From linear algebra to Feynman integrals

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Mainly based on works done with:
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1711.09572, 1801.10523, 1912.09294, 2107.01864, 2201.11669, 2201.11637, ...

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北京大學

Outline

I. Introduction

II. Linear space of FIs

III. Kinematics

IV. Spacetime and loops

V. Phenomenology

Why study Feynman integrals

1) Fundamental

- QFT: theoretical foundation of physics at current and future
- Amplitudes: linear combinations of FIs with rational coefficients
- FIs: study of QFT, phenomenology

2) Challenging

- One-loop calculation: satisfactory approach existed as early as 1970s

’t Hooft, Veltman, NPB (1979); Passarino, Veltman, NPB (1979); Oldenborgh, Vermaseren (1990)

Britto, Cachazo, Feng, 0412103; Ossola, Papadopoulos, Pittau, 0609007; Giele, Kunszt, Melnikov, 0801.2237

- 40 years later, no satisfactory method for multi-loop calculation

3) Fun

- Plenty of ideas: large dimension/mass expansion, finite field, algebraic geometry, unitarity cut, intersection theory, uniform transcendental, symbol, …

Difficulties of FIs

➢ A family of Feynman integrals

$$I_{\vec{\nu}}(D,\vec{s}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_\alpha = A_{\alpha ij}\ell_i \cdot \ell_j + B_{\alpha ij}\ell_i \cdot p_j + C_\alpha$$

- $\ell_1, \dots, \ell_L$: loop momenta; $p_1, \dots, p_E$: external momenta;
- A, B: integers; C: linear combination of $\vec{s}$ (including masses)
- $\mathcal{D}_1, \dots, \mathcal{D}_K$: inverse propagators; $\nu_1, \dots, \nu_K$: integers
- $\mathcal{D}_{K+1}, \dots, \mathcal{D}_N$: irreducible scalar products; $\nu_{K+1}, \dots, \nu_N$: non-negative integers

➢ Difficulties of calculating FIs

- Analytical: known special functions are insufficient to express FIs
- Numerical: UV, IR, integrable singularities, …

Linear algebra

- **An ancient topic**
  - 《鸡兔同笼》**(chickens and rabbits in the same cage)**
  - 《九章算术·方程》**(Nine Chapters on Mathematical Art · Equations)**
- **Well studied**

$$M\,\vec{x} = \vec{c}$$

  - **Vector, matrix, determinant, rank**
  - **Gaussian elimination**
  - …

FIs are completely determined by linear algebra???

The law of conservation of mistery!

IBP equations

- **Dimensional regularization: vanish at boundary**

'‘t Hooft, Veltman, NPB (1972)
Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(q_k^\mu \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}} \right) = 0, \qquad \forall \vec{\nu}, j, k$$

$$\overrightarrow{q^\mu} = (\ell_1^\mu, \cdots, \ell_L^\mu, p_1^\mu, \cdots, p_E^\mu)$$

$$\Downarrow$$

- **Linear equation:** $\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D, \vec{s}) I_{\vec{\nu}'}(D, \vec{s}) = 0$
- Q: **polynomials in** $D, \vec{s}$
- **Plenty of linear equations can be easily obtained by varying:** $\vec{\nu}, j, k$

Warning: IBP is insensitive to Feynman prescription $\mathrm{i}0^+$**, suppressed**

IBP reduction

- # of equations grows faster than # of FIs

Laporta, Remiddi, 9602417, Gehrmann, Remiddi, 9912329

- A family of FIs form a FINITE-dim. linear space

Proved by: Smirnov, Petukhov, 1004.4199

  - Bases of the linear space called master integrals (MIs)
  - IBPs reduce tens of thousands of FIs to much less MIs

- Laporta's algorithm

Laporta, 0102033

  - Solving IBP eqs. automaticaly, to any-loop order
  - Public codes: AIR, FIRE, LiteRed, Reduze, Kira, FiniteFlow,…
  - Many more private codes
  - Warning: time-consuming for complicated problems

Since 90s'

FIs $\triangleq$ Linear algebra $\oplus$ Master integrals

Input:

The same kinematics

The same spacetime dimension

The same number of loops

Traditional differential eqs method

➢ Step 1: Set up $\vec{s}$-DEs of MIs

- Differentiate MIs w.r.t. invariants $\vec{s}$, such as $m^2, p \cdot q$

Kotikov, PLB(1991)

- Solving IBP relations: $\frac{\partial}{\partial s_i}\vec{I}(D,\vec{s}) = A_i(D,\vec{s})\vec{I}(D,\vec{s})$

➢ Step 2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2
- Case by case, not systematic, maybe still hard!

➢ Step 3: Solve DEs

- Systematic, not hard (explain later)

Auxiliary mass terms

Liu, YQM, Wang, 1711.09572

➢ Auxiliary FIs

$$I_{\vec{\nu}}^{\text{aux}}(D,\vec{s},\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \lambda_1 \eta + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K - \lambda_K \eta + \mathrm{i}0^+)^{\nu_K}}$$

- $\lambda_i \geq 0$ **(typically 0 or 1), an auxiliary mass if $\lambda_i > 0$**
- **Analytical function of η**
- **Physical result obtained by (correct Feynman prescription)**

$$I_{\vec{\nu}}(D,\vec{s}) \equiv \lim_{\eta \to \mathrm{i}0^-} I_{\vec{\nu}}^{\text{aux}}(D,\vec{s},\eta)$$

- **1) Setup η-DEs; 2) Calculate boundary conditions; 3) Solve η-DEs**

➢ Why not proposed in the past a few decades?

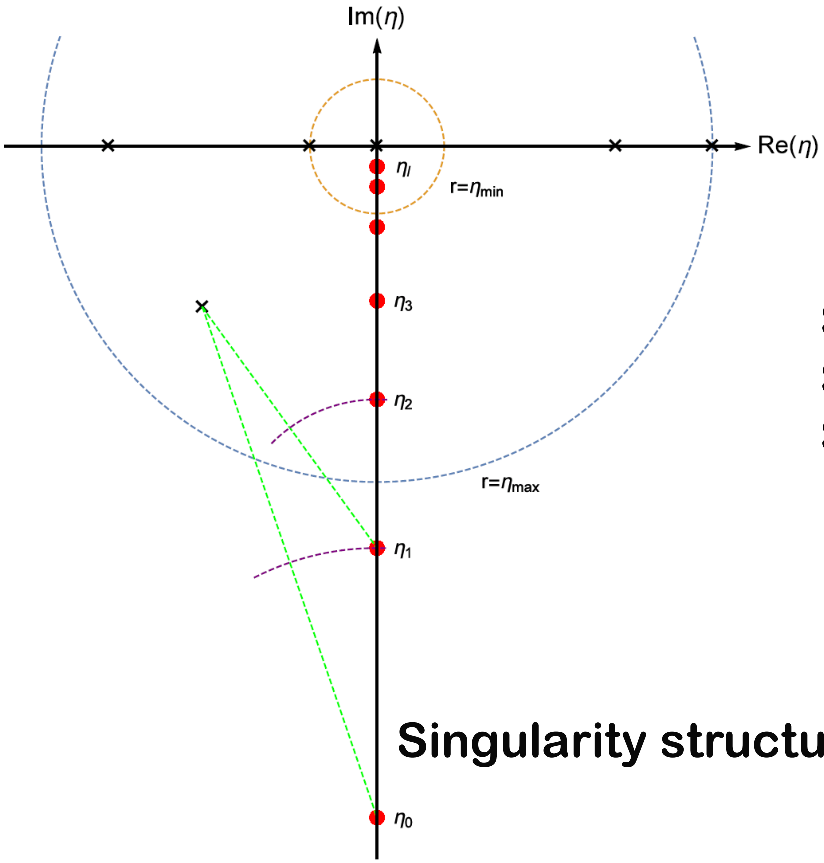
- **Auxiliary FIs always have massive propagators**
- **Stereotype in the community: harder to calculate**

(it is right unless using the method to be explained)

Flow of auxiliary mass

➢ **Solve ODEs of MIs**

$$\frac{\partial}{\partial \eta}\vec{I}^{\text{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta)\vec{I}^{\text{aux}}(D,\vec{s},\eta)$$



Singularity structure

If $\vec{I}^{aux}(D,\vec{s},\infty)$ is known , solving ODEs numerically to obtain $\vec{I}^{aux}(D,\vec{s},\mathrm{i}0^-)$ is a well-studied mathematical problem:

Step1: Asymptotic expansion at $\eta = \infty$
Step2: Taylor expansion at analytical points
Step3: Asymptotic expansion at $\eta = 0$

Efficient to get high precision :
ODEs, known singularity structure

Boundary values at $\eta \to \infty$

➢ Nonzero integration regions as $\eta \to \infty$

- Linear combinations of loop momenta: $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$

Beneke, Smirnov, 9711391
Smirnov, 9907471

➢ Simplify propagators at $\eta \to \infty$

- ℓ_L is the $\mathcal{O}(\sqrt{|\eta|})$ part of loop momenta
- ℓ_S is the $\mathcal{O}(1)$ part of loop momenta
- p is linear combination of external momenta

$$\frac{1}{(\ell_\mathrm{L}+\ell_\mathrm{S}+p)^2-m^2-\kappa\,\eta} \sim \frac{1}{\ell_\mathrm{L}^2-\kappa\,\eta}$$

- Unchange if $\ell_L = 0$ and $\kappa = 0$

➢ Boundary FIs after simplification

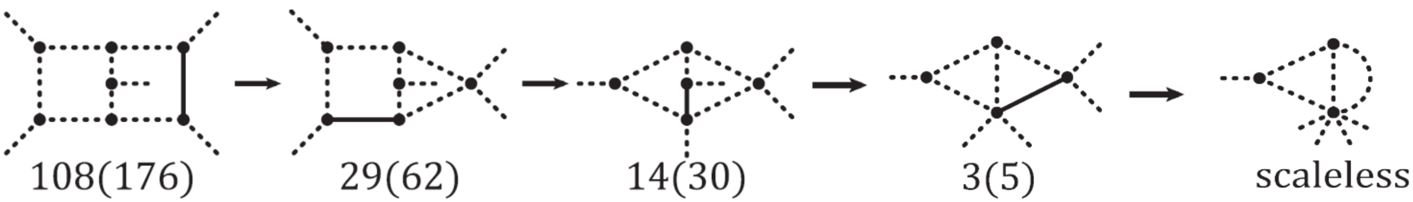
1. Simpler FIs with less denominators, if all loop momenta are $\mathcal{O}(1)$
2. Vacuum integrals

Iterative strategy

➢ For boundary FIs with less denominators:

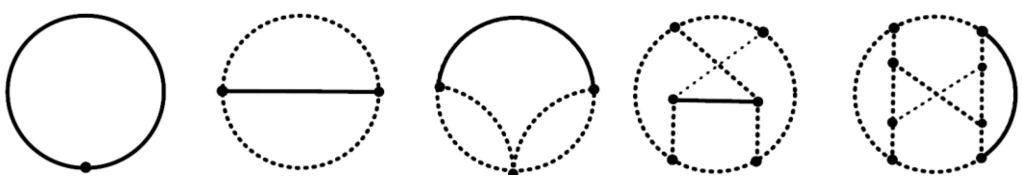
- Calculate them again use AMF method, even simpler boundary FIs as input (besides vacuum integrals)

Liu, YQM, 2107.01864



- Eventually, leaving only (single-mass) vacuum integrals as input
- Kinematic information can be recovered by linear algebra!

➢ Typical single-mass vacuum MIs



Baikov, Chetyrkin, 1004.1153
Lee, Smirnov, Smirnov, 1108.0732
Georgoudis, et. al., 2104.08272

- Much simpler to be calculated
- The same number of loops.

2017-2021

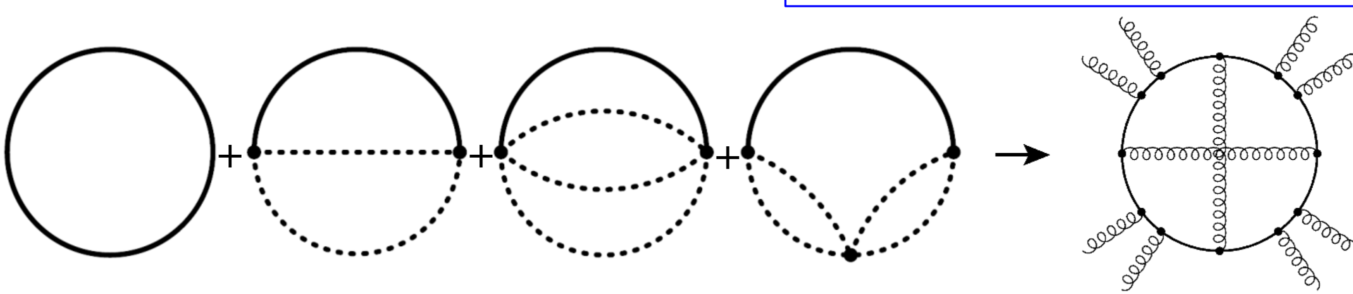
FIs $\triangleq$ Linear algebra $\oplus$ Vacuum integrals

Input:

No kinematics (no external legs)

The same spacetime dimension

The same number of loops



Is this the end of the story?

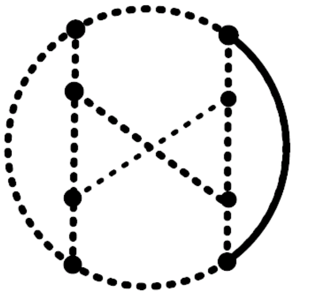
From p-integrals to vacuum integrals

Liu, YQM, 2201.11637

➢ A family of single-mass vacuum integrals

$$I_{\vec{\nu}}(D, m^2) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_1 = \ell_1^2 - m^2 + \mathrm{i}0^+$$



- m^2: the only scale. Can choose $m^2 = 1$

➢ Propagator (p-)integrals

$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}$$

$$\vec{\nu}' = (\nu_2, \cdots, \nu_N)$$
$$\nu = \sum_{i=1}^{N} \nu_i$$

- As ℓ_1^2 is the only scale: $\widehat{I}_{\vec{\nu}'}(\ell_1^2) = (-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1} \widehat{I}_{\vec{\nu}'}(-1)$

- L-loop single-mass vacuum integral expressed by $(L-1)$-loop p-integral

$$I_{\vec{\nu}} = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(\ell_1^2 - 1 + \mathrm{i}0^+)^{\nu_1}} \widehat{I}_{\vec{\nu}'}(-1) = \frac{\Gamma(\nu - LD/2)\Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)} \widehat{I}_{\vec{\nu}'}(-1)$$

From vacuum integrals to p-integrals

Liu, YQM, 2201.11637

➢ Apply AMF method on $(L-1)$-loop p-integral

1) IBP to setup η-DEs

2) Single-mass vacuum integrals no more than $(L-1)$ loops as input

Single-mass vacuum integrals with L loops are determined by that with no more than $(L-1)$ loops (besides IBP)

- Boundary: 0-loop p-integrals equal 1

➢ Only IBPs are needed to determine FIs!

- IBPs: linear algebra, exact (in $D, \vec{s}$) relations between FIs
- Loop integrations are completely avoided!

Workflow

➢ **The 'FICalc' to calculate FIs can be defined as (any given nonsingular D and $\vec{s}$):**

Liu, YQM, 2201.11637

① If it is a 0-loop p-integral, return 1;

② If it is a single-mass vacuum integral, express it by a p-integral, and call 'FICalc' to calculate the p-integral;

③ Otherwise:

a) Introduce η to one propagator (if the mass mode is not possible)

b) Setup η-DEs using IBP as input

c) Call 'FICalc' to calculate boundary FIs at $\eta \to \infty$

d) Numerically solve η-DEs with given BCs to obtain $\eta \to i0^-$

FIs $\triangleq$ Linear algebra

No other input:

No kinematics!

No spacetime dimension!

No loops!

Package: AMFlow

Liu, YQM, 2201.11669

➢ Download

Link: https://gitlab.com/multiloop-pku/amflow

Name	Last commit	Last update
diffeq_solver	update	4 hours ago
examples	update	4 hours ago
ibp_interface	test	6 days ago
AMFlow.m	update	4 hours ago
CHANGELOG.md	update	4 hours ago
FAQ.md	test	6 days ago
LICENSE.md	test	6 days ago
README.md	update	4 hours ago
options_summary	update	4 hours ago

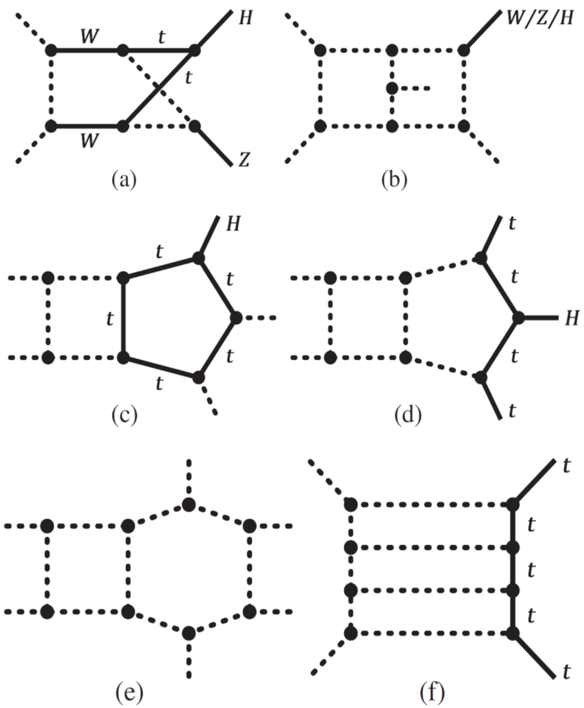
➢ Description

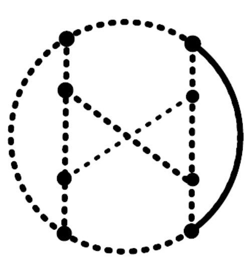
- **The first (method and) package that can calculate any FI (with any number of loops, any D and $\vec{s}$) to arbitrary precision, *given sufficient resource***

Examples using AMF

Liu, YQM, 2107.01864
Liu, YQM, 2201.11637

➢ Cutting-edge problems





Family	dp	a	b	c	d	e	f
T_{setup}	6	20	18	8	1	25	30
T_{solve}	7	11	15	6	3	15	42
P_1	95%	99%	96%	99%	98%	94%	93%
$T_{\vec{s}}$	2	916	64	1305	30	1801	63

Time to setup DEs (CPU core hours)

- Results: 16-digit precision, to $\mathcal{O}(\epsilon^4)$
- First step of iteration: cost most time
- All results in (a)-(f) are new, very challenging for all other methods!
- Highly nontrivially checked!
- IBP reduction (bottleneck): C++
- Solve η-DEs: Mathematica. Can be significantly improved

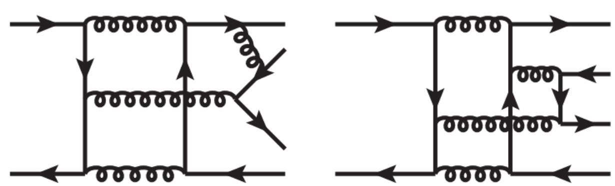
Pheno. applications of AMF

- **Two ways to use AMF**
  - Use AMF to calculate each phase-space point
  - Use AMF to generate BCs of $\vec{s}$-DEs
- **Wide range of applications**
  - Linear propagators; Phase space integrals; Complex mass; QCD sum rules; Electroweak corrections; Quarkonia; Higgs; …

Zhang, et.al., 1810.07656
Yang, et.al., 2005.11010
Brønnum-Hansen, et. al., 2108.09222
Baranowski, et. al., 2111.13594
Wu, et. al., 2201.11714
Sang, et. al., 2202.11615
Tao, et. al., 2204.06385
Armadillo, et. al., 2205.03345
Chaubey, et. al., 2205.06339
Zhang, et. al., 2205.06124
Abreu, et. al., 2206.03848
Bonciani, et. al., 2206.10490
…

- **Example**

$$\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$$



Zhang, Sang, Zhang, 2205.06124

  - Two-loop six external legs, massive particles
  - Very challenging for other methods

Other methods to calculate FIs (1)

➢ Sector decomposition

- Using Monte Carlo: time-consuming
- Hard for non-Euclidean kinematic points

Hepp, (1966)
Binoth, Heinrich, 0004013

➢ Mellin-Barnes representation

- Using Monte Carlo: time-consuming
- Hard for non-planar diagrams

Usyukina (1975)
Smirnov, 9905323

➢ Difference equations

- Depends on reduction and BCs
- Hard to solve difference equations: BCs, convergence

Laporta, 0102033
Lee, 0911.0252

Other methods to calculate FIs (2)

➢ Loop-Tree duality (under development)

- Using Monte Carlo: time-consuming

Catani, et. al., 0804.3170
…
Lotty: Bobadilla, 2103.09237

	Planar triangle		Non-planar triangle	
$\frac{s}{m^2}$	LTD (10^{-6})	SecDec 3.0 (10^{-6})	LTD (10^{-6})	SecDec 3.0 (10^{-6})
$-\frac{1}{4}$	9.48(5)	9.4647(9)	4.461(3)	4.4606(4)
-1	8.10(5)	8.0885(8)	4.101(3)	4.1012(4)
$-\frac{9}{4}$	6.49(3)	6.4760(6)	3.627(5)	3.6276(3)
-4	5.02(2)	5.0188(5)	3.15(5)	3.1334(3)
$+\frac{1}{4}$	10.68(6)	10.651(1)	4.743(3)	4.7436(4)
1	13.11(8)	13.070(1)	5.259(3)	5.2590(5)
$+\frac{9}{4}$	20.81(1)	20.748(2)	6.533(3)	6.5331(6)
$+\frac{25}{16}$	15.74(9)	15.700(1)	5.748(3)	5.7474(6)

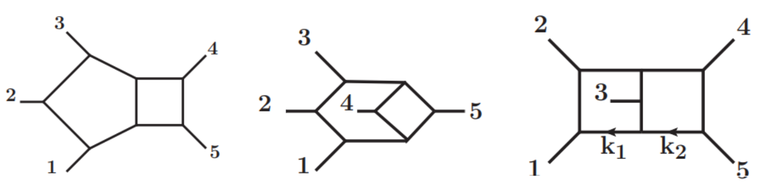
No real phenomenological applications yet

Other methods to calculate FIs (3)

➢ (Traditional) differential equations

- Depends on reduction and BCs — Kotikov, PLB (1991)
- For some cases, ϵ-form exists ⇒ analytical — Henn, 1304.1806; Chen, Yang, Zhang, …
- The frontier: MIs for 2→3 massless processes at two loops

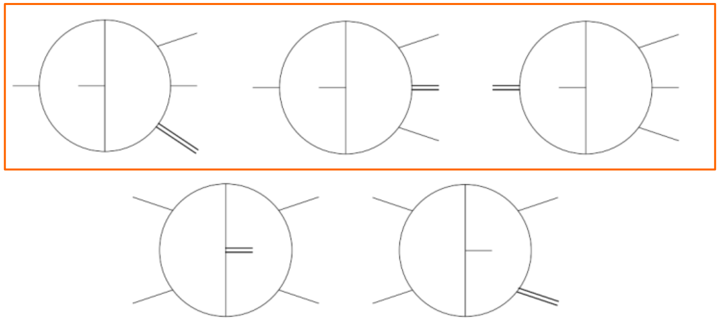
Onshell: Badger, et. al., 1812.11160
Chicherin, Sotnikov, 2009.07803



(a) penta-box (b) hexa-box (c) double-pentagon

- All MIs are known analytically to $\mathcal{O}(1)$
- AMF (numerical): known easily to $\mathcal{O}(\epsilon^4)$

One offshell: Kardos, et. al., 2201.07509



- Hexa-box MIs are known analytically to $\mathcal{O}(1)$
- AMF (numerical): all MIs are known easily to $\mathcal{O}(\epsilon^4)$

State-of-the-art computation

- 2→2 process with massive particles at two-loop order: almost done $g+g\to t+\bar{t}$, $g+g\to H+H(g)$

- Frontier in the following decade:
  - 2→3 processes at two loops (3j/γ, V/H+2j $t\bar{t}$+j, $t\bar{t}H$,…)
  - 2→2 processes at three loops (2j/γ, V/H+j, $t\bar{t}$, HH, …)
  - 2→1 processes at four loops (j, V/H)

- Very challenging (without new development)
  - Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved

    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349

  - Four-loop $g+g\to H$ (NNLP in HTL): 860 days (wall time!)

    Davies, Herren, Steinhauser, 1911.10214

Summary and outlook

- Feynman integrals form a finite-dim. linear space
- AMF: Feynman integrals can be completely determined once relations in the linear space is clear
- Results in a powerful method to calculate FIs: for the first time, any FI can calculated to high precision

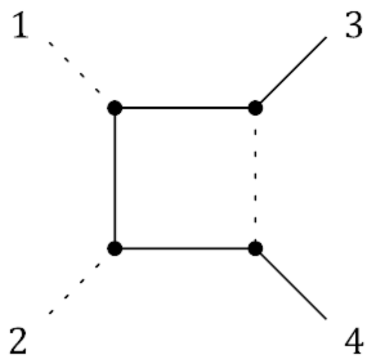
$$\text{Impossible} \overset{2022}{\Longrightarrow} \text{possible} \overset{\text{future}}{\Longrightarrow} \text{efficiency}$$

- Perturbative QFT in the new era: stay tune

Thank you!

Infrared Divergences

➢ Example: one-loop four-point integral



$s = 10, t = -3, m^2 = 1$

- **dim-reg** $I[1,1,1,1] = \dfrac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i).$
- **eta-reg:** $I[1,1,1,1](\eta) \sim (0.0665971 - 0.101394i)\log(\eta) + 0.0250704 + 0.22933i.$
- **both:** $I[1,1,1,1](\eta) \sim \eta^{-\epsilon} f_1 + f_2 + \eta^{1/2-\epsilon} f_3,$

$$f_1 = \frac{-0.0665971 + 0.101394i}{\epsilon} + (0.0384409 - 0.0585265i),$$
$$f_2 = \frac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i),$$
$$f_3 = 0.1309.$$

- **take $\eta \to 0$ first, only f_2 survives;**
- **take $\epsilon \to 0$ first, $1/\epsilon$ cancels between f_1 and f_2**

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments: unitarity, recurrence relation, CHY, …

2. Calculate Feynman loop integrals (FIs)

- Amplitudes: linear combinations of FIs with rational coefficients

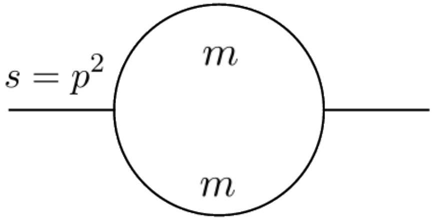
3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity (if no jet)

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\left(\frac{\mathrm{i}}{p^2+\mathrm{i}0^+} + \frac{-\mathrm{i}}{p^2-\mathrm{i}0^+}\right)$$

Differential equations: example

➢ Due to IBP: DEs of MIs w.r.t. $\vec{s}$



$$I_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)^{\nu_1}[(\ell+p)^2 - m^2]^{\nu_2}}$$

$$\begin{cases} \dfrac{\partial}{\partial m^2} I_{11} = I_{21} + I_{12} \overset{\text{IBP}}{=} \dfrac{2(D-3)}{4m^2 - s} I_{11} - \dfrac{D-2}{m^2(4m^2-s)} I_{10} \\ \dfrac{\partial}{\partial m^2} I_{10} = I_{20} \overset{\text{IBP}}{=} \dfrac{D-2}{2m^2} I_{10} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial s} I_{11} = \dfrac{p^\mu}{2s} \dfrac{\partial}{\partial p^\mu} I_{11} = -\dfrac{1}{2s} \displaystyle\int \dfrac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \dfrac{2(\ell+p)\cdot p}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]^2} \\ \qquad\quad = -\dfrac{sI_{12} + I_{11} - I_{02}}{2s} \overset{\text{IBP}}{=} a_{11} I_{11} + a_{10} I_{10} \\ \dfrac{\partial}{\partial s} I_{10} = 0 \end{cases}$$

➢ Boundary Condition

$$\begin{cases} I_{11}|_{m^2 \to 0} = (-s)^{D/2-2} \Gamma(2 - D/2) \dfrac{\Gamma(D/2-1)^2}{\Gamma(D-2)} \\ I_{10} \end{cases}$$

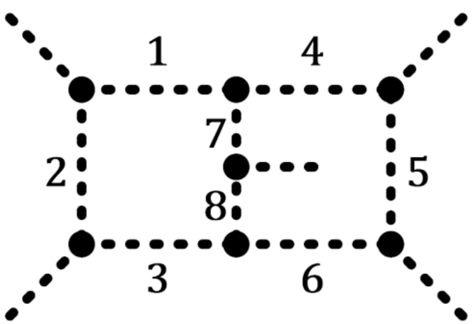
Set up DEs w.r.t. η

- η-DEs for MIs in auxiliary family using IBP

$$\frac{\partial}{\partial \eta}\vec{I}^{\text{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta)\vec{I}^{\text{aux}}(D,\vec{s},\eta)$$

- To minimize #MIs: usually the propagator mode

Liu, YQM, 2107.01864



Massless two-loop double-pentagon integrals (108 MIs)

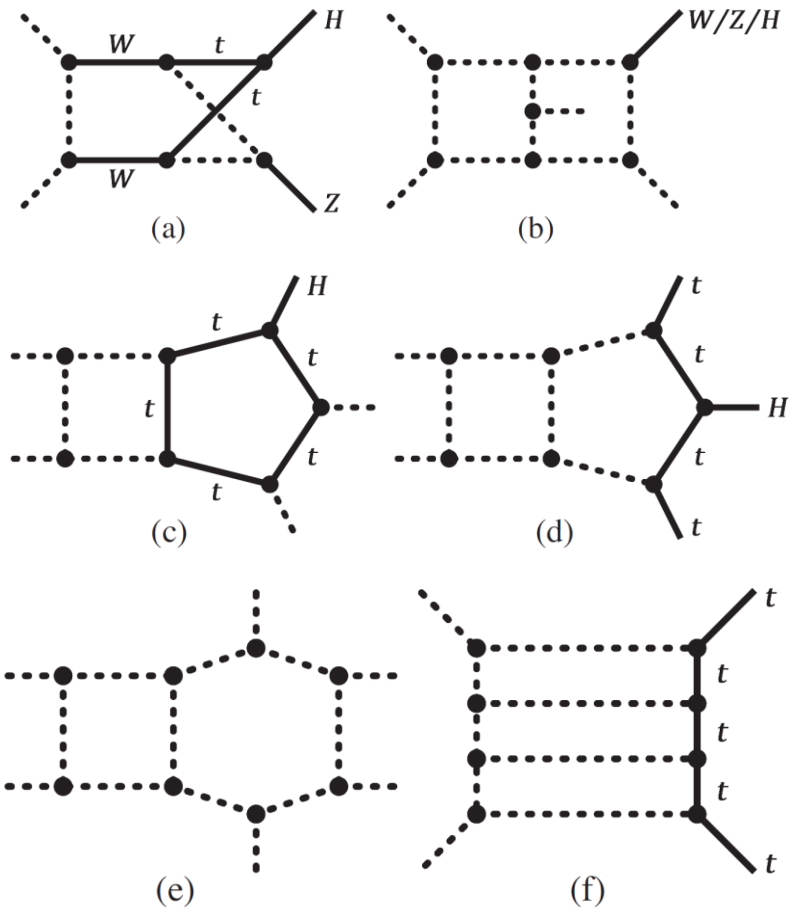
Mode	Propagators	Number of MIs
All	{1,2,3,4,5,6,7,8}	476
Loop	{4,5,6,7,8}	305
	{1,2,3,4,5,6}	319
Branch	{4,5,6}	233
	{7,8}	234
Propagator	{4}	178
	{5}	176
	{7}	220
Mass	...	...

- η-DEs are easier to set up if there are less MIs

η-DEs V.S. $\vec{s}$-DEs

Liu, YQM, 2107.01864

➢ Test for various cutting-edge problems



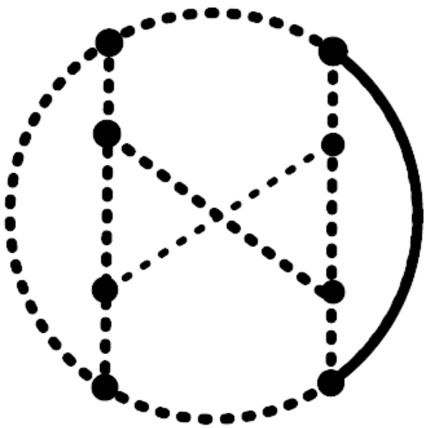
Family	dp	(a)	(b)	(c)	(d)	(e)	(f)
$T_{\eta\text{-DEs}}$	6	20	18	8	1	25	30
$T_{\vec{s}\text{-DEs}}$	2	916	64	1305	30	1801	63

Time to setup DEs (CPU core hours)

- Use propagator mode: easier to set up η-DEs for the auxiliary family than to set up $\vec{s}$-DEs for the original family!
- Differentiate with η: only increase power of denominator by one
- Differentiate with $\vec{s}$: increase powers of both numerator and denominator by one. Harder to do IBP reduction

A five-loop example

Liu, YQM, 2201.11637



$$= \begin{aligned}&- 2.07385551028674\epsilon^{-2} - 7.81275531259013\epsilon^{-1} \\ &- 17.2588286494587 + 717.680884549214\epsilon \\ &+ 8190.876448160049\epsilon^{2} + 78840.29598046500\epsilon^{3} \\ &+ 566649.1116484678\epsilon^{4} + 3901713.802716081\epsilon^{5} \\ &+ 23702384.71086095\epsilon^{6} + 142142936.8205112\epsilon^{7},\end{aligned}$$

- IBP relations are the only input!
- Terms up to $\mathcal{O}(\epsilon^4)$ agree with literature; Others are new ($D = 4 - 2\epsilon$)

  Lee, Smirnov, Smirnov, 1108.0732
- An arbitrary dimension D = 4/7, challenging for other methods

-9.79311209704864932180879598006911164642818254746542833061469472644431
5160318306100566682423418773094010322939010045743194940172060911582447082246541938856806619503723720902111961684999664025920163632 1*10^7

with about 130 significant digits

Difficulty of IBP reduction

➢ Solve IBP equations

Laporta's algorithm, 0102033

$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

- Very large scale of linear equations (can be billions of) E.g., Laporta 1910.01248
- Equations are coupled

× Explicit solution for multi-scale problem: hard to get, expression can be too large

× Numerical solution at each floating phase space point : too slow

➢ Cutting-edge problems

- Hundreds GB RAM
- Months of runtime using super computer

Solve IBP system over finite field

➢ Usage of FF is common in computer algebra

$$a^{-1} \equiv b \textbf{ mod } p \Leftrightarrow (ab) \equiv 1 \textbf{ mod } p$$

$$7 \equiv 2 \textbf{ mod } 5$$

$$2^{-1} \equiv 3 \textbf{ mod } 5$$

➢ A better way to solve IBP systems

Manteuffel, Schabinger, 1406.4513
FireFly: Klappert, Lange, 1904.00009
FiniteFlow: Peraro, 1905.08019

- Solving linear system numerically and then reconstruct analytical solution (using Chinese remainder theorem)
- Avoid intermediate expression swell
- It is now a standard technique in FIs reduction

Trim IBP system

➢ Remove irrelevant FIs

Gluza, Kajda, Kosower, 1009.0472
Schabinger, 1111.4220

- FIs with double propagator usually not show up in amplitude
- Can be removed by combining IBPs, constrained by syzygy equations

➢ Solving syzs using module intersection

- IBPs in Baikov representation. P: Baikov polynomial; z_i: denominator

Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{m} \frac{\partial}{\partial z_i}\left(a_i P^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_m^{\nu_m}}\right)$$

$$= \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{m} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-L-E-1}{2P} a_i \frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i}\right) P^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_m^{\nu_m}}$$

- Polynomials list $(a_1, \dots, a_m)$ forms a module (generalization of ideal)
- No dimensional shift, module M_1 from syzs: $\left(\sum_{i=1}^{m} a_i \frac{\partial P}{\partial z_i}\right) + bP = 0$
- No double propagators, module M_2 from syzs: $a_i = b_i z_i\,, \quad i = 1, \dots, k$
- Module intersection $M_1 \cap M_2$ calculable using algebraic geometry

Very promising. No publicly available code yet

Module reconstruction

Liu, Ma, 1801.10523, Guan, Liu, Ma, 1912.09294

➢ IBP system as a module

$$\sum_{\vec{\nu}'} Q^{\vec{\nu}jk}_{\vec{\nu}'}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

- Taking all FIs as bases, coefficient vectors form a module (different module from previous page)
- Need to know its Gröebner basis (or simplest generators) with polynomial ordering: position over term, degree ordered
- Result: block-triangular form, smallest polynomial degree

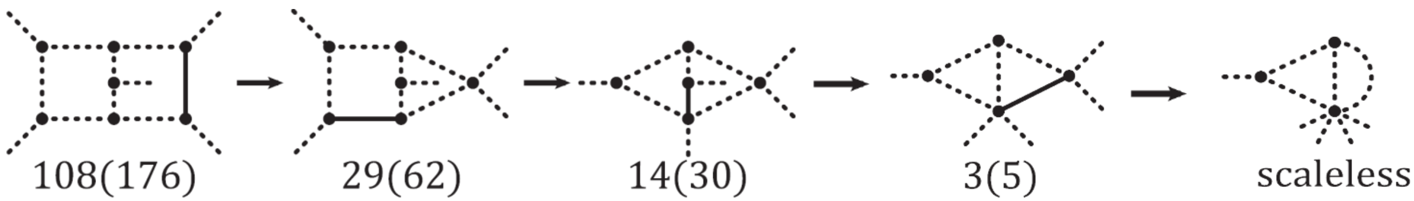
➢ Construct simplest generators

- Linear independent subset of Gröebner basis, minimal system
- Input linear system, e.g., from IBPs, trimmed IBPs, or other ways
- One method: sampling and fit. A public code will be released soon!

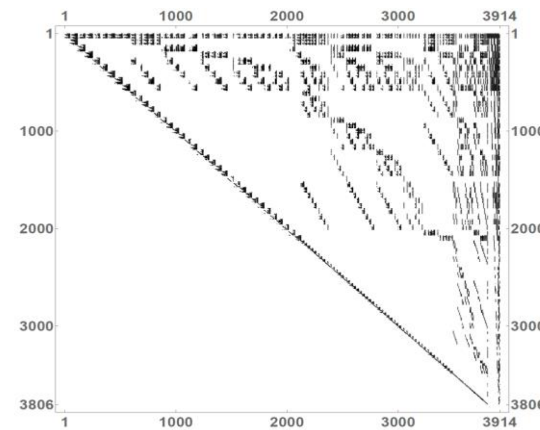
Application of module reconstruction

➢ Example: two-loop double-pentagon

Liu, YQM, 2107.01864



- Construct DEs: 3000 points
- Block-triangular system: 40 points



- Time =6h=(40*5s+3000*0.05s)*45+…
- Set DEs:90%; solve: 10%.
- New reduction strategy: 100× faster

➢ Typically faster by 2 orders of magnitude

Family	dp	(a)	(b)	(c)	(d)	(e)	(f)
T_{η}-DEs	6	20	18	8	1	25	30
$T_{\vec{s}}$-DEs	2	916	64	1305	30	1801	63

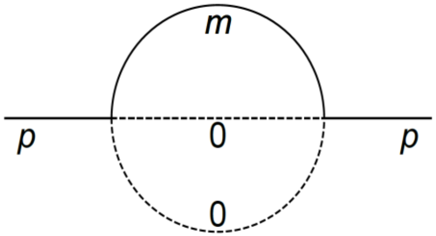
Time to setup DEs (CPU core hours)

Ways to bypass IBPs

- **$1/D$ expansion and matching**

Baikov, Chetyrkin, Kuhn, 0108197
Baikov, NPB (2003)
Baikov, 0507053

m=0: $I_{111} = -\left(\frac{4}{p^4}\right)^{-d/2}\left(1+\frac{13}{4d}+\frac{281}{32d^2}+\frac{2823}{128d^3}+\cdots\right)$



- **$1/\eta$ expansion and matching**

Guan, Liu, Ma, 1801.10523, 1912.09294
Wang, Li, Basat, 1901.09390, 2102.08225

$$I_{111}^{\rm aux} = \eta^{D-3}\left\{\left[\frac{(D-2)^2}{3D}\frac{p^2}{\eta}\right]I_{2,1}^{\rm bub} + \left[1+\frac{D-3}{3}\frac{m^2}{\eta} - \frac{(D+4)(D-3)}{9D}\frac{p^2}{\eta}\right]I_{2,2}^{\rm bub} + \mathcal{O}(\eta^{-2})\right\}$$

- **Intersection theory**

Frellesvig, et. al., 1901.11510, 1907.02000
Yang,..

- **FIs** $I_{a_1,a_2,\ldots,a_N} \equiv K\int_{\mathcal{C}} u\,\varphi \equiv K\,\langle\varphi|\mathcal{C}]_\omega$

$$\varphi \equiv \hat{\varphi}\, d^N\mathbf{z}\,, \qquad \hat{\varphi} \equiv \frac{1}{z_1^{a_1} z_2^{a_2}\cdots z_N^{a_N}}\,, \qquad d^N\mathbf{z} \equiv dz_1\wedge dz_2\wedge\cdots\wedge dz_N\,.$$

- **Intersection number** $\langle\varphi_L|\varphi_R\rangle_\omega = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}\big(\psi_p\,\varphi_R\big)$