{n},$$

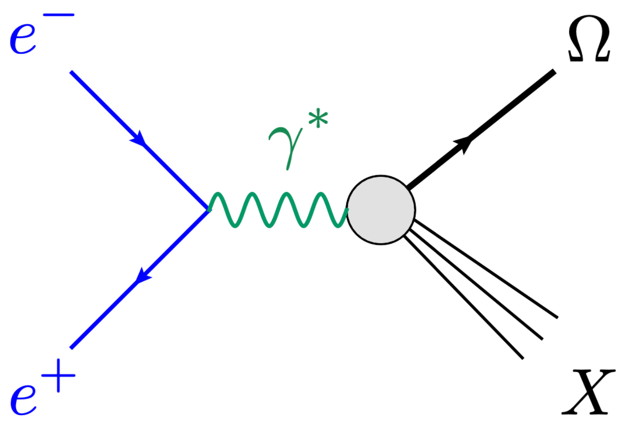
$$\Delta_{LT}(z) = \frac{1}{4} H_{1LT}\left(z\right) \sigma_{\mu\nu} S_{LT}^\mu n^\nu,$$

$$\boxed{\begin{aligned}\Delta_{LLL}(z) &= \frac{1}{4} G_{1LLL}\left(z\right) S_{LLL}\gamma_5 \not{n},\\ \Delta_{LLT}(z) &= \frac{1}{4} H_{1LLT}\left(z\right) i\sigma_{\mu\nu}\gamma_5 n^\mu S_{LLT}^\nu\end{aligned}}$$

After integration, the terms with inhomogeneous k_T -dependence all vanish.

Inclusive production of the Ω in e^+e^- collisions

Decomposition of the hadronic tensor



$$e^-(l_1) + e^+(l_2) \to \Omega(P) + X(P_X)$$

$$P^0 \frac{d\sigma}{d^3 \boldsymbol{P}} = \frac{\alpha^2}{2Q^6} \boldsymbol{L_{\mu\nu}} \boldsymbol{W^{\mu\nu}}$$

$$L_{\mu\nu}(l_1, l_2) = 2[l_{1\mu} l_{2\nu} + l_{1\nu} l_{2\mu} - g_{\mu\nu}(l_1 \cdot l_2)]$$

Leptons are unpolarized

$$W^{\mu\nu}(q; P, S, T, R) = \frac{1}{(2\pi)} \sum_X \langle 0 | J^\mu(0) | P_X; P, S, T, R \rangle \langle P_X; P, S, T, R | J^\nu(0) | 0 \rangle$$
$$\times (2\pi)^4 \delta^{(4)}(q - P_X - P)$$

The hadronic tensor must satisfy:

Hermiticity: $W^{*\mu\nu}(q; P, S, T, R) = W^{\nu\mu}(q; P, S, T, R)$

Parity invariance : $W^{\mu\nu}(q; P, S, T, R) = W_{\mu\nu}(q; \bar{P}, -\bar{S}, \bar{T}, -\bar{R})$

Gauge invariance: $q_\mu W^{\mu\nu} = W^{\mu\nu} q_\nu = 0$

Inclusive production of the Ω in e^+e^- collisions

Hadronic tensor $W^{\mu\nu}$ → basis tensors multiplied by structure functions（基底张量）

$$W^{\mu\nu}(P,S,T,R)$$

9 independent basis tensors:

Unpolarized: $$t_U^{\mu\nu}=\left\{\tilde{g}^{\mu\nu},\tilde{P}^{\mu}\tilde{P}^{\nu}\right\},$$

Vector polarized: $$t_V^{\mu\nu}=\epsilon^{SPq\{\mu}\tilde{P}^{\nu\}},$$

Rank-2 tensor polarized: $$t_T^{\mu\nu}=\left\{T^{qq}\tilde{g}^{\mu\nu},T^{qq}\tilde{P}^{\mu}\tilde{P}^{\nu},\tilde{T}^{q\{\mu}\tilde{P}^{\nu\}},\tilde{T}^{\mu\nu}\right\},$$

Rank-3 tensor polarized: $$t_R^{\mu\nu}=\left\{\epsilon^{R^{qq}Pq\{\mu}\tilde{P}^{\nu\}},R_{\rho}{}^{q\{\mu}\epsilon^{\nu\}\rho Pq}\right\}$$

$$W^{\mu\nu}=\sum_{i=1}^{2}V_{U,i}t_{U,i}^{\mu\nu}+\sum_{i=1}^{1}V_{V,i}t_{V,i}^{\mu\nu}+\sum_{i=1}^{4}V_{T,i}t_{T,i}^{\mu\nu}+\sum_{i=1}^{2}V_{R,i}t_{R,i}^{\mu\nu}$$

Subscript U, V, T, R: hadron polarization
V_i: structure functions

$$\tilde{g}^{\mu\nu}=g^{\mu\nu}-\frac{q^{\mu}q^{\nu}}{q^2},$$

$$\tilde{P}^{\mu}=P^{\mu}-\frac{P\cdot q}{q^2}q^{\mu},$$

$$\tilde{T}^{\mu q}=T^{\mu q}-\frac{T^{qq}}{q^2}q^{\mu},$$

$$\tilde{T}^{\mu\nu}=T^{\mu\nu}-\frac{T^{\mu q}q^{\nu}}{q^2}-\frac{T^{\nu q}q^{\mu}}{q^2}+\frac{T^{qq}q^{\mu}q^{\nu}}{q^4},$$

$$\tilde{R}^{\mu\nu q}=R^{\mu\nu q}-\frac{R^{\mu qq}q^{\nu}}{q^2}-\frac{R^{\nu qq}q^{\mu}}{q^2}+\frac{R^{qqq}q^{\mu}q^{\nu}}{q^4}$$

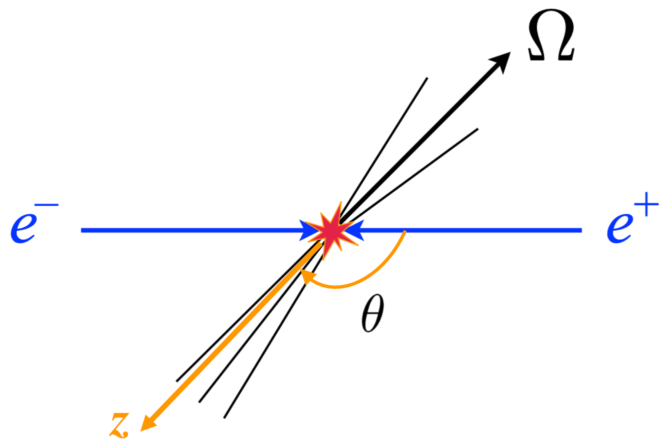
↓

will vanish when contract with q

A short-handed notation: $A^{\mu q}\equiv A^{\mu\nu}q_{\nu}$

Inclusive production of the Ω in e^+e^- collisions

The cross section in terms of structure functions



calculate $L_{\mu\nu}W^{\mu\nu}$ → The general form of the cross section

It is convenient to specify a reference frame to obtain a general angular distribution of this cross section.

The cross section is expressed in terms of **9 structure functions:**

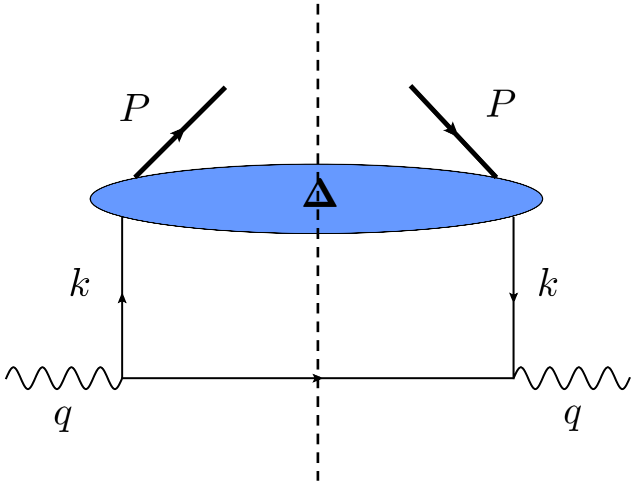
$$P^0\frac{d\sigma}{d^3\boldsymbol{P}}=\frac{\alpha^2}{Q^4}\Big\{\left(1+\cos^2\theta\right)F_U^T+\left(1-\cos^2\theta\right)F_U^L+|S_T|\sin\phi_T\sin 2\theta F_T^{\sin\phi_T}$$
$$+S_{LL}\left[\left(1+\cos^2\theta\right)F_{LL}^T+\left(1-\cos^2\theta\right)F_{LL}^L\right]$$
$$+|S_{LT}|\cos\phi_{LT}\sin 2\theta F_{LT}^{\cos\phi_{LT}}+|S_{TT}|\cos 2\phi_{TT}\sin^2\theta F_{TT}^{\cos 2\phi_{TT}}$$
$$\boxed{+|S_{LLT}|\sin\phi_{LLT}\sin 2\theta F_{LLT}^{\sin\phi_{LLT}}+|S_{LTT}|\sin 2\phi_{LTT}\sin^2\theta F_{LTT}^{\sin 2\phi_{LTT}}}\Big\}$$

Structure functions $F(z_h, Q^2)$

Superscript T, L: virtual photon polarization
Other superscripts: azimuthal modulations
Subscript: hadron polarization

Inclusive production of the Ω in e^+e^- collisions

■ The structure functions in parton model



According to the factorization,

$$W^{\mu\nu}(q;P,S,T,R) = \frac{1}{2\pi}\int d^4k\,\mathrm{Tr}[\Delta(k;P,S,T,R)H^{\mu\nu}(k,q)]$$

With collinear approximation, $k^- = \frac{P^-}{z}$

$$W^{\mu\nu}(q;P,S,T,R) = \frac{2}{z}\mathrm{Tr}\left[\Delta(z)\gamma^\mu \not{n}\gamma^\nu\right]$$

Helicity conservation ⟹ **Only chiral-even FFs contribute**

The cross section in terms of collinear FFs as

$$P^0\frac{d\sigma}{d^3\boldsymbol{P}} = N_c\sum_q e_q^2\frac{\alpha^2}{Q^4}\frac{1}{z_h}\left(1+\cos^2\theta\right)\left[D_{1,q}(z_h) + D_{1LL,q}(z_h)S_{LL}\right]$$

⟹

$$F_U^T(z_h,Q^2) = \frac{N_c}{z_h}\sum_q e_q^2 D_{1,q}(z_h,Q^2),$$

$$F_{LL}^T(z_h,Q^2) = \frac{N_c}{z_h}\sum_q e_q^2 D_{1LL,q}(z_h,Q^2)$$

$N_c = 3$ is the number of colors

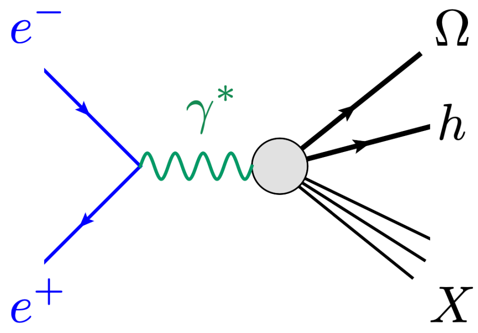
None of the newly defined rank-3 tensor polarized FFs contributes.

⇧

High order or high twist effects

Semi-inclusive production of the Ω in e^+e^- collisions

Decomposition of the hadronic tensor



$$e^-(l_1) + e^+(l_2) \to \Omega(P_1) + h(P_2) + X(P_X)$$

$$\frac{P_1^0 P_2^0 d\sigma}{d^3 \boldsymbol{P}_1 d^3 \boldsymbol{P}_2} = \frac{\alpha^2}{4Q^6} \boldsymbol{L_{\mu\nu} W^{\mu\nu}}$$

$$W^{\mu\nu}(q; P_1, S, T, R; P_2) = \frac{1}{(2\pi)^4} \sum_X (2\pi)^4 \delta^4(q - P_X - P_1 - P_2)$$

$$\times \langle 0 | J^\mu(0) | P_X; P_1, S, T, R; P_2 \rangle \langle P_X; P_1, S, T, R; P_2 | J^\nu(0) | 0 \rangle$$

Unpolarized: P_1^μ, P_2^μ, q^μ $\quad t_U^{\mu\nu} = \left\{ \tilde{g}^{\mu\nu}, \tilde{P}_1^\mu \tilde{P}_1^\nu, \tilde{P}_2^\mu \tilde{P}_2^\nu, \tilde{P}_1^{\{\mu} \tilde{P}_2^{\nu\}} \right\}$ $\quad t_U^{\mathcal{P},\mu\nu} = \left\{ \tilde{P}_1^{\{\mu} \epsilon^{\nu\} q P_1 P_2}, \tilde{P}_2^{\{\mu} \epsilon^{\nu\} q P_1 P_2} \right\}$

Vector polarized: $P_1^\mu, P_2^\mu, q^\mu, S^\mu$ $\quad t_V^{\mu\nu} = \{S \cdot q, S \cdot P_2\} t_U^{\mathcal{P},\mu\nu} \ldots$

Rank-2 tensor polarized: $P_1^\mu, P_2^\mu, q^\mu, S^\mu, T^{\mu\nu}$ $\quad t_T^{\mu\nu} = \{T^{P_2 P_2}, T^{P_2 q}, T^{qq}\} t_U^{\mu\nu} \ldots$

Rank-3 tensor polarized: $P_1^\mu, P_2^\mu, q^\mu, S^\mu, T^{\mu\nu}, R^{\mu\nu\rho}$ $\quad t_R^{\mu\nu} = \{R^{P_2 P_2 P_2}, R^{qqq}, R^{P_2 P_2 q}, R^{P_2 qq}\} t_U^{\mathcal{P},\mu\nu} \ldots$

$$W^{\mu\nu} = \sum_{i=1}^{4} V_{U,i} t_{U,i}^{\mu\nu} + \sum_{i=1}^{8} V_{V,i} t_{V,i}^{\mu\nu} + \sum_{i=1}^{16} V_{T,i} t_{T,i}^{\mu\nu} + \sum_{i=1}^{20} V_{R,i} t_{R,i}^{\mu\nu}$$

A total of 48 basis tensors.

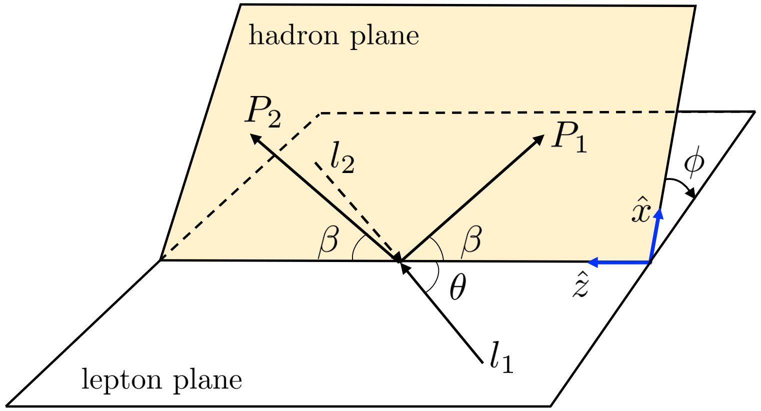
superscript $\mathcal{P}$: parity non-conserved

Semi-inclusive production of the Ω in e^+e^- collisions

Reference frames and the cross section in terms of structure functions

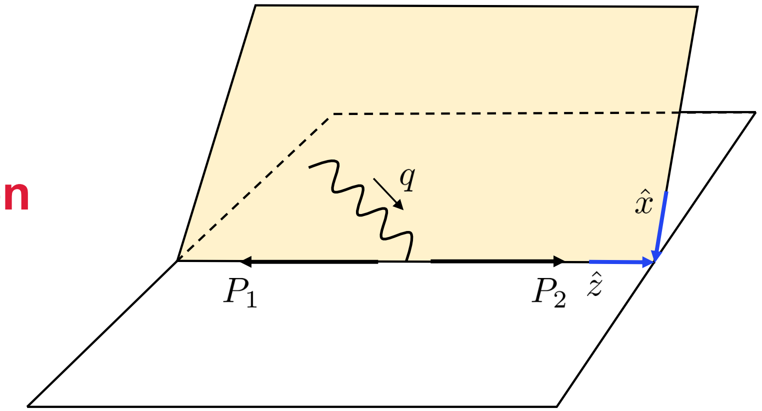
calculate $L_{\mu\nu}W^{\mu\nu}$ → The general form of the cross section

It is convenient to specify a reference frame to obtain a general angular distribution of this cross section.



Collins-Soper frame (CS frame)

Lorentz transformation



Hadronic center-of-mass frame (CM frame)

J. C. Collins and D. E. Soper, Phys. Rev. D 16, 2219 (1977).

CS frame is more convenient to describe the angular distributions of the produced hadrons.

The spin components are easier to be defined in the CM frame.

Semi-inclusive production of the Ω in e^+e^- collisions

$$\frac{P_1^0 P_2^0\, d\sigma}{d^3\boldsymbol{P}_1\, d^3\boldsymbol{P}_2} = \frac{\alpha^2}{4Q^4}\times\Big\{\Big[(1+\cos^2\theta)\,F_{U,U}^T + (1-\cos^2\theta)\,F_{U,U}^L + (\sin 2\theta\cos\phi)F_{U,U}^{\cos\phi} + (\sin^2\theta\cos 2\phi)\,F_{U,U}^{\cos 2\phi}\Big]$$

Unpolarized 4

$$+ S_L\Big[(\sin^2\theta\sin 2\phi)\,F_{L,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{L,U}^{\sin\phi}\Big]$$
$$+ |S_T|\Big[\sin\phi_T\Big((1+\cos^2\theta)\,F_{T,U}^T + (1-\cos^2\theta)\,F_{T,U}^L + (\sin 2\theta\cos\phi)F_{T,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{T,U}^{\cos 2\phi}\Big) + \cos\phi_T\Big((\sin^2\theta\sin 2\phi)\,F_{T,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{T,U}^{\sin\phi}\Big)\Big]$$

Vector polarized 8

$$+ S_{LL}\Big[(1+\cos^2\theta)\,F_{LL,U}^T + (1-\cos^2\theta)\,F_{LL,U}^L + (\sin 2\theta\cos\phi)F_{LL,U}^{\cos\phi} + (\sin^2\theta\cos 2\phi)\,F_{LL,U}^{\cos 2\phi}\Big]$$
$$+ |S_{LT}|\Big[\cos\phi_{LT}\Big((1+\cos^2\theta)\,F_{LT,U}^T + (1-\cos^2\theta)\,F_{LT,U}^L + (\sin 2\theta\cos\phi)F_{LT,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{LT,U}^{\cos 2\phi}\Big) + \sin\phi_{LT}\Big((\sin^2\theta\sin 2\phi)\,F_{LT,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{LT,U}^{\sin\phi}\Big)\Big]$$
$$+ |S_{TT}|\Big[\cos 2\phi_{TT}\Big((1+\cos^2\theta)\,F_{TT,U}^T + (1-\cos^2\theta)\,F_{TT,U}^L + (\sin 2\theta\cos\phi)F_{TT,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{TT,U}^{\cos 2\phi}\Big) + \sin 2\phi_{TT}\Big((\sin^2\theta\sin 2\phi)\,F_{TT,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{TT,U}^{\sin\phi}\Big)\Big]$$

Rank-2 tensor polarized 16

$$+ S_{LLL}\Big[(\sin^2\theta\sin 2\phi)\,F_{LLL,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{LLL,U}^{\sin\phi}\Big]$$
$$+ |S_{LLT}|\Big[\sin\phi_{LLT}\Big((1+\cos^2\theta)\,F_{LLT,U}^T + (1-\cos^2\theta)\,F_{LLT,U}^L + (\sin 2\theta\cos\phi)F_{LLT,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{LLT,U}^{\cos 2\phi}\Big) + \cos\phi_{LLT}\Big((\sin^2\theta\sin 2\phi)\,F_{LLT,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{LLT,U}^{\sin\phi}\Big)\Big]$$
$$+ |S_{LTT}|\Big[\sin 2\phi_{LTT}\Big((1+\cos^2\theta)\,F_{LTT,U}^T + (1-\cos^2\theta)\,F_{LTT,U}^L + (\sin 2\theta\cos\phi)F_{LTT,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{LTT,U}^{\cos 2\phi}\Big) + \cos 2\phi_{LTT}\Big((\sin^2\theta\sin 2\phi)\,F_{LTT,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{LTT,U}^{\sin\phi}\Big)\Big]$$
$$+ |S_{TTT}|\Big[\sin 3\phi_{TTT}\Big((1+\cos^2\theta)\,F_{TTT,U}^T + (1-\cos^2\theta)\,F_{TTT,U}^L + (\sin 2\theta\cos\phi)F_{TTT,U}^{\cos\phi}$$
$$+ (\sin^2\theta\cos 2\phi)\,F_{TTT,U}^{\cos 2\phi}\Big) + \cos 3\phi_{TTT}\Big((\sin^2\theta\sin 2\phi)\,F_{TTT,U}^{\sin 2\phi} + (\sin 2\theta\sin\phi)F_{TTT,U}^{\sin\phi}\Big)\Big]\Big\}$$

Rank-3 tensor polarized 20

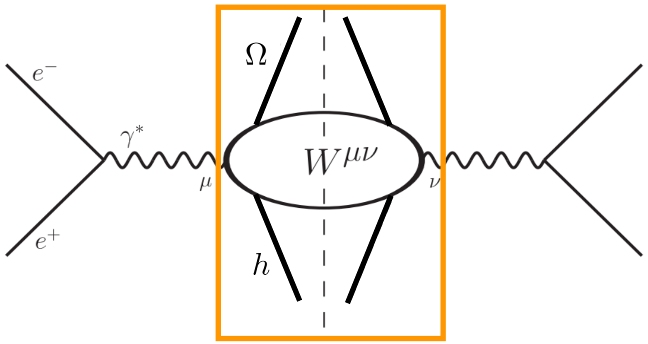
48 structure functions

$$F(z_{h1}, z_{h2}, Q^2, \boldsymbol{q}_T^2)$$

$$z_{h1} = \frac{2P_1\cdot q}{Q^2}, \quad z_{h2} = \frac{2P_2\cdot q}{Q^2}$$

Semi-inclusive production of the Ω in e^+e^- collisions

■ The structure functions in parton model



Hadronic tensor

$$W^{\mu\nu} = N_c z_1 z_2 \sum_q e_q^2 \int d^2\boldsymbol{k}_{1T} d^2\boldsymbol{k}_{2T} \delta^{(2)}(\boldsymbol{k}_{1T} + \boldsymbol{k}_{2T} - \boldsymbol{q}_T) \mathrm{Tr}[\Delta^{\Omega/q}\gamma^\mu \Delta^{h/\bar{q}}\gamma^\nu]$$

The correlation function is parametrized in terms of leading-twist TMD FFs as

$$\Delta(z_1, k_{1T}) = \frac{1}{4}\left\{\Delta^{[\gamma^-]}(z_1, k_{1T})\gamma^+ - \Delta^{[\gamma^-\gamma_5]}(z_1, k_{1T})\gamma^+\gamma_5 + \Delta^{[i\sigma^{i-}\gamma_5]}(z_1, k_{1T}) i\sigma_{-i}\gamma_5\right\}$$

$$\Delta(z_2, k_{2T}) = \frac{1}{4}\left\{D_1(z_2, k_{2T})\gamma^- - \frac{\epsilon_T^{jl} k_{2Tl}}{M_2} H_1^\perp(z_2, k_{2T}) i\sigma_{+j}\gamma_5\right\}$$

h: unpolarized hadron

For the helicity conservation of massless quarks, the chiral-odd TMD FFs must couple to chiral-odd function.

For conciseness, we introduce the transverse momentum convolution notation

$$\mathcal{C}\left[w_a(k_{1T}, k_{2T}) D(z_1, k_{1T}^2) D(z_2, k_{2T}^2)\right]$$
$$\equiv \frac{1}{4} N_c z_1 z_2 \sum_q e_q^2 \int d^2\boldsymbol{k}_{1T} d^2\boldsymbol{k}_{2T} \delta^{(2)}(\boldsymbol{k}_{1T} + \boldsymbol{k}_{2T} - \boldsymbol{q}_T) w_a(k_{1T}, k_{2T}) D_q(z_1, k_{1T}^2) D_{\bar{q}}(z_2, k_{2T}^2)$$

dimensionless scalar functions

TMD FF for the first hadron Ω

TMD FF for the second hadron h

Semi-inclusive production of the Ω in e^+e^- collisions

At leading twist, 24 structure functions have nontrivial expressions.

Unpolarized state: 2

$$F^{T}_{U,U} = \mathcal{C}\left[D_1(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\cos 2\phi}_{U,U} = \mathcal{C}\left[w_3 H^{\perp}_1(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

Vector polarized states: 4

$$F^{\sin 2\phi}_{L,U} = -\mathcal{C}\left[w_3 H^{\perp}_{1L}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{T,U} = \mathcal{C}\left[w_1 D^{\perp}_{1T}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+\phi_T)}_{T,U} = \mathcal{C}\left[w_4 H^{\perp}_{1T}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-\phi_T)}_{T,U} = -\mathcal{C}\left[w_2 H_{1T}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

Rank-2 tensor polarized states: 8

$$F^{T}_{LL,U} = \mathcal{C}\left[D_{1LL}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\cos 2\phi}_{LL,U} = -\mathcal{C}\left[w_3 H^{\perp}_{1LL}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{LT,U} = \mathcal{C}\left[w_1 D^{\perp}_{1LT}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+\phi_{LT})}_{LT,U} = -\mathcal{C}\left[w_2 H_{1LT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-\phi_{LT})}_{LT,U} = -\mathcal{C}\left[w_4 H^{\perp}_{1LT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{TT,U} = \mathcal{C}\left[w_5 D^{\perp}_{1TT}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+2\phi_{TT})}_{TT,U} = \mathcal{C}\left[w_7 H^{\perp}_{1TT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-2\phi_{TT})}_{TT,U} = \mathcal{C}\left[w_6 H^{\perp\perp}_{1TT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

Rank-3 tensor polarized states: 10

$$F^{\sin 2\phi}_{LLL,U} = -\mathcal{C}\left[w_3 H^{\perp}_{1LLL}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{LLT,U} = \mathcal{C}\left[w_1 D^{\perp}_{1LLT}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+\phi_{LLT})}_{LLT,U} = \mathcal{C}\left[w_4 H^{\perp}_{1LLT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-\phi_{LLT})}_{LLT,U} = -\mathcal{C}\left[w_2 H_{1LLT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{LTT,U} = -\mathcal{C}\left[w_5 D^{\perp}_{1LTT}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+2\phi_{LTT})}_{LTT,U} = -\mathcal{C}\left[w_6 H^{\perp\perp}_{1LTT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-2\phi_{LTT})}_{LTT,U} = \mathcal{C}\left[w_7 H^{\perp}_{1LTT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{T}_{TTT,U} = -\mathcal{C}\left[w_8 D^{\perp}_{1TTT}(z_1,k^2_{1T})D_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi+3\phi_{TTT})}_{TTT,U} = -\mathcal{C}\left[w_9 H^{\perp\perp}_{1TTT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

$$F^{\sin(2\phi-3\phi_{TTT})}_{TTT,U} = -\mathcal{C}\left[w_{10} H^{\perp}_{1TTT}(z_1,k^2_{1T})H^{\perp}_1(z_2,k^2_{2T})\right]$$

Study the TMD FFs for spin-3/2 hadrons

The other 24 structure functions only arise at high twist or high order.

Summary

- We describe the spin state of a spin-3/2 hadron with the spin density matrix ρ.

  16 independent components: 1 for unpolarized state, 3 for vector polarized states,

  5 for rank-2 tensor polarized states, 7 for rank-3 tensor polarized states.

- We obtain 32 leading-twist TMD FFs via the parametrization of the quark-quark correlation function.

  32 leading-twist TMD FFs, 14 for rank-3 tensor polarized states.

- For $e^+e^- \to \Omega X$, the cross section is expressed in terms of 9 structure functions.

  In parton model: only 2 of the structure functions are nonzero at leading twist.

- For $e^+e^- \to \Omega h X$, the cross section is expressed in terms of 48 structure functions.

  In parton model: half of the structure functions are nonzero at leading twist.

  10 are contributions from rank-3 tensor polarized hadron states.

Thank you!

Back up

The description of spin-3/2 particles

7 rank-3 tensor polarization basis:

$$\Sigma^{ijk} = \frac{1}{6}\Sigma^{\{i}\Sigma^{j}\Sigma^{k\}} - \frac{41}{60}\left(\delta^{ij}\Sigma^{k} + \delta^{jk}\Sigma^{i} + \delta^{ki}\Sigma^{j}\right)$$
$$= \frac{1}{3}\left(\Sigma^{ij}\Sigma^{k} + \Sigma^{jk}\Sigma^{i} + \Sigma^{ki}\Sigma^{j}\right) - \frac{4}{15}\left(\delta^{ij}\Sigma^{k} + \delta^{jk}\Sigma^{i} + \delta^{ki}\Sigma^{j}\right)$$

$$\boxed{\begin{aligned}\Sigma^{xxz} + \Sigma^{yyz} + \Sigma^{zzz} &= 0,\\ \Sigma^{xxy} + \Sigma^{yyy} + \Sigma^{zzy} &= 0,\\ \Sigma^{xxx} + \Sigma^{yyx} + \Sigma^{zzx} &= 0.\end{aligned}}$$

$$\mathrm{Tr}[\Sigma^{i}\Sigma^{jk}] = \mathrm{Tr}[\Sigma^{i}\Sigma^{jkl}] = \mathrm{Tr}[\Sigma^{ij}\Sigma^{klm}] = 0$$ orthogonal relation

$$\boldsymbol{\rho} = \tfrac{1}{4}\left(\mathbf{1} + \tfrac{4}{5}S^{i}\boldsymbol{\Sigma}^{i} + \tfrac{2}{3}T^{ij}\boldsymbol{\Sigma}^{ij} + \tfrac{8}{9}R^{ijk}\boldsymbol{\Sigma}^{ijk}\right)$$

In the rest frame

$$S^i, T^{ij}, R^{ijk}$$

$\Longrightarrow$

Lorentz covariant form

$$S^\mu, T^{\mu\nu}, R^{\mu\nu\rho}$$

$$\boxed{P_\mu S^\mu = 0, \quad P_\mu T^{\mu\nu} = 0, \quad P_\mu R^{\mu\nu\rho} = 0}$$

Light-cone coordinate: $v^\mu = (v^+, v^-, \boldsymbol{v}_\perp)$ $\quad v^\pm = (v^0 \pm v^3)/\sqrt{2}$

Two null vectors: $n^\mu = (0, 1, \boldsymbol{0}_\perp)$ $\quad \bar{n}^\mu = (1, 0, \boldsymbol{0}_\perp)$ $\quad P^\mu = \dfrac{M^2}{2P^-}\bar{n}^\mu + P^- n^\mu$

$$S^\mu = S_L\left(\frac{M}{2P\cdot\bar{n}}\bar{n}^\mu - \frac{P\cdot\bar{n}}{M}n^\mu\right) + \boxed{S_T^\mu},$$

$$\begin{aligned}
T^{\mu\nu} = \frac{1}{2}\Bigg\{ & S_{LL}\left[\frac{1}{2}\left(\frac{M}{P\cdot\bar{n}}\right)^2 \bar{n}^\mu\bar{n}^\nu + 2\left(\frac{P\cdot\bar{n}}{M}\right)^2 n^\mu n^\nu - \bar{n}^{\{\mu}n^{\nu\}} + g_T^{\mu\nu}\right] \\
& + \frac{1}{2}\left(\frac{M}{P\cdot\bar{n}}\right)\bar{n}^{\{\mu}\boxed{S_{LT}^{\nu\}}} - \left(\frac{P\cdot\bar{n}}{M}\right)n^{\{\mu}\boxed{S_{LT}^{\nu\}}} + \boxed{S_{TT}^{\mu\nu}}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
R^{\mu\nu\rho} = \frac{1}{4}\Bigg\{ & S_{LLL}\left[\frac{1}{2}\left(\frac{M}{P\cdot\bar{n}}\right)^3 \bar{n}^\mu\bar{n}^\nu\bar{n}^\rho - \frac{1}{2}\left(\frac{M}{P\cdot\bar{n}}\right)\left(\bar{n}^{\{\mu}\bar{n}^\nu n^{\rho\}} - \bar{n}^{\{\mu}g_T^{\nu\rho\}}\right)\right. \\
& \left. + \left(\frac{P\cdot\bar{n}}{M}\right)\left(\bar{n}^{\{\mu}n^\nu n^{\rho\}} - n^{\{\mu}g_T^{\nu\rho\}}\right) - 4\left(\frac{P\cdot\bar{n}}{M}\right)^3 n^\mu n^\nu n^\rho\right] \\
& + \frac{1}{2}\left(\frac{M}{P\cdot\bar{n}}\right)^2 \bar{n}^{\{\mu}\bar{n}^\nu\boxed{S_{LLT}^{\rho\}}} + 2\left(\frac{P\cdot\bar{n}}{M}\right)^2 n^{\{\mu}n^\nu\boxed{S_{LLT}^{\rho\}}} - 2\bar{n}^{\{\mu}n^\nu\boxed{S_{LLT}^{\rho\}}} + \frac{1}{2}\boxed{S_{LLT}^{\{\mu}}g_T^{\nu\rho\}} \\
& + \frac{1}{4}\left(\frac{M}{P\cdot\bar{n}}\right)\bar{n}^{\{\mu}\boxed{S_{LTT}^{\nu\rho\}}} - \frac{1}{2}\left(\frac{P\cdot\bar{n}}{M}\right)n^{\{\mu}\boxed{S_{LTT}^{\nu\rho\}}} + \boxed{S_{TTT}^{\mu\nu\rho}}\Bigg\},
\end{aligned}$$

The transverse components of

$$S_T^\mu, S_{LT}^\mu, S_{TT}^{\mu\nu}, S_{LLT}^\mu, S_{LTT}^{\mu\nu}, S_{TTT}^{\mu\nu\rho}$$

$$S_T^i = (S_T^x, S_T^y), \quad S_{LT}^i = (S_{LT}^x, S_{LT}^y), \quad S_{LLT}^i = (S_{LLT}^x, S_{LLT}^y),$$

$$S_{TT}^{ij} = \begin{pmatrix} S_{TT}^{xx} & S_{TT}^{xy} \\ S_{TT}^{xy} & -S_{TT}^{xx} \end{pmatrix}, \quad S_{LTT}^{ij} = \begin{pmatrix} S_{LTT}^{xx} & S_{LTT}^{xy} \\ S_{LTT}^{xy} & -S_{LTT}^{xx} \end{pmatrix},$$

$$S_{TTT}^{ijk} = \left[\begin{pmatrix} S_{TTT}^{xxx} & S_{TTT}^{yxx} \\ S_{TTT}^{yxx} & -S_{TTT}^{xxx} \end{pmatrix}, \begin{pmatrix} S_{TTT}^{yxx} & -S_{TTT}^{xxx} \\ -S_{TTT}^{xxx} & -S_{TTT}^{yxx} \end{pmatrix}\right],$$

TMD fragmentation functions

How to pick out leading-twist terms?

The Sudakov decomposition of the quark momentum:

$$k^\mu = \frac{z\left(k^2 + \boldsymbol{k}_T^2\right)}{2P^-}\bar{n}^\mu + \frac{P^-}{z}n^\mu + k_T^\mu$$

The k_T -unintegrated quark-quark correlation function:

$$\Delta\left(z, k_T\right) = \frac{1}{4z}\int \mathrm{d}k^+ \Delta\left(k, P, S, T, R\right)\Big|_{k^- = \frac{P^-}{z}}$$

P^- as a large momentum component and the leading-twist TMD FFs can be projected out from the correlator by the Dirac matrices.

$\boldsymbol{\gamma^-, \gamma^-\gamma_5, i\sigma^{i-}\gamma_5}$

$$\Delta = \Delta_U + \Delta_L + \Delta_T + \Delta_{LL} + \Delta_{LT} + \Delta_{TT} + \Delta_{LLL} + \Delta_{LLT} + \Delta_{LTT} + \Delta_{TTT}$$

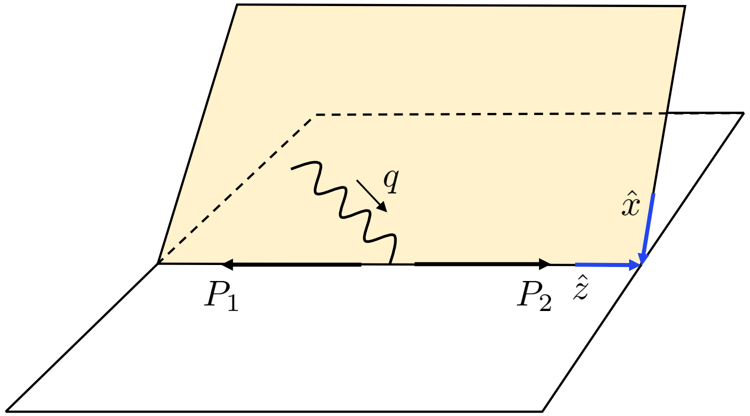
$$\Delta^{[\Gamma]}\left(z, k_T\right) = \mathrm{Tr}\left[\Delta\left(z, k_T\right)\Gamma\right]$$

$$\longrightarrow$$

$$\boxed{\begin{aligned}
&\Delta_{LLL}^{[\gamma^-]}\left(z, k_T\right) = 0,\\
&\Delta_{LLT}^{[\gamma^-]}\left(z, k_T\right) = \left(\epsilon_T^{\mu\nu} S_{LLT\nu}\frac{k_{T\mu}}{M}D_{1LLT}^{\perp}\right),\\
&\Delta_{LTT}^{[\gamma^-]}\left(z, k_T\right) = \left(\epsilon_{T\nu}^{\mu} S_{LTT}^{\nu\rho}\frac{k_{T\mu\rho}}{M^2}D_{1LTT}^{\perp}\right),\\
&\Delta_{TTT}^{[\gamma^-]}\left(z, k_T\right) = \left(\epsilon_{T\nu}^{\mu} S_{TTT}^{\nu\rho\sigma}\frac{k_{T\mu\rho\sigma}}{M^3}D_{1TTT}^{\perp}\right),\\
&\quad\vdots
\end{aligned}}$$

Semi-inclusive production of the Ω in e^+e^- collisions

The basis vectors in CM frame:



$$\hat{t}^\mu = \frac{P_1^\mu + P_2^\mu}{\sqrt{M_1^2 + M_2^2 + 2P_1 \cdot P_2}}$$

$$\hat{z}^\mu = \frac{P_2^\mu \left(M_1^2 + P_1 \cdot P_2\right) - P_1^\mu \left(M_2^2 + P_1 \cdot P_2\right)}{\sqrt{\left(P_1 \cdot P_2\right)^2 - M_1^2 M_2^2}\sqrt{M_1^2 + M_2^2 + 2P_1 \cdot P_2}}$$

$$\hat{x}^\mu = \frac{g_T^{\mu\nu} q_\nu}{\sqrt{-g_T^{\mu\nu} q_\mu q_\nu}}$$

$$\hat{y}^\mu = \epsilon_T^{\mu\nu} \hat{x}_\nu$$

Transverse metric:

$$g_T^{\mu\nu} = g^{\mu\nu} - \frac{\left(P_1 \cdot P_2\right)\left(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu\right)}{\left(P_1 \cdot P_2\right)^2 - M_1^2 M_2^2} + \frac{M_1^2 P_2^\mu P_2^\nu + M_2^2 P_1^\mu P_1^\nu}{\left(P_1 \cdot P_2\right)^2 - M_1^2 M_2^2}$$

Transverse antisymmetric tensor:

$$\epsilon_T^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \frac{P_{2\rho} P_{1\sigma}}{\sqrt{\left(P_1 \cdot P_2\right)^2 - M_1^2 M_2^2}}$$

$$S_L = \langle \Sigma^z \rangle, \quad S_T^x = \langle \Sigma^x \rangle, \quad S_T^y = \langle \Sigma^y \rangle.$$

$$S_{LL} = \langle \Sigma^{zz} \rangle, \quad S_{LT}^x = 2\langle \Sigma^{xz} \rangle, \quad S_{LT}^y = 2\langle \Sigma^{yz} \rangle,$$
$$S_{TT}^{xy} = 2\langle \Sigma^{xy} \rangle, \quad S_{TT}^{xx} = \langle \Sigma^{xx} - \Sigma^{yy} \rangle.$$

$$S_{LLL} = \langle \Sigma^{zzz} \rangle, \quad S_{LLT}^x = \langle \Sigma^{xzz} \rangle, \quad S_{LLT}^y = \langle \Sigma^{yzz} \rangle, \quad S_{LTT}^{xy} = 4\langle \Sigma^{xyz} \rangle,$$
$$S_{LTT}^{xx} = 2\langle \Sigma^{xxz} - \Sigma^{yyz} \rangle, \quad S_{TTT}^{xxx} = \langle \Sigma^{xxx} - 3\Sigma^{xyy} \rangle, \quad S_{TTT}^{yxx} = \langle 3\Sigma^{yxx} - \Sigma^{yyy} \rangle.$$

$$
\begin{aligned}
k_T^{ij} &= k_T^i k_T^j - \frac{1}{2} k_T^2 g_T^{ij}, \\
k_T^{ijk} &= k_T^i k_T^j k_T^k - \frac{1}{4} k_T^2 \left(g_T^{ij} k_T^k + g_T^{ik} k_T^j + g_T^{jk} k_T^i \right),
\end{aligned}
$$

$$
\begin{aligned}
k_T^{ijkl} &= k_T^i k_T^j k_T^k k_T^l \\
&\quad - \frac{1}{6} k_T^2 \left(g_T^{ij} k_T^{kl} + g_T^{ik} k_T^{jl} + g_T^{il} k_T^{jk} + g_T^{jk} k_T^{il} + g_T^{jl} k_T^{ik} + g_T^{kl} k_T^{ij} \right) \\
&\quad - \frac{1}{8} \left(k_T^2 \right)^2 \left(g_T^{ij} g_T^{kl} + g_T^{ik} g_T^{jl} + g_T^{il} g_T^{jk} \right),
\end{aligned}
$$

$$
g_{Tij} k_T^{ij} = g_{Tij} k_T^{ijk} = g_{Tij} k_T^{ijkl} = 0.
$$

$$
\begin{aligned}
\Delta_U\left(z,k_T\right)=&\frac{1}{4}\left\{D_1\left(z,k_T^2\right)\not{n}+\left(H_1^{\perp}\left(z,k_T^2\right)\sigma_{\mu\nu}\frac{k_T^{\mu}}{M}n^{\nu}\right)\right\},\\
\Delta_L\left(z,k_T\right)=&\frac{1}{4}\left\{G_{1L}\left(z,k_T^2\right)S_L\gamma_5\not{n}+H_{1L}^{\perp}\left(z,k_T^2\right)S_L i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu}}{M}\right\},\\
\Delta_T\left(z,k_T\right)=&\frac{1}{4}\left\{G_{1T}^{\perp}\left(z,k_T^2\right)\frac{\boldsymbol{S}_T\cdot\boldsymbol{k}_T}{M}\gamma_5\not{n}+H_{1T}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}S_T^{\nu}\right.\\
&\left.-H_{1T}^{\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu\rho}}{M^2}S_{T\rho}+\left(D_{1T}^{\perp}\left(z,k_T^2\right)\epsilon_T^{\mu\nu}\frac{k_{T\mu}}{M}S_{T\nu}\not{n}\right)\right\},\\
\Delta_{LL}\left(z,k_T\right)=&\frac{1}{4}\left\{D_{1LL}\left(z,k_T^2\right)S_{LL}\not{n}-\left(H_{1LL}^{\perp}\left(z,k_T^2\right)S_{LL}\sigma_{\mu\nu}\frac{k_T^{\mu}}{M}n^{\nu}\right)\right\},\\
\Delta_{LT}\left(z,k_T\right)=&\frac{1}{4}\left\{D_{1LT}^{\perp}\left(z,k_T^2\right)\frac{\boldsymbol{S}_{LT}\cdot\boldsymbol{k}_T}{M}\not{n}-\left(G_{1LT}^{\perp}\left(z,k_T^2\right)\epsilon_T^{\mu\nu}\frac{k_{T\mu}}{M}S_{LT\nu}\gamma_5\not{n}\right)\right.\\
&\left.-\left(H_{1LT}\left(z,k_T^2\right)\sigma_{\mu\nu}S_{LT}^{\mu}n^{\nu}\right)+\left(H_{1LT}^{\perp}\left(z,k_T^2\right)\sigma_{\mu\nu}\frac{k_T^{\mu\rho}}{M^2}S_{LT\rho}n^{\nu}\right)\right\},\\
\Delta_{TT}\left(z,k_T\right)=&\frac{1}{4}\left\{D_{1TT}^{\perp}\left(z,k_T^2\right)S_{TT\mu\nu}\frac{k_T^{\mu\nu}}{M^2}\not{n}+\left(G_{1TT}^{\perp}\left(z,k_T^2\right)\epsilon_{T\nu}^{\mu}\frac{k_{T\mu\rho}}{M_h^2}S_{TT}^{\nu\rho}\gamma_5\not{n}\right)\right.\\
&\left.-\left(H_{1TT}^{\perp}\left(z,k_T^2\right)\sigma_{\mu\nu}S_{TT}^{\mu\rho}\frac{k_{T\rho}}{M_h}n^{\nu}\right)-\left(H_{1TT}^{\perp\perp}\left(z,k_T^2\right)\sigma_{\mu\nu}\frac{k_T^{\mu\rho\sigma}}{M^3}S_{TT\rho\sigma}n^{\nu}\right)\right\},\\
\Delta_{LLL}(z,k_T)=&\frac{1}{4}\left\{G_{1LLL}\left(z,k_T^2\right)S_{LLL}\gamma_5\not{n}+H_{1LLL}^{\perp}\left(z,k_T^2\right)S_{LLL}i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu}}{M}\right\},\\
\Delta_{LLT}(z,k_T)=&\frac{1}{4}\left\{G_{1LLT}^{\perp}\left(z,k_T^2\right)\frac{\boldsymbol{S}_{LLT}\cdot\boldsymbol{k}_T}{M}\gamma_5\not{n}-H_{1LLT}^{\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu\rho}}{M^2}S_{LLT\rho}\right.\\
&\left.+H_{1LLT}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}S_{LLT}^{\nu}+\left(D_{1LLT}^{\perp}\left(z,k_T^2\right)\epsilon_T^{\mu\nu}\frac{k_{T\mu}}{M}S_{LLT\nu}\not{n}\right)\right\},\\
\Delta_{LTT}(z,k_T)=&\frac{1}{4}\left\{G_{1LTT}^{\perp}\left(z,k_T^2\right)S_{LTT\mu\nu}\frac{k_T^{\mu\nu}}{M^2}\gamma_5\not{n}+H_{1LTT}^{\perp\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu\rho\sigma}}{M^3}S_{LTT\rho\sigma}\right.\\
&\left.+H_{1LTT}^{\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}S_{LTT}^{\nu\rho}\frac{k_{T\rho}}{M}+\left(D_{1LTT}^{\perp}\left(z,k_T^2\right)\epsilon_{T\nu}^{\mu}\frac{k_{T\mu\rho}}{M^2}S_{LTT}^{\nu\rho}\not{n}\right)\right\},\\
\Delta_{TTT}(z,k_T)=&\frac{1}{4}\left\{G_{1TTT}^{\perp}\left(z,k_T^2\right)S_{TTT\mu\nu\rho}\frac{k_T^{\mu\nu\rho}}{M^3}\gamma_5\not{n}+H_{1TTT}^{\perp\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}\frac{k_T^{\nu\rho\sigma\tau}}{M^4}S_{TTT\rho\sigma\tau}\right.\\
&\left.+H_{1TTT}^{\perp}\left(z,k_T^2\right)i\sigma_{\mu\nu}\gamma_5 n^{\mu}S_{TTT}^{\nu\rho\sigma}\frac{k_{T\rho\sigma}}{M^2}+\left(D_{1TTT}^{\perp}\left(z,k_T^2\right)\epsilon_{T\nu}^{\mu}\frac{k_{T\mu\rho\sigma}}{M^3}S_{TTT}^{\nu\rho\sigma}\not{n}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
S_L &= S^\mu \hat{L}_\mu, \\
S_T^x &= |S_T| \cos\phi_T = -S^\mu \hat{x}_\mu, \\
S_T^y &= |S_T| \sin\phi_T = -S^\mu \hat{y}_\mu, \\
S_{LL} &= T^{\mu\nu} \hat{L}_\mu \hat{L}_\nu, \\
S_{LT}^x &= |S_{LT}| \cos\phi_{LT} = -2T^{\mu\nu} \hat{L}_\mu \hat{x}_\nu, \\
S_{LT}^y &= |S_{LT}| \sin\phi_{LT} = -2T^{\mu\nu} \hat{L}_\mu \hat{y}_\nu, \\
S_{TT}^{xx} &= |S_{TT}| \cos 2\phi_{TT} = 2T^{\mu\nu} \hat{x}_\mu \hat{x}_\nu + T^{\mu\nu} \hat{L}_\mu \hat{L}_\nu, \\
S_{TT}^{xy} &= |S_{TT}| \sin 2\phi_{TT} = 2T^{\mu\nu} \hat{x}_\mu \hat{y}_\nu, \\
S_{LLL} &= R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{L}_\rho, \\
S_{LLT}^x &= |S_{LLT}| \cos\phi_{LLT} = -R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{x}_\rho, \\
S_{LLT}^y &= |S_{LLT}| \sin\phi_{LLT} = -R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{y}_\rho, \\
S_{LTT}^{xx} &= |S_{LTT}| \cos 2\phi_{LTT} = 4R^{\mu\nu\rho} \hat{L}_\mu \hat{x}_\nu \hat{x}_\rho + 2R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{L}_\rho, \\
S_{LTT}^{xy} &= |S_{LTT}| \sin 2\phi_{LTT} = 4R^{\mu\nu\rho} \hat{L}_\mu \hat{x}_\nu \hat{y}_\rho, \\
S_{TTT}^{xxx} &= |S_{TTT}| \cos 3\phi_{TTT} = -4R^{\mu\nu\rho} \hat{x}_\mu \hat{x}_\nu \hat{x}_\rho - 3R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{x}_\rho, \\
S_{TTT}^{yxx} &= |S_{TTT}| \sin 3\phi_{TTT} = -4R^{\mu\nu\rho} \hat{y}_\mu \hat{x}_\nu \hat{x}_\rho - R^{\mu\nu\rho} \hat{L}_\mu \hat{L}_\nu \hat{y}_\rho.
\end{aligned}
$$

$$
\cos\theta = \frac{2y-1}{\sqrt{1-\gamma_h^2}}, \quad \sin\theta = \sqrt{\frac{4y-4y^2-\gamma_h^2}{1-\gamma_h^2}},
$$

Basis tensors:

$$t_V^{\mu\nu} = \left\{\epsilon^{SqP_1P_2}\right\} t_U^{\mu\nu}, \{S\cdot q, S\cdot P_2\}\, t_U^{\mathcal{P},\mu\nu},$$

$$t_T^{\mu\nu} = \{T^{P_2P_2}, T^{P_2q}, T^{qq}\}\, t_U^{\mu\nu}, \left\{\epsilon^{T^{P_2}P_1P_2q}, \epsilon^{T^qP_1P_2q}\right\} t_U^{\mathcal{P},\mu\nu},$$

$$t_R^{\mu\nu} = \left\{\epsilon^{R^{P_2P_2}P_1P_2q}, \epsilon^{R^{P_2q}P_1P_2q}, \epsilon^{R^{qq}P_1P_2q}\right\} t_U^{\mu\nu}, \{R^{P_2P_2P_2}, R^{qqq}, R^{P_2P_2q}, R^{P_2qq}\}\, t_U^{\mathcal{P},\mu\nu}.$$

In CS frame:

$$
\begin{aligned}
l_1^\mu &= \frac{Q}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\
l_2^\mu &= \frac{Q}{2}(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta), \\
P_1^\mu &= \frac{z_{h1}Q}{2}\left(1, \sqrt{1-\gamma_{h1}^2}\sin\beta, 0, -\sqrt{1-\gamma_{h1}^2}\cos\beta\right), \\
P_2^\mu &= \frac{z_{h2}Q}{2}\left(1, \sqrt{1-\gamma_{h2}^2}\sin\beta, 0, \sqrt{1-\gamma_{h2}^2}\cos\beta\right), \\
q^\mu &= (Q, 0, 0, 0),
\end{aligned}
$$

$$
\begin{aligned}
\cos 2\beta &= \frac{2\xi - z_{h1}z_{h2}}{z_{h1}z_{h2}\sqrt{(1-\gamma_{h1}^2)}\sqrt{(1-\gamma_{h2}^2)}} \approx \frac{2\xi - z_{h1}z_{h2}}{z_{h1}z_{h2}}, \\
\cos\theta &= \frac{1-2y_2}{2\sqrt{1-\gamma_{h2}^2}\cos\beta} - \frac{1-2y_1}{2\sqrt{1-\gamma_{h1}^2}\cos\beta} \approx \frac{y_1 - y_2}{\cos\beta}, \\
\cos\phi &= \frac{1-2y_2}{2\sqrt{1-\gamma_{h2}^2}\sin\beta\sin\theta} + \frac{1-2y_1}{2\sqrt{1-\gamma_{h1}^2}\sin\beta\sin\theta} \approx \frac{1-y_1-y_2}{\sin\beta\sin\theta}
\end{aligned}
$$

The dimensionless scalar functions:

$$w_1 = -\frac{\hat{q}_T \cdot k_{1T}}{M_1}, \quad w_2 = -\frac{\hat{q}_T \cdot k_{2T}}{M_2}, \quad w_3 = \frac{2(\hat{q}_T \cdot k_{1T})(\hat{q}_T \cdot k_{2T}) + k_{1T} \cdot k_{2T}}{M_1 M_2},$$

$$w_4 = \frac{k_{1T}^{ij}\hat{q}_{Ti}k_{2Tj} + 2k_{1T}^{ij}\hat{q}_{Ti}\hat{q}_{Tj}(\hat{q}_T \cdot k_{2T})}{M_1^2 M_2}, \quad w_5 = \frac{2k_{1T}^{ij}\hat{q}_{Ti}\hat{q}_{Tj}}{M_1^2},$$

$$w_6 = \frac{2\left[k_{1T}^{ijl}\hat{q}_{Ti}\hat{q}_{Tj}k_{2Tl} + 2k_{1T}^{ijl}\hat{q}_{Ti}\hat{q}_{Tj}\hat{q}_{Tl}(k_{2T} \cdot \hat{q}_T)\right]}{M_1^3 M_2}, \quad w_7 = -\frac{k_{1T} \cdot k_{2T}}{M_1 M_2}, \quad w_8 = \frac{4k_{1T}^{ijl}\hat{q}_{Ti}\hat{q}_{Tj}\hat{q}_{Tl}}{M_1^3}$$

$$w_9 = \frac{4\left[k_{1T}^{ijlm}\hat{q}_{Ti}\hat{q}_{Tj}\hat{q}_{Tl}k_{2Tm} + 2k_{1T}^{ijlm}\hat{q}_{Ti}\hat{q}_{Tj}\hat{q}_{Tl}\hat{q}_{Tm}(k_{2T} \cdot \hat{q}_T)\right]}{M_1^4 M_2}, \quad w_{10} = \frac{2k_{1T}^{ij}\hat{q}_{Ti}k_{2Tj}}{M_1^2 M_2},$$

$\hat{q}_T^\mu \equiv g_T^{\mu\nu} q_\nu / \sqrt{\boldsymbol{q}_T^2}$ is the direction of the virtual photon transverse momentum in the CM frame.

$$\rho = \frac{1}{4}\begin{pmatrix} \rho_{\frac{3}{2}\frac{3}{2}} & \rho_{\frac{3}{2}\frac{1}{2}} & \rho_{\frac{3}{2}-\frac{1}{2}} & \rho_{\frac{3}{2}-\frac{3}{2}} \\ \rho_{\frac{1}{2}\frac{3}{2}} & \rho_{\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{3}{2}} \\ \rho_{-\frac{1}{2}\frac{3}{2}} & \rho_{-\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{3}{2}} \\ \rho_{-\frac{3}{2}\frac{3}{2}} & \rho_{-\frac{3}{2}\frac{1}{2}} & \rho_{-\frac{3}{2}-\frac{1}{2}} & \rho_{-\frac{3}{2}-\frac{3}{2}} \end{pmatrix},$$

$$\begin{aligned}
\rho_{\frac{3}{2}\frac{3}{2}} &= 1 + \frac{6}{5}S_L + S_{LL} + \frac{2}{3}S_{LLL}, \\
\rho_{\frac{1}{2}\frac{1}{2}} &= 1 + \frac{2}{5}S_L - S_{LL} - 2S_{LLL}, \\
\rho_{-\frac{1}{2}-\frac{1}{2}} &= 1 - \frac{2}{5}S_L - S_{LL} + 2S_{LLL}, \\
\rho_{-\frac{3}{2}-\frac{3}{2}} &= 1 - \frac{6}{5}S_L + S_{LL} - \frac{2}{3}S_{LLL}, \\
\rho_{\frac{3}{2}\frac{1}{2}} &= \frac{2\sqrt{3}}{5}(S_T^x - iS_T^y) + \frac{\sqrt{3}}{3}(S_{LT}^x - iS_{LT}^y) + \frac{2\sqrt{3}}{3}(S_{LLT}^x - iS_{LLT}^y), \\
\rho_{\frac{1}{2}\frac{3}{2}} &= \frac{2\sqrt{3}}{5}(S_T^x + iS_T^y) + \frac{\sqrt{3}}{3}(S_{LT}^x + iS_{LT}^y) + \frac{2\sqrt{3}}{3}(S_{LLT}^x + iS_{LLT}^y), \\
\rho_{\frac{1}{2}-\frac{1}{2}} &= \frac{4}{5}(S_T^x - iS_T^y) - 2(S_{LLT}^x - iS_{LLT}^y), \\
\rho_{-\frac{1}{2}\frac{1}{2}} &= \frac{4}{5}(S_T^x + iS_T^y) - 2(S_{LLT}^x + iS_{LLT}^y), \\
\rho_{-\frac{1}{2}-\frac{3}{2}} &= \frac{2\sqrt{3}}{5}(S_T^x - iS_T^y) - \frac{\sqrt{3}}{3}(S_{LT}^x - iS_{LT}^y) + \frac{2\sqrt{3}}{3}(S_{LLT}^x - iS_{LLT}^y), \\
\rho_{-\frac{3}{2}-\frac{1}{2}} &= \frac{2\sqrt{3}}{5}(S_T^x + iS_T^y) - \frac{\sqrt{3}}{3}(S_{LT}^x + iS_{LT}^y) + \frac{2\sqrt{3}}{3}(S_{LLT}^x + iS_{LLT}^y), \\
\rho_{\frac{3}{2}-\frac{1}{2}} &= \frac{\sqrt{3}}{3}(S_{TT}^{xx} - iS_{TT}^{xy}) + \frac{\sqrt{3}}{3}(S_{LTT}^{xx} - iS_{LTT}^{xy}), \\
\rho_{-\frac{1}{2}\frac{3}{2}} &= \frac{\sqrt{3}}{3}(S_{TT}^{xx} + iS_{TT}^{xy}) + \frac{\sqrt{3}}{3}(S_{LTT}^{xx} + iS_{LTT}^{xy}), \\
\rho_{\frac{1}{2}-\frac{3}{2}} &= \frac{\sqrt{3}}{3}(S_{TT}^{xx} - iS_{TT}^{xy}) - \frac{\sqrt{3}}{3}(S_{LTT}^{xx} - iS_{LTT}^{xy}), \\
\rho_{\frac{1}{2}-\frac{3}{2}} &= \frac{\sqrt{3}}{3}(S_{TT}^{xx} + iS_{TT}^{xy}) - \frac{\sqrt{3}}{3}(S_{LTT}^{xx} + iS_{LTT}^{xy}), \\
\rho_{\frac{3}{2}-\frac{3}{2}} &= \frac{2}{3}(S_{TTT}^{xxx} + iS_{TTT}^{yxx}), \\
\rho_{-\frac{3}{2}\frac{3}{2}} &= \frac{2}{3}(S_{TTT}^{xxx} - iS_{TTT}^{yxx}).
\end{aligned}$$

$$\Sigma^i \hat{n}_i = \Sigma_x \cos\theta \cos\phi + \Sigma_y \cos\theta \sin\phi + \Sigma_z \sin\theta$$

$$P\left(m_{(\theta,\phi)}\right) = \mathrm{Tr}\left[\rho | m_{(\theta,\phi)}\rangle\langle m_{(\theta,\phi)}|\right]$$

$$\begin{aligned}
S_L =& \frac{3}{2}\left[P\left(\frac{3}{2}_{(0,0)}\right) - P\left(-\frac{3}{2}_{(0,0)}\right)\right] + \frac{1}{2}\left[P\left(\frac{1}{2}_{(0,0)}\right) - P\left(-\frac{1}{2}_{(0,0)}\right)\right], \\
S_T^x =& \frac{3}{2}\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},0)}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{2},0)}\right)\right] + \frac{1}{2}\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},0)}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{2},0)}\right)\right], \\
S_T^y =& \frac{3}{2}\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right] + \frac{1}{2}\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right].
\end{aligned}$$

$$\begin{aligned}
S_{LL} =& \left[P\left(\frac{3}{2}_{(0,0)}\right) + P\left(-\frac{3}{2}_{(0,0)}\right)\right] - \left[P\left(\frac{1}{2}_{(0,0)}\right) + P\left(-\frac{1}{2}_{(0,0)}\right)\right], \\
S_{LT}^x =& 2\left\{\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},0)}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{4},0)}\right)\right] - \left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},0)}\right) + P\left(-\frac{3}{2}_{(-\frac{\pi}{4},0)}\right)\right]\right\}, \\
S_{LT}^y =& 2\left\{\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right] - \left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right) + P\left(-\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\}, \\
S_{TT}^{xx} =& 2\left\{\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},0)}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{2},0)}\right)\right] - \left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]\right\}, \\
S_{TT}^{xy} =& 2\left\{\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right] - \left[P\left(\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right) + P\left(-\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)\right]\right\}.
\end{aligned}$$

$$
\begin{aligned}
S_{LLL} =& \frac{3}{10}\left[P\left(\frac{3}{2}_{(0,0)}\right) - P\left(-\frac{3}{2}_{(0,0)}\right)\right] - \frac{9}{10}\left[P\left(\frac{1}{2}_{(0,0)}\right) - P\left(-\frac{1}{2}_{(0,0)}\right)\right], \\
S^{x}_{LLT} =& -\frac{1}{60}\left\{129\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},0)}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{2},0)}\right)\right] + 23\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},0)}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{2},0)}\right)\right]\right\} \\
&+ \frac{\sqrt{2}}{24}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},0)}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{4},0)}\right)\right] + \left[P\left(\frac{1}{2}_{(\frac{\pi}{4},0)}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{4},0)}\right)\right]\right\} \\
&+ \frac{\sqrt{2}}{24}\left\{27\left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},0)}\right) - P\left(-\frac{3}{2}_{(-\frac{\pi}{4},0)}\right)\right] + \left[P\left(\frac{1}{2}_{(-\frac{\pi}{4},0)}\right) - P\left(-\frac{1}{2}_{(-\frac{\pi}{4},0)}\right)\right]\right\}, \\
S^{y}_{LLT} =& -\frac{1}{60}\left\{129\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right] + 23\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]\right\} \\
&+ \frac{\sqrt{2}}{24}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right) - P\left(-\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right] + \left[P\left(\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right) - P\left(-\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\} \\
&+ \frac{\sqrt{2}}{24}\left\{27\left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right) - P\left(-\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)\right] + \left[P\left(\frac{1}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right) - P\left(-\frac{1}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
S_{LTT}^{xx} =& \frac{\sqrt{2}}{12}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},0)}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{4},0)}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{4},0)}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{4},0)}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{12}\left\{27\left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},0)}\right)-P\left(-\frac{3}{2}_{(-\frac{\pi}{4},0)}\right)\right]+\left[P\left(\frac{1}{2}_{(-\frac{\pi}{4},0)}\right)-P\left(-\frac{1}{2}_{(-\frac{\pi}{4},0)}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{12}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\} \\
&+\frac{\sqrt{2}}{12}\left\{27\left[P\left(\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{3}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)\right]+\left[P\left(\frac{1}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{1}{2}_{(-\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
S_{LTT}^{xy} =& \frac{1}{12}\left\{27\left[P\left(\frac{3}{2}(0,0)\right)-P\left(-\frac{3}{2}(0,0)\right)\right]+\left[P\left(\frac{1}{2}(0,0)\right)-P\left(-\frac{1}{2}(0,0)\right)\right]\right\} \\
&+\frac{1}{12}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},0)}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},0)}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},0)}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},0)}\right)\right]\right\} \\
&+\frac{1}{12}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{6}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},0)}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{4},0)}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{4},0)}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{4},0)}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{6}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{4},\frac{\pi}{2})}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{6}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right]\right\} \\
&+\frac{\sqrt{3}}{4}\left\{27\left[P\left(\frac{3}{2}_{(\theta_{xyz},\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\theta_{xyz},\frac{\pi}{4})}\right)\right]+\left[P\left(\frac{1}{2}_{(\theta_{xyz},-\frac{\pi}{4})}\right)-\left(-\frac{1}{2}_{(\theta_{xyz},-\frac{\pi}{4})}\right)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
S_{TTT}^{xxx} =& \frac{1}{4}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},0)}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},0)}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},0)}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},0)}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{8}\left\{27\left[\right]P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\Bigg]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{8}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
S_{TTT}^{yxx} =& -\frac{1}{4}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{2})}\right)\right]\right\} \\
&+\frac{\sqrt{2}}{8}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},\frac{\pi}{4})}\right)\right]\right\} \\
&-\frac{\sqrt{2}}{8}\left\{27\left[P\left(\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)-P\left(-\frac{3}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)\right]+\left[P\left(\frac{1}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)-P\left(-\frac{1}{2}_{(\frac{\pi}{2},-\frac{\pi}{4})}\right)\right]\right\}
\end{aligned}
$$