The mass spectrum and wave functions of the Bc system

Guo-Li Wang
Hebei University

In collaboration with

Tianhong Wang, Qiang Li, Chao-Hsi Chang

JEHP 05 (2022) 006

outline

- Motivation
- Meson and its description–3 categories
- Wave functions and partial waves
- Mass spectra
- Summary

Motivation

- Bc(2S) and B*c(2S) are already detected,

  $M(B_c^*) - M(B_c)$ > $M(B_c^*(2S)) - M(B_c(2S))$

  Atlas PRL 113 (2014) 212004, CMS PRL 122 (2019) 132001

- Bc(3S) can be found via their strong decays, R. Ding, B.D. Wan, Z.Q. Chen, G.L. Wang, C.F. Qiao, PLB 816 (2021) 136277

- S-D mixing

$$|\psi(3770)\rangle = |1^3D_1\rangle\cos\theta + |2^3S_1\rangle\sin\theta,$$

$$|\psi(3686)\rangle = -|1^3D_1\rangle\sin\theta + |2^3S_1\rangle\cos\theta$$

- ${}^1P_1 - {}^3P_1$ mixing

$$|D_1(2420)\rangle = |\frac{3}{2}\rangle = \cos\theta|{}^1P_1\rangle + \sin\theta|{}^3P_1\rangle,$$

$$|D_1'(2430)\rangle = |\frac{1}{2}\rangle = -\sin\theta|{}^1P_1\rangle + \cos\theta|{}^3P_1\rangle.$$

- In a relativistic method, e.g. solving the Bethe-Salpeter equation or Salpeter equation, how about upper mixings

Meson and its description

- Usually using $^{2S+1}L_J$ or $J^{P(C)}$ to describe a meson
- Non-relativistic $^{2S+1}L_J$

	$S=0$	$S=1$	$S=1$	$S=1$
S	$^1S_0$ (0^{-+})	$^3S_1$ (1^{--})		
P	$^1P_1$ (1^{+-})	$^3P_0$ (0^{++})	$^3P_1$ (1^{++})	$^3P_2$ (2^{++})
D	$^1D_2$ (2^{-+})	$^3D_1$ (1^{--})	$^3D_2$ (2^{--})	$^3D_3$ (3^{--})
F	$^1F_3$ (3^{+-})	$^3F_2$ (2^{++})	$^3F_3$ (3^{++})	$^3F_4$ (4^{++})
G	$^1G_4$ (4^{-+})	$^3G_3$ (3^{--})	$^3G_4$ (4^{--})	$^3G_5$ (5^{--})

Meson and its description

- Relativistic $J^{P(C)}$

$J=0$	$0^{-(+)}\ (^1S_0)$	$0^{+(+)}\ (^3P_0)$		
$J=1$	$1^{-(-)}\ (^3S_1,\ ^3S_1-{}^3D_1,\ ^3D_1)$	$1^{++}\ (^3P_1)$	$1^{+-}\ (^1P_1)$	$1^{+}\ (^3P_1-{}^1P_1)$
$J=2$	$2^{+(+)}\ (^3P_2,\ ^3P_2-{}^3F_2,\ ^3F_2)$	$2^{--}\ (^3D_2)$	$2^{-+}\ (^1D_2)$	$2^{-}\ (^3D_2-{}^1D_2)$
$J=3$	$3^{-(-)}\ (^3D_3,\ ^3D_3-{}^3G_3,\ ^3G_3)$	$3^{++}\ (^3F_3)$	$3^{+-}\ (^1F_3)$	$3^{+}\ (^3F_3-{}^1F_3)$

- Three categories

1. 0^- and 0^+
2. Natural parity $1^-,\ 2^+$ and 3^-
3. Unnatural parity $1^+,\ 2^-$ and 3^+

Wave function of 0^- state and its partial waves

- Wave function

$$\varphi_P^{0^-}(q_\perp) = \left(a_1 M + a_2 \not{P} + a_3 \not{q}_\perp + a_4 \frac{\not{q}_\perp \not{P}}{M} \right) \gamma^5$$

$$\varphi_P^{0^-}(q_\perp) = M \left(a_1 + a_2 \frac{\not{P}}{M} - a_1 x_- \not{q}_\perp + a_2 x_+ \frac{\not{q}_\perp \not{P}}{M} \right) \gamma^5$$

$$x_+ = \frac{\omega_1 + \omega_2}{m_1 \omega_2 + m_2 \omega_1}, \quad x_- = \frac{\omega_1 - \omega_2}{m_1 \omega_2 + m_2 \omega_1}$$

$$\omega_1 = \sqrt{m_1^2 - q_\perp^2}, \ \omega_2 = \sqrt{m_2^2 - q_\perp^2}$$

Wave function of 0^- state and its partial waves

- J^{PC}

$$\varphi_P(q) = \eta_P \gamma_0 \varphi_{P'}(q')\gamma_0,\ \ P' = (P_0, -\vec{P}) \text{ and } q' = (q_0, -\vec{q})$$

$$\varphi_P(q) = \eta_C C \varphi_P^T(-q) C^{-1},\ \ C\gamma_5^T C^{-1} = \gamma_5 \text{ and } C\gamma_\mu^T C^{-1} = -\gamma_\mu$$

- Partial waves

$$\varphi_P^{0^-}(q_\perp) = \sqrt{4\pi}\left[M Y_{00}\left(a_1 + a_2\gamma^0\right) - \frac{|\vec{q}|}{\sqrt{3}}(Y_{1-1}\gamma^+ + Y_{11}\gamma^- - Y_{10}\gamma^3)(a_3 + a_4\gamma^0)\right]\gamma^5$$

a1 and a2 terms are S waves, non-relativistic

a3 and a4 terms are P waves, relativistic correction

$Wave\ function\ of\ 0^{-}\ \ state\ and\ its\ partial\ waves$

- Non-relativistic

$$\varphi_{P}^{^1S_0}(q_{\perp}) = (a_1 M + a_2 \not{P})\gamma^5$$

- Normalizaiton $\propto S^2$

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{2Ma_1a_2(\omega_1 m_2 + \omega_2 m_1)}{\omega_1\omega_2} = 1$$

- Full normalization $\propto (S+P)^2$

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8M\omega_1\omega_2 a_1 a_2}{(\omega_1 m_2 + \omega_2 m_1)} = 1$$

- S: P=1: 0.082; 1: 0.091; 1: 0.097 for 1S, 2S, 3S Bc states

Wave function of 0^- state and its partial waves

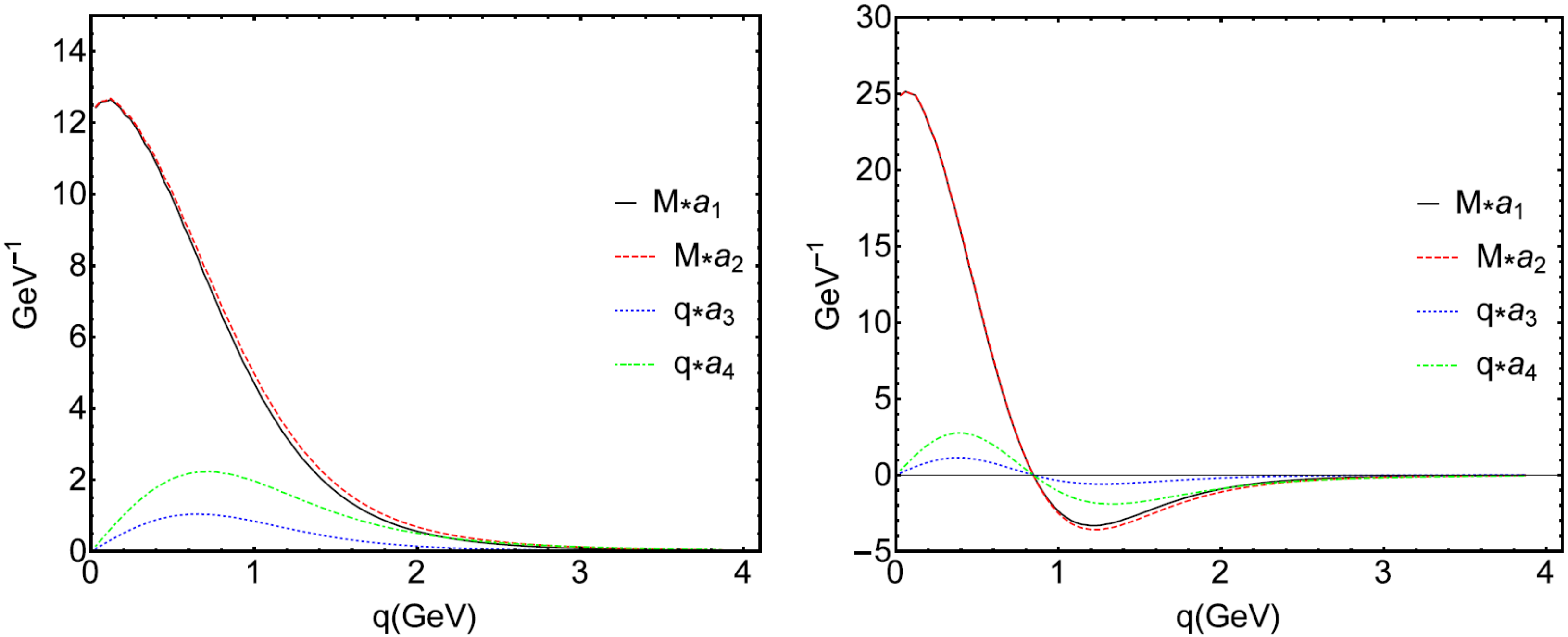

Figure 1. The 0^- wave functions of the ground state $B_c(1S)$ (left) and the first excited state $B_c(2S)$ (right). a_1 and a_2 terms are S waves; a_3 and a_4 terms are P waves.

$Wave\ function\ of\ 0^+\ state\ and\ its\ partial\ waves$

$$\varphi_P^{0^+}(q_\perp) = b_1\ \not{q}_\perp + b_2\ \frac{\not{P}\not{q}_\perp}{M} + b_3\ M + b_4\ \not{P}$$

$$b_3 = \frac{b_1\ q_\perp^2\, x_+}{M}, \quad b_4 = \frac{b_2\ q_\perp^2\, x_-}{M}.$$

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^2 b_1 b_2}{M(m_1\omega_2 + m_2\omega_1)} = 1$$

$$\varphi_P^{^3P_0}(q_\perp) = b_1\ \not{q}_\perp + b_2\ \frac{\not{P}\not{q}_\perp}{M} \qquad \int \frac{d\vec{q}}{(2\pi)^3} \frac{2\vec{q}^2 b_1 b_2 (m_1\omega_2 + m_2\omega_1)}{M\omega_1\omega_2} = 1$$

- P: S = 1: 0.097; 1: 0.10; 1: 0.11 for 1P, 2P, 3P Bc

Wave function of a 1^- state and its partial waves

- zero-q is S wave, one-q P wave, two is D, three is F…

$$\varphi_P^{1^-}(q_\perp) = \epsilon\cdot q_\perp\left[c_1 + \frac{\not P}{M}c_2 + \frac{\not q_\perp}{M}c_3 + \frac{\not P\not q_\perp}{M^2}c_4\right] + M\not\epsilon\, c_5 + \not\epsilon\not P\, c_6 + (\not q_\perp\not\epsilon - \epsilon\cdot q_\perp)\, c_7 + \frac{1}{M}(\not P\not\epsilon\not q_\perp - \not P\epsilon\cdot q_\perp)\, c_8,$$

- So it is S-P-D mixing state

- Normalization

$$\int\frac{d\vec{q}}{(2\pi)^3}\frac{8M\omega_1\omega_2}{3(\omega_1 m_2+\omega_2 m_1)}\left[-3c_5c_6 + \frac{\vec{q}^2}{M^2}\left(-c_4c_5 + c_3c_6 + c_3c_4\frac{\vec{q}^2}{M^2}\right)\right] = 1$$

It is $\propto (S+P+D)^2$

$Wave\ function\ of\ a\ 1^{-}\ state\ and\ its\ partial\ waves$

- Pure S wave $\varphi_P^{^3S_1}(q_\perp) = M \not{\epsilon}\, c_5 + \not{P} \not{\epsilon}\, c_6,$

with $-\int \frac{d\vec{q}}{(2\pi)^3} \frac{2Mc_5c_6(\omega_1 m_2 + \omega_2 m_1)}{\omega_1\omega_2} = 1.$ $\propto (S)^2$

- Pure D wave $\varphi_P^{^3D_1}(q_\perp) = \epsilon \cdot q_\perp \left(\frac{\not{q}_\perp}{M}\, c_3 + \frac{\not{P}\not{q}_\perp}{M^2}\, c_4 \right),$

With $\int \frac{d\vec{q}}{(2\pi)^3} \frac{2c_3c_4\vec{q}^4(\omega_1 m_2 + \omega_2 m_1)}{3M^3\omega_1\omega_2} = 1$ $\propto (D)^2$

So S: P: D=1: 0.09: 0.037; 1: 0.097: 0.044; -0.576: 0.48: 1 for 1S, 2S, 1D dominant states. If delete P wave, then obtain the S-D mixing angle for 1D dominant state: $\theta = 30^\circ$

Wave function of a 1^- *state and its partial waves*

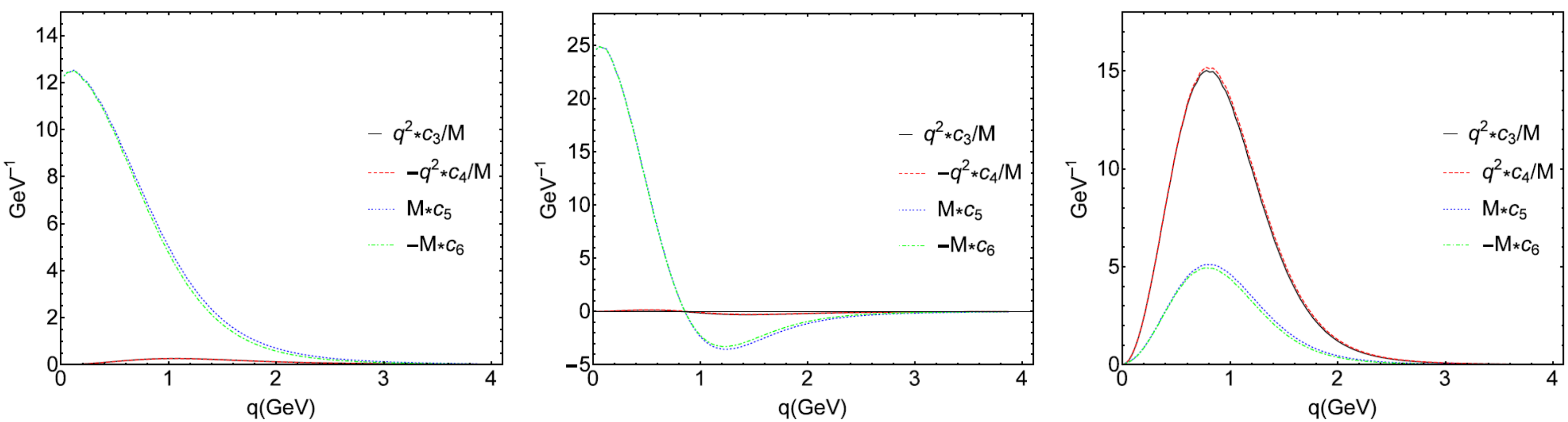

Figure 3. The 1^- wave functions of the $B_c^*(1S)$ (left), $B_c^*(2S)$ (middle) and the $S-P-D$ mixture $B_c^*(1D)$ (right). c_3 and c_4 terms are D waves; c_5 and c_6 terms are S waves.

$Wave\ function\ of\ a\ 2^{+}\ state\ and\ its\ partial\ waves$

$$\varphi^{2^+}_{P}(q_\perp) = \epsilon_{\mu\nu} q^\mu_\perp \left\{ q^\nu_\perp \left[d_1 + \frac{\not{P}}{M} d_2 + \frac{\not{q}_\perp}{M} d_3 + \frac{\not{P}\not{q}_\perp}{M^2} d_4 \right] + \gamma^\nu (M d_5 + \not{P} d_6) + (\not{q}_\perp \gamma^\nu - q^\nu_\perp) d_7 + \frac{(\gamma^\nu \not{q}_\perp - q^\nu_\perp)\not{P}}{M} d_8 \right\},$$

- It is P-D-F mixing state, with normalization condition

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8M\omega_1\omega_2\vec{q}^2}{15(\omega_1 m_2 + \omega_2 m_1)} \left[-5d_5 d_6 + \frac{2\vec{q}^2}{M^2} \left(-d_4 d_5 + d_3 d_6 + d_3 d_4 \frac{\vec{q}^2}{M^2} \right) \right] = 1.$$

- Pure P wave $\varphi^{^3P_2}_{P}(q_\perp) = \epsilon_{\mu\nu} q^\mu_\perp \gamma^\nu (M d_5 + \not{P} d_6),$

$$-\int \frac{d\vec{q}}{(2\pi)^3} \frac{2d_5 d_6 \vec{q}^2 M(\omega_1 m_2 + \omega_2 m_1)}{3\omega_1\omega_2} = 1.$$

$Wave\ function\ of\ a\ 2^{+}\ state\ and\ its\ partial\ waves$

- Pure F wave

$$\varphi_P^{^3F_2}(q_\perp) = \epsilon_{\mu\nu} q_\perp^\mu q_\perp^\nu \left[\frac{\not{q}_\perp}{M} d_3 + \frac{\not{P}\not{q}_\perp}{M^2} d_4 \right],$$

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{4 d_3 d_4 \vec{q}^6 (\omega_1 m_2 + \omega_2 m_1)}{15 M^3 \omega_1 \omega_2} = 1.$$

- P: D: F=1: 0.10: 0.039; 1: 0.11: 0.049; -0.633: 0.50: 1 for 1P, 2P, 1F dominant states, respectively

Wave function of a 3^- state and its partial waves

$$\varphi_P^{3^-}(q_\perp) = \epsilon_{\mu\nu\alpha} q_\perp^\mu q_\perp^\nu \left[q_\perp^\alpha \left(e_1 + \frac{\not{P}}{M} e_2 + \frac{\not{q}_\perp}{M} e_3 + \frac{\not{P}\not{q}_\perp}{M^2} e_4 \right) + M\gamma^\alpha \left(e_5 + \frac{\not{P}}{M} e_6 + \frac{\not{q}_\perp}{M} e_7 + \frac{\not{P}\not{q}_\perp}{M^2} e_8 \right) \right]$$

- It is D-F-G mixing state
- D: F: G=1: 0.11: 0.043; 1:0.13: 0.052; -0.654: 0.505: 1 for 1D, 2D and 1G dominant states

Wave function of a 1^+ state and its partial waves

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} + f_3\frac{\not q_\perp}{M} + f_4\frac{\not q_\perp \not P}{M^2}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} + g_3\frac{\not q_\perp}{M} + g_4\frac{\not q_\perp \not P}{M^2}\right),$$

f_i terms are 1^{+-} $({}^1P_1)$, g_i are 1^{++} $({}^3P_1)$, so it is ${}^1P_1 -{}^3 P_1$ mixing state.

$$\varphi_P^{1^+}(q_\perp) = \epsilon\cdot q_\perp\left(f_1 + f_2\frac{\not P}{M} - f_1 x_- \not q_\perp + f_2 x_+\frac{\not q_\perp \not P}{M}\right)\gamma^5$$
$$+\frac{i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu q_\perp^\rho \epsilon^\sigma}{M}\left(g_1 + g_2\frac{\not P}{M} - g_1 x_- \not q_\perp + g_2 x_+\frac{\not q_\perp \not P}{M}\right).$$

Wave function of a 1^+ *state and its partial waves*

- Normalization

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^2}{3M(m_1\omega_2 + m_2\omega_1)}(f_1 f_2 - 2g_1 g_2) \equiv \cos^2\theta + \sin^2\theta = 1.$$

where the mixing angle is defined by wave function

- There is also P and D partial waves, f1,f2,g1,g2 terms are pure P waves, with normalization condition

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{2(m_1\omega_2 + m_2\omega_1)\vec{q}^2}{3M\omega_1\omega_2}(f_1 f_2 - 2g_1 g_2) \equiv \cos^2\varphi + \sin^2\varphi = 1.$$

Wave function of a 1^+ *state and its partial waves*

- Solutions appear in pairs, first and second are 1P, third and fourth are 2P, etc
- First two $1^1P_1 : 1^3P_1 = 0.284 : 0.716$ and $0.716 : 0.284$
  with mixing angle $\theta_{1P} = -57.8^\circ$ or 32.2°
- 3, 4 $2^1P_1 : 2^3P_1 = 0.263 : 0.737$ and $0.737 : 0.263$,

$$\theta_{2P} = -59.1^\circ \text{ or } 30.9^\circ.$$

- Pure P wave: $\varphi_{nP} = \theta_{nP}$
- $P : D = 1 : 0.0971$ for two 1P
  $P : D = 1 : 0.0936$ for two 2P

Wave function of a 1^+ state and its partial waves

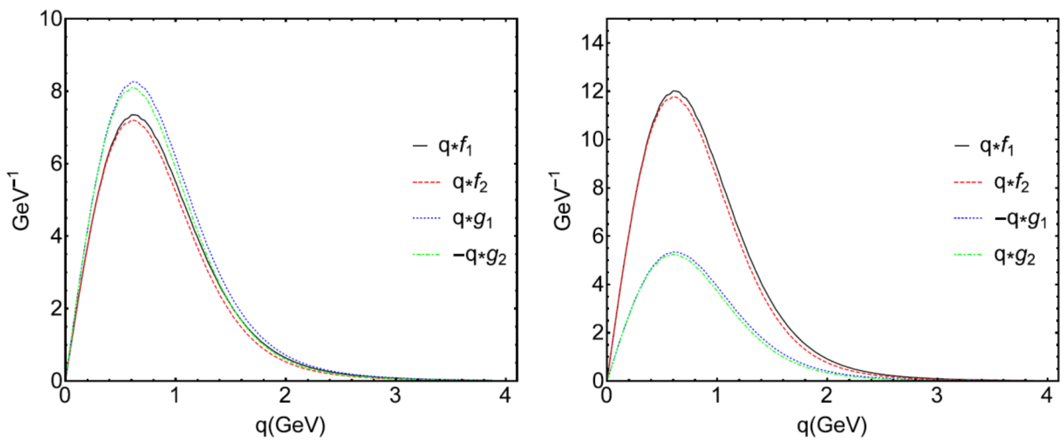

Figure 6. The 1^+ wave functions of the $1^1P_1 - 1^3P_1$ mixing states $B_{c1}(1P)$ (left) and $B'_{c1}(1P)$ (right). f_1 and f_2 terms are $^1P_1$ waves; g_1 and g_2 terms are $^3P_1$ waves.

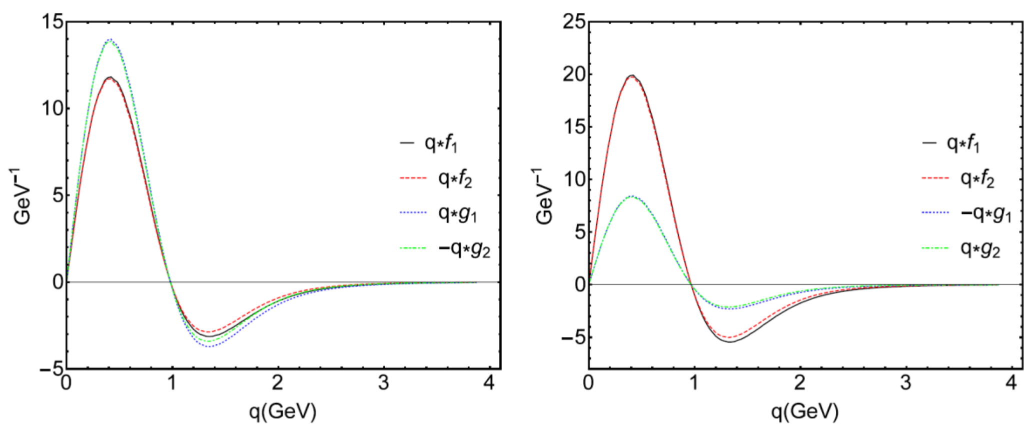

Figure 7. The 1^+ wave functions of the $2^1P_1 - 2^3P_1$ mixing states $B_{c1}(2P)$ (left) and $B'_{c1}(2P)$ (right). f_1 and f_2 terms are $^1P_1$ waves; g_1 and g_2 terms are $^3P_1$ waves.

Wave function of a 2^- state and its partial waves

$$\varphi_P^{2^-}(q_\perp) = \epsilon_{\mu\nu} q_\perp^\mu q_\perp^\nu \left(h_1 + \frac{\not P}{M} h_2 + \frac{\not q_\perp}{M} h_3 + \frac{\not P \not q_\perp}{M^2} h_4 \right) \gamma^5$$
$$+ \frac{i\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu P^\nu q_\perp^\alpha \epsilon^{\beta\delta} q_{\perp\delta}}{M} \left(i_1 + \frac{\not P}{M} i_2 + \frac{\not q_\perp}{M} i_3 + \frac{\not P \not q_\perp}{M^2} i_4 \right)$$

2^- state is a ${}^1D_2 - {}^3D_2$ mixture

- Normalization and mixing angle

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^4}{15M(m_1\omega_2 + m_2\omega_1)}(2h_1h_2 - 3i_1i_2) \equiv \cos^2\theta + \sin^2\theta = 1$$

Wave function of a 2^- *state and its partial waves*

- First and second solutions

$$1^1D_2 : 1^3D_2 = 0.277 : 0.723 \text{ and } 0.723 : 0.277$$

$$\theta_{1D} = -58.2^\circ \text{ or } 31.8^\circ \qquad D : F = 1 : 0.106$$

- Third and fourth solutions

$$2^1D_2 : 2^3D_2 = 0.291 : 0.709 \text{ and } 0.709 : 0.291$$

$$\theta_{2D} = -57.4^\circ \text{ or } 32.6^\circ \qquad D : F = 1 : 0.108$$

Mass spectra

$n\ ^{2S+1}L_J$	J^P	ours	[38]	[39, 40]	[41]	[42]	[55]	Exp
$1\ ^1S_0$	0^-	6277 (input)	6271	6272	6275	6271 (input)	6276	6274.9±0.8 [56]
$1\ ^3S_1$	1^-	6332 (input)	6338	6333	6329	6326 (input)	6331	6333 [27]
$2\ ^1S_0$	0^-	6867	6855	6842	6867	6871 (input)		6871.6±1.1 [56]
$2\ ^3S_1$	1^-	6911	6887	6882	6898	6890		6900.1 [27]
$3\ ^1S_0$	0^-	7228	7250	7226	7254	7239		
$3\ ^3S_1$	1^-	7272	7272	7258	7280	7252		
$1\ ^3P_0$	0^+	6705 (input)	6706	6699	6693	6714	6712	
$1\ P_1$	1^+	6739 (input)	6741	6743	6731	6757	6736	
$1\ P_1'$	1^+	6748	6750	6750	6739	6776		
θ_{1P}		$-57.8°(32.2°)$	22.4°	20.5°	18.7°	35.5°	$33.4 \pm 1.5°$ [37]	
$1\ ^3P_2$	2^+	6762 (input)	6768	6761	6751	6787		
$2\ ^3P_0$	0^+	7112	7122	7094	7105	7107		
$2\ P_1$	1^+	7144	7145	7134	7136	7134		
$2\ P_1'$	1^+	7149	7150	7147	7144	7150		
θ_{2P}		$-59.1°(30.9°)$	18.9°	23.2°	21.2°	38.0°		
$2\ ^3P_2$	2^+	7163	7164	7157	7155	7160		

Mass spectra

$n\ ^{2S+1}L_J$	J^P	ours	[38]	[39, 40]	[41]	[42]
$1\ ^3D_1$	1^-	7014 $(S-P-D)$	7028	7021	7007	7020
$2\ ^3D_1$	1^-	7335 $(S-P-D)$		7392	7347	7336
$1\ ^3F_2$	2^+	7239 $(P-D-F)$	7269	7232	7234	7235
$2\ ^3F_2$	2^+	7508 $(P-D-F)$		7618		7518
$1\ ^3D_3$	3^-	7035 (input)	7045	7029	7011	7030
$1\ D_2$	2^-	7025 (input)	7036	7025	7006	7024
$1\ D_2'$	2^-	7029	7041	7026	7016	7032
θ_{1D}		$-58.2°(31.8°)$	$44.5°$	$-35.9°$	$-49.2°$	$45.0°$
$2\ ^3D_3$	3^-	7355		7405	7351	7348
$2\ D_2$	2^-	7345		7399	7339	7343
$2\ D_2'$	2^-	7349		7400	7359	7347
θ_{2D}		$-57.4°(32.6°)$			$-40.3°$	$45.0°$

Summary

- There are three categories of mesons.
- All states have different partial waves.
- The mixing of different waves naturally appears in relativistic method, not manmade, only one wave function is needed.
- In a relativistic method, the mixing angle can be calculated by wave function, not potential.

Thank you