Classification of the Near Threshold Behaviors for a Symmetrical Coupled-Channel System

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In cooperation with X.-Y. Zhang and F.-K. Guo.

Background

X.-K. Dong, F.-K. Guo, B.-S. Zou
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Two Coupled Channels: $\mathrm{CH1}: J/\Psi\pi^- \ (m_{1,1}=m_{J/\Psi}, m_{1,2}=m_\pi), \mathrm{CH2}: D^0D^{*-} \ (m_{2,1}=m_{D^0}, m_{2,2}=m_D), \ \Sigma_i = m_{i,1}+m_{i,2}$

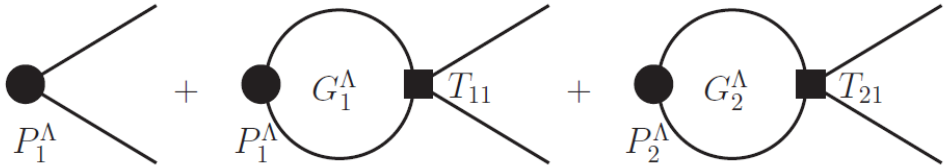
LO T-Matrix: $k_1 = \frac{1}{2\sqrt{s}}\sqrt{[s-(m_{1,1}+m_{12})^2][s-(m_{1,1}-m_{12})^2]} = \frac{1}{2\Sigma_2}\sqrt{\delta(\delta+2m_{1,1})(\delta+2m_{1,2})(\Sigma_1+\Sigma_2)}+\mathcal{O}(E) \quad \sqrt{s}=\Sigma_2+E$

$$\delta = \Sigma_2 - \Sigma_1$$

$$T(E) = 8\pi\Sigma_2\begin{pmatrix} -\frac{1}{a_{11}}+ik_1 & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & -\frac{1}{a_{22}}-\sqrt{-2\mu_2E-i\epsilon}\end{pmatrix}^{-1} = -\frac{8\pi\Sigma_2}{\det}\begin{pmatrix} \frac{1}{a_{22}}+\sqrt{-2\mu_2E-i\epsilon} & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & \frac{1}{a_{11}}-ik_1\end{pmatrix} \quad \det = (\frac{1}{a_{11}}-ik_1)(\frac{1}{a_{22}}+\sqrt{-2\mu_2E-i\epsilon})-\frac{1}{a_{12}^2}$$

$$\mu_i = \frac{m_{i,1}m_{i,2}}{\Sigma_i}$$

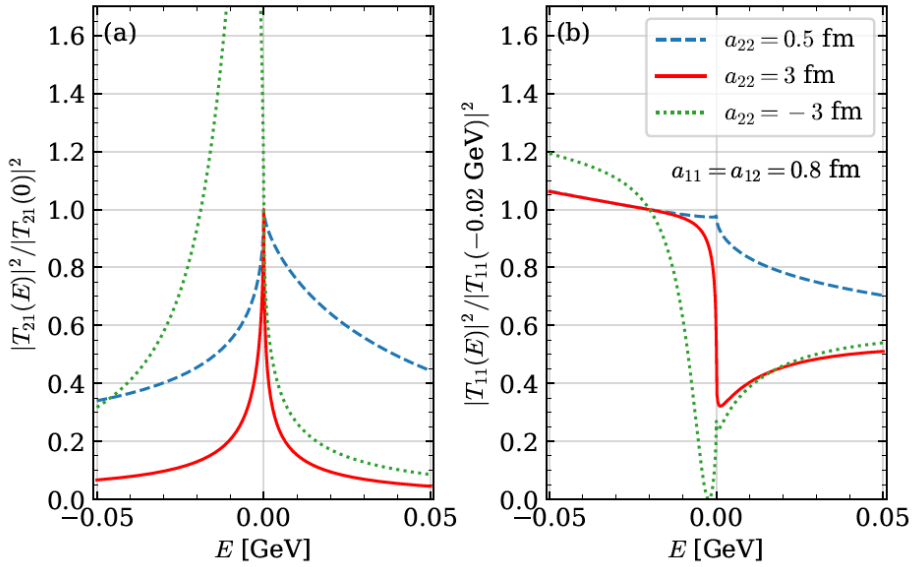
Production in CH1: $P_{1,\mathrm{tot}} = P_1T_{11}(E)+P_2T_{21}(E)$



Threshold Behaviors:

Case 1: $|P_1| \gg |P_2|, |T_{11}(E)|$ **driving**

Case 2: $|P_1| \ll |P_2|, |T_{21}(E)|$ **driving**



Three Coupled Channels with Symmetry

Three Coupled Channels: $\mathrm{CH1}: J/\Psi\pi^0(m_{1,1},m_{1,2}), \mathrm{CH2}: D^0\bar{D}^{*0}(m_{2,1},m_{2,2}), \mathrm{CH3}: D^+D^{*-}(m_{3,1},m_{3,2}), \Sigma_i = \sum_{j=1}^{3} m_{i,j}$

LO T-Matrix: $k_1 = \frac{1}{2\sqrt{s}}\sqrt{[s-(m_{1,1}+m_{12})^2][s-(m_{1,1}-m_{12})^2]} = \frac{1}{2\Sigma_2}\sqrt{\delta(\delta+2m_{1,1})(\delta+2m_{1,2})(\Sigma_1+\Sigma_2)} + \mathcal{O}(E) \quad \sqrt{s} = \Sigma_2 + E$

$$\mathbf{T}(E) = 8\pi\Sigma_2 \begin{pmatrix} -\frac{1}{a_{11}} + ik_1 & \frac{1}{a_{12}} & \frac{1}{a_{13}} \\ \frac{1}{a_{12}} & -\frac{1}{a_{22}} - \sqrt{-2\mu_2 E - i\epsilon} & \frac{1}{a_{23}} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & -\frac{1}{a_{33}} - \sqrt{-2\mu_3(E-\Delta) - i\epsilon} \end{pmatrix}^{-1} \quad \delta = \Sigma_2 - \Sigma_1 \quad \Delta = \Sigma_3 - \Sigma_2 \quad \mu_i = \frac{m_{i,1}m_{i,2}}{\Sigma_i}$$

$$\mathbf{T}(E) = \frac{8\pi\Sigma_2}{\det} \begin{pmatrix} \left[\frac{1}{a_{22}}+\kappa_2\right]\left[\frac{1}{a_{33}}+\kappa_3\right] - \frac{1}{a_{23}^2}, & \frac{1}{a_{12}}\left[\frac{1}{a_{33}}+\kappa_3\right] + \frac{1}{a_{13}a_{23}}, & \frac{1}{a_{13}}\left[\frac{1}{a_{22}}+\kappa_2\right] + \frac{1}{a_{12}a_{23}} \\ \frac{1}{a_{12}}\left[\frac{1}{a_{33}}+\kappa_3\right] + \frac{1}{a_{13}a_{23}}, & \left[\frac{1}{a_{11}}-ik_1\right]\left[\frac{1}{a_{33}}+\kappa_3\right] - \frac{1}{a_{13}^2}, & \frac{1}{a_{23}}\left[\frac{1}{a_{11}}-ik_1\right] + \frac{1}{a_{12}a_{13}} \\ \frac{1}{a_{13}}\left[\frac{1}{a_{22}}+\kappa_2\right] + \frac{1}{a_{12}a_{23}}, & \frac{1}{a_{23}}\left[\frac{1}{a_{11}}-ik_1\right] + \frac{1}{a_{12}a_{13}}, & \left[\frac{1}{a_{11}}-ik_1\right]\left[\frac{1}{a_{22}}+\kappa_2\right] - \frac{1}{a_{12}^2} \end{pmatrix}$$

symmetry

$a_{22} = a_{33},$

$a_{13} = \pm a_{12}$

$$\det = -\left[\frac{1}{a_{11}}-ik_1\right]\left[\frac{1}{a_{22}}+\kappa_2\right]\left[\frac{1}{a_{33}}+\kappa_3\right] + \frac{2}{a_{12}a_{13}a_{23}} + \left[\frac{1}{a_{11}}-ik_1\right]\frac{1}{a_{23}^2} + \left[\frac{1}{a_{22}}+\kappa_2\right]\frac{1}{a_{13}^2} + \left[\frac{1}{a_{33}}+\kappa_3\right]\frac{1}{a_{12}^2}$$

$$\kappa_2 = \sqrt{-2M_2E - i\epsilon} \equiv -ik_2$$

$$\kappa_3 = \sqrt{-2M_3(E-\Delta) - i\epsilon} \equiv -ik_3$$

Assumption and Simplification

Assume the couplings between CH1 and CH2,3 are weak: $|a_{12}|\sqrt{2M_3\Delta} \gg 1$

Reduced determinant and T matrix elements: $\det \simeq -\left[\frac{1}{a_{11}} - ik_1\right]\left\{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}\right\}$

$$\frac{\mathbf{T}_{11}(E)}{8\pi\Sigma_2} = \frac{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}{\det} \simeq -\frac{1}{\frac{1}{a_{11}} - ik_1}$$

$$\frac{\mathbf{T}_{22}(E)}{8\pi\Sigma_2} \simeq \frac{-\left[\frac{1}{a_{22}} + \kappa_3\right]}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{21}(E)}{8\pi\Sigma_2} = -\frac{1}{a_{12}\left[\frac{1}{a_{11}} - ik_1\right]} \frac{\frac{1}{a_{22}} \pm \frac{1}{a_{23}} + \kappa_3}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{32}(E)}{8\pi\Sigma_2} \simeq \frac{-\frac{1}{a_{23}}}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{31}(E)}{8\pi\Sigma_2} = -\frac{1}{\pm a_{12}\left[\frac{1}{a_{11}} - ik_1\right]} \frac{\frac{1}{a_{22}} \pm \frac{1}{a_{23}} + \kappa_2}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{[\mathbf{T}_{22} - \mathbf{T}_{32}](E)}{8\pi\Sigma_2} \simeq -\frac{\frac{1}{a_{22}} - \frac{1}{a_{23}} + \kappa_3}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{[\mathbf{T}_{21} - \mathbf{T}_{31}](E)}{8\pi\Sigma_2} = \begin{cases} \frac{\frac{1}{a_{12}}[\kappa_2 - \kappa_3]}{\det}, & a_{12} = a_{13}; \\ \frac{\frac{1}{a_{12}}[\frac{2}{a_{22}} - \frac{2}{a_{23}} + \kappa_2 + \kappa_3]}{\det}, & a_{12} = -a_{13} \end{cases}$$

$$\kappa_2 = \sqrt{-2M_2E - i\epsilon} \equiv -ik_2$$

$$\kappa_3 = \sqrt{-2M_3(E - \Delta) - i\epsilon} \equiv -ik_3$$

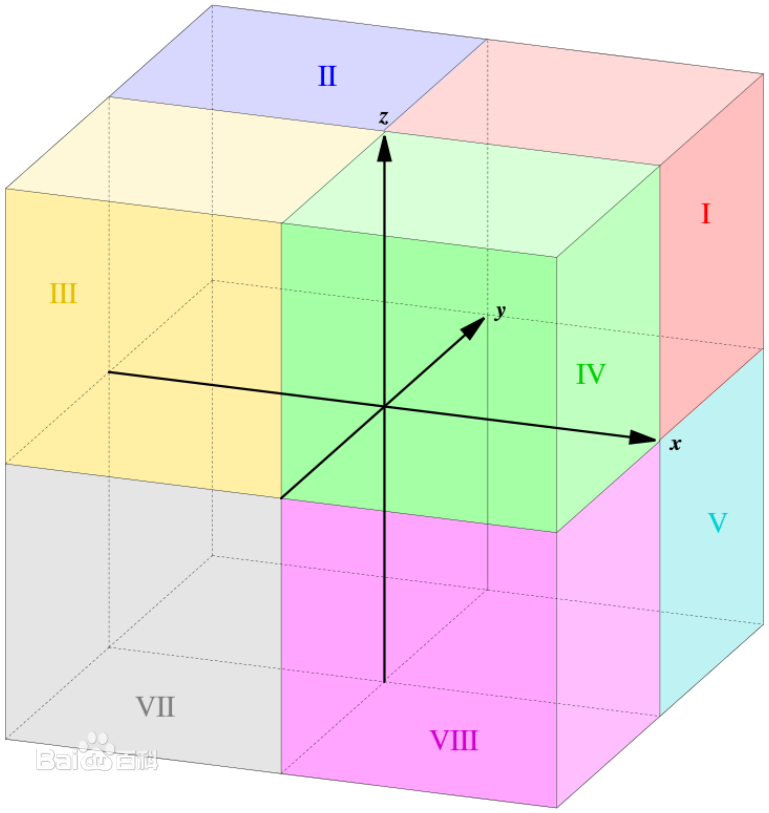
symmetry

$$a_{22} = a_{33},$$

$$a_{13} = \pm a_{12}$$

Labels of different Riemann sheets

Different Riemann sheets are labeled by the signs of Imk_1, Imk_2, Imk_3, and the labels are same as the labels of the octants



Labels of the octants

$\mathrm{sgn}[\mathrm{Im}k_1]$	$\mathrm{sgn}[\mathrm{Im}k_2]$	$\mathrm{sgn}[\mathrm{Im}k_3]$	Number of the Riemann sheet
$+$	$+$	$+$,	$1^{\mathrm{st}}(+++)$
$-$	$+$	$+$,	$2^{\mathrm{nd}}(-++)$
$-$	$-$	$+$,	$3^{\mathrm{rd}}(--+)$
$+$	$-$	$+$,	$4^{\mathrm{th}}(+-+)$
$+$	$+$	$-$,	$5^{\mathrm{th}}(++-)$
$-$	$+$	$-$,	$6^{\mathrm{th}}(-+-)$
$-$	$-$	$-$,	$7^{\mathrm{th}}(---)$
$+$	$-$	$-$,	$8^{\mathrm{th}}(+--)$

Pole Positions for Different a_{22} and $|a_{23}|$

The poles of the T matrix are the zeros of the determinant:

$$\det = -\left[\frac{1}{a_{11}} - ik_1\right]\left\{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}\right\} = 0$$

The pole positions depend on a_{22} and $|a_{23}|$

TABLE II. Pole positions for different a_{22} and $|a_{23}|$ near $E = 0$ and $E = \Delta$.

a_{22}	$\lvert a_{23}\rvert$	Pole position	
		Near $E = 0$	Near $E = \Delta$
$a_{22}\sqrt{2M_3\Delta} > 1$	$\lvert a_{23}\rvert \approx \lvert a_{22}\rvert$	$3^{\text{rd}}\text{RS}(--+)$, $4^{\text{th}}\text{RS}(+-+)$ $7^{\text{th}}\text{RS}(---)$, $8^{\text{th}}\text{RS}(+--)$	$5^{\text{th}}\text{RS}(++-)$, $6^{\text{th}}\text{RS}(-+-)$ $7^{\text{th}}\text{RS}(---)$, $8^{\text{th}}\text{RS}(+--)$
$0 < a_{22}\sqrt{2M_3\Delta} < 1$	$\lvert a_{23}\rvert \approx \lvert a_{22}\rvert$	$3^{\text{rd}}\text{RS}(--+)$, $4^{\text{th}}\text{RS}(+-+)$ $5^{\text{th}}\text{RS}(++-)$, $6^{\text{th}}\text{RS}(-+-)$	$5^{\text{th}}\text{RS}(++-)$, $6^{\text{th}}\text{RS}(-+-)$ $7^{\text{th}}\text{RS}(---)$, $8^{\text{th}}\text{RS}(+--)$
$a_{22}\sqrt{2M_3\Delta} < -1$	$\lvert a_{23}\rvert \approx \lvert a_{22}\rvert$	$2^{\text{nd}}\text{RS}(-++)$ $5^{\text{th}}\text{RS}(++-)$, $6^{\text{th}}\text{RS}(-+-)$	$2^{\text{nd}}\text{RS}(-++)$ $3^{\text{rd}}\text{RS}(--+)$, $4^{\text{th}}\text{RS}(+-+)$
$-1 < a_{22}\sqrt{2M_3\Delta} < 0$	$\lvert a_{23}\rvert \approx \lvert a_{22}\rvert$	$3^{\text{rd}}\text{RS}(--+)$, $4^{\text{th}}\text{RS}(+-+)$ $5^{\text{th}}\text{RS}(++-)$, $6^{\text{th}}\text{RS}(-+-)$	$2^{\text{nd}}\text{RS}(-++)$ $3^{\text{rd}}\text{RS}(--+)$, $4^{\text{th}}\text{RS}(+-+)$

$$|a_{22}| \approx |a_{23}|$$

Pole Positions for Different a_{22} and $|a_{23}|$

The poles of the T matrix are the zeros of the determinant:

$$\det = -\left[\frac{1}{a_{11}} - ik_1\right]\left\{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}\right\} = 0$$

The pole positions depend on a_{22} and $|a_{23}|$

TABLE I. Pole positions for different a_{22} and $|a_{23}|$ near $E = 0$ and $E = \Delta$.

a_{22}	$\lvert a_{23}\rvert$	Pole position	
		Near $E = 0$	Near $E = \Delta$
$\lvert a_{22}\rvert\sqrt{2M_3\Delta} \gg 1$	$\lvert a_{23}\rvert\sqrt{2M_3\Delta} < 1$	no pole	no pole
$0 < a_{22}\sqrt{2M_3\Delta} < 1$	$\lvert a_{23}\rvert\sqrt{2M_3\Delta} \gg 1$	$3^{\text{rd}}\text{RS}(--+), 4^{\text{th}}\text{RS}(+-+)$ $5^{\text{th}}\text{RS}(++-), 6^{\text{th}}\text{RS}(-+-)$ $7^{\text{th}}\text{RS}(---)$(two poles) $8^{\text{th}}\text{RS}(+--)$(two poles)	no pole
$-1 < a_{22}\sqrt{2M_3\Delta} < 0$	$\lvert a_{23}\rvert\sqrt{2M_3\Delta} \gg 1$	$2^{\text{nd}}\text{RS}(-++)$(two poles) $3^{\text{rd}}\text{RS}(--+), 4^{\text{th}}\text{RS}(+-+)$ $5^{\text{th}}\text{RS}(++-), 6^{\text{th}}\text{RS}(-+-)$	no pole

$|a_{22}| \gg |a_{23}|$

$|a_{22}| \ll |a_{23}|$

Threshold Behavior of the Line Shapes

The line shapes are classified by the pole positions and sgn[a_{23}]

Weak coupling limit and symmetry: $|a_{12}|\sqrt{2M_3\Delta} \gg 1, a_{22} = a_{23}, a_{12} = \pm a_{13}$

Reduced determinant and T matrix elements:

$$\det \simeq -\left[\frac{1}{a_{11}} - ik_1\right]\left\{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}\right\}$$

Case a: $a_{23} > 0$, **Case b:** $a_{23} < 0$; $(a_{12} = a_{13})$

Case a $\Leftrightarrow$ **Case b;** $(a_{12} = -a_{13})$

$$\frac{\mathbf{T}_{11}(E)}{8\pi\Sigma_2} = \frac{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}{\det} \simeq -\frac{1}{\frac{1}{a_{11}} - ik_1}$$

$$\frac{\mathbf{T}_{21}(E)}{8\pi\Sigma_2} = -\frac{1}{a_{12}\left[\frac{1}{a_{11}} - ik_1\right]}\frac{\frac{1}{a_{22}} \pm \frac{1}{a_{23}} + \kappa_3}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{31}(E)}{8\pi\Sigma_2} = -\frac{1}{\pm a_{12}\left[\frac{1}{a_{11}} - ik_1\right]}\frac{\frac{1}{a_{22}} \pm \frac{1}{a_{23}} + \kappa_2}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{22}(E)}{8\pi\Sigma_2} \simeq \frac{-\left[\frac{1}{a_{22}} + \kappa_3\right]}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{\mathbf{T}_{32}(E)}{8\pi\Sigma_2} \simeq \frac{-\frac{1}{a_{23}}}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{[\mathbf{T}_{22} - \mathbf{T}_{32}](E)}{8\pi\Sigma_2} \simeq -\frac{\frac{1}{a_{22}} - \frac{1}{a_{23}} + \kappa_3}{\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right] - \frac{1}{a_{23}^2}}$$

$$\frac{[\mathbf{T}_{21} - \mathbf{T}_{31}](E)}{8\pi\Sigma_2} = \begin{cases} \frac{\frac{1}{a_{12}}[\kappa_2 - \kappa_3]}{\det}, & a_{12} = a_{13}; \\ \frac{\frac{1}{a_{12}}[\frac{2}{a_{22}} - \frac{2}{a_{23}} + \kappa_2 + \kappa_3]}{\det}, & a_{12} = -a_{13} \end{cases}$$

$$\kappa_2 = \sqrt{-2M_2E - i\epsilon} \equiv -ik_2 \qquad \kappa_3 = \sqrt{-2M_3(E - \Delta) - i\epsilon} \equiv -ik_3$$

Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

Case 1: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are weak.

Threshold behaviors are similar to the two coupled channel case:

X.-K. Dong, F.-K. Guo, B.-S. Zou
Phys.Rev.Lett.126,152001(2021)

$$\frac{\mathbf{T}_{11}(E)}{8\pi\Sigma_2} \simeq -\frac{1}{\frac{1}{a_{11}} - ik_1} \qquad \det \simeq -\left[\frac{1}{a_{11}} - ik_1\right]\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right]$$

$$\frac{\mathbf{T}_{21}(E)}{8\pi\Sigma_2} \simeq -\frac{1}{a_{12}\left[\frac{1}{a_{11}} - ik_1\right]}\frac{1}{\left[\frac{1}{a_{22}} + \kappa_2\right]} \qquad \frac{\mathbf{T}_{22}(E)}{8\pi\Sigma_2} \simeq -\frac{1}{\left[\frac{1}{a_{22}} + \kappa_2\right]}$$

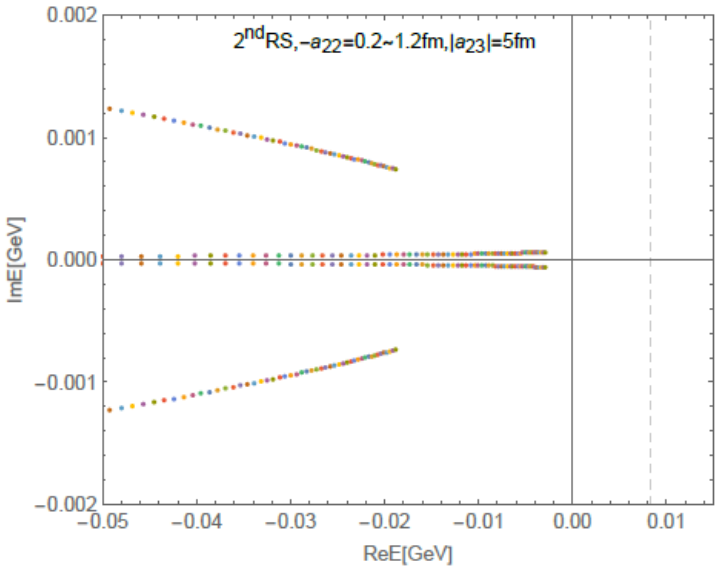
$$\frac{\mathbf{T}_{31}(E)}{8\pi\Sigma_2} \simeq -\frac{1}{a_{12}\left[\frac{1}{a_{11}} - ik_1\right]}\frac{1}{\left[\frac{1}{a_{22}} + \kappa_3\right]} \qquad \frac{\mathbf{T}_{32}(E)}{8\pi\Sigma_2} = -\frac{1}{a_{23}\left[\frac{1}{a_{22}} + \kappa_2\right]\left[\frac{1}{a_{22}} + \kappa_3\right]}$$

Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

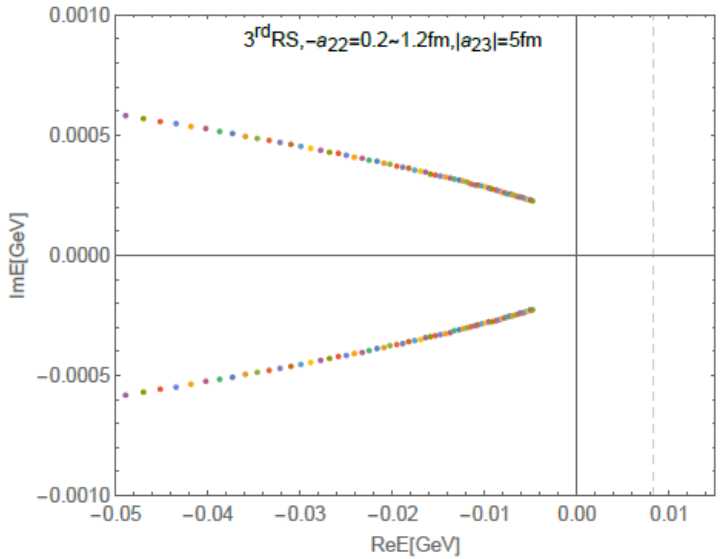
The pole trajectories for $-a_{22} = 0.2\sim1.2$ fm & $|a_{23}| = 5$ fm: $a_{11} = 0.6$ fm, $a_{12} = 5$ fm

The pole trajectories on the 2nd, 3rd, 6th Riemann sheet

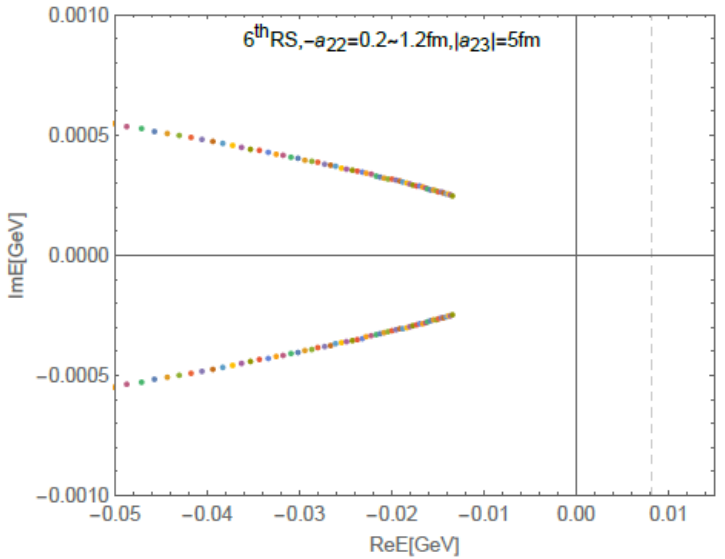
$T_1 = -0.64$ GeV, $T_2 = 0$ GeV,
$T_3 = \Delta = 0.0082$ GeV



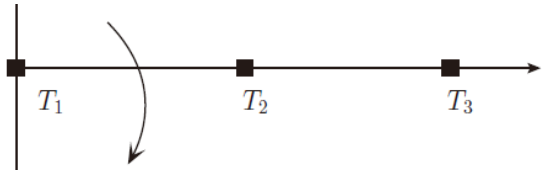
(a)Poles on the 2nd RS

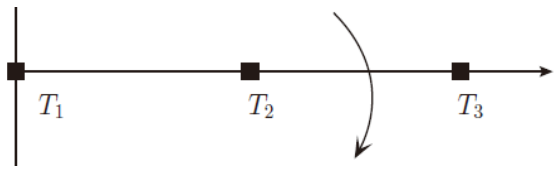


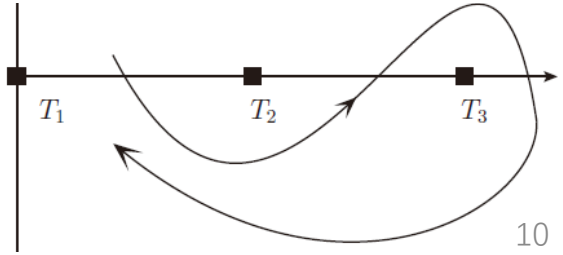
(b)Poles on the 3rd RS



(c)Poles on the 6th RS





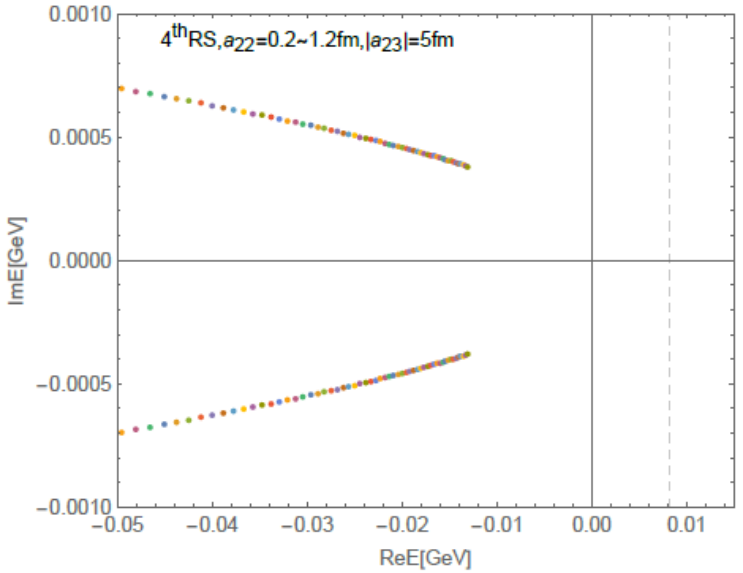


Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

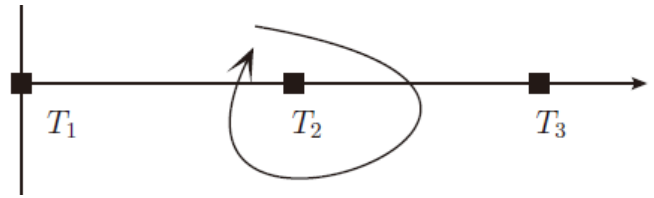
The pole trajectories for $a_{22} = 0.2\sim1.2$ fm & $|a_{23}| = 5$ fm: $a_{11} = 0.6$ fm, $a_{12} = 5$ fm

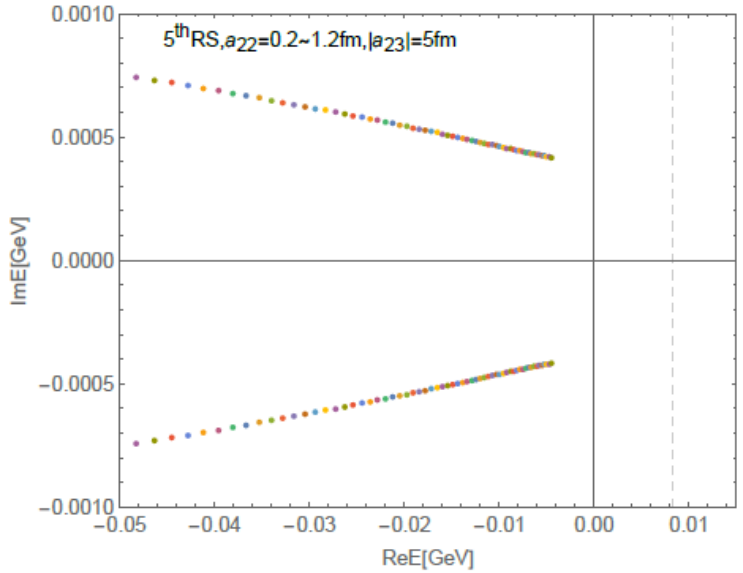
The pole trajectories on the 4th, 5th, 8th Riemann sheet

$T_1 = -0.64$ GeV, $T_2 = 0$ GeV,
$T_3 = \Delta = 0.0082$ GeV

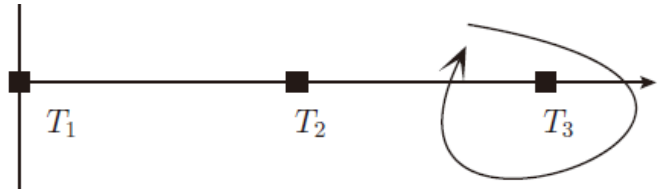


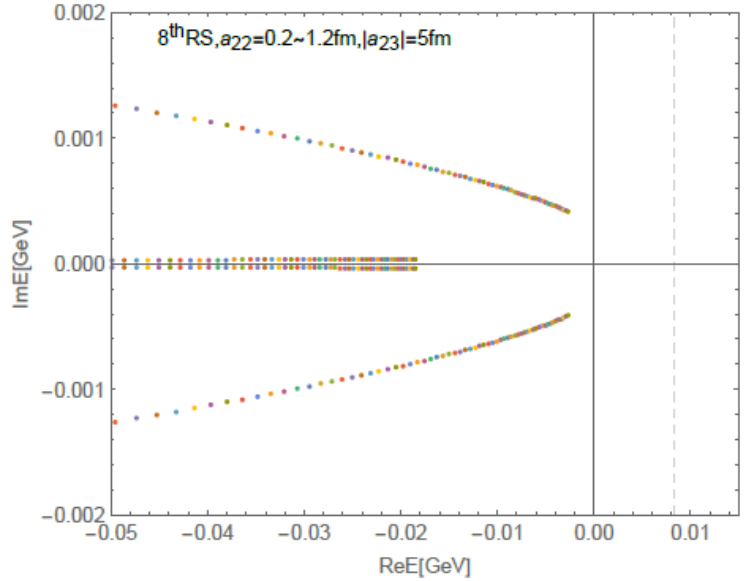
(a)Poles on the 4th RS



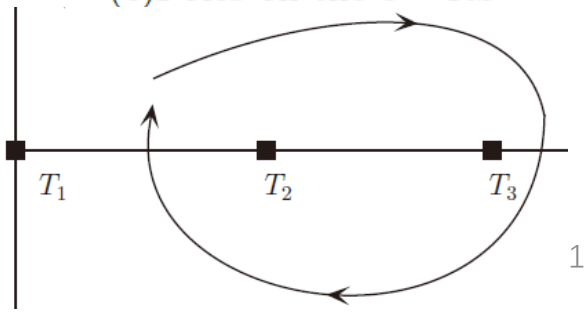


(b)Poles on the 5th RS





(c)Poles on the 8th RS



Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

Case 1:

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

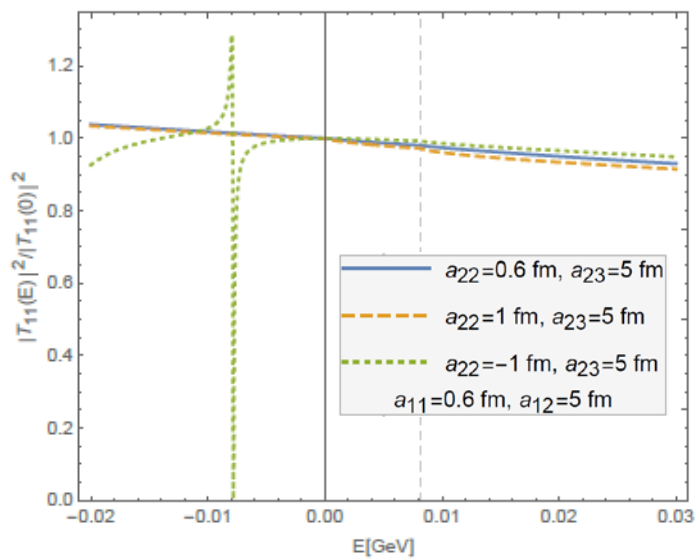
$a_{22} = 0.6$ fm, $a_{23} = 5$ fm

$a_{22} = 1$ fm, $a_{23} = 5$ fm

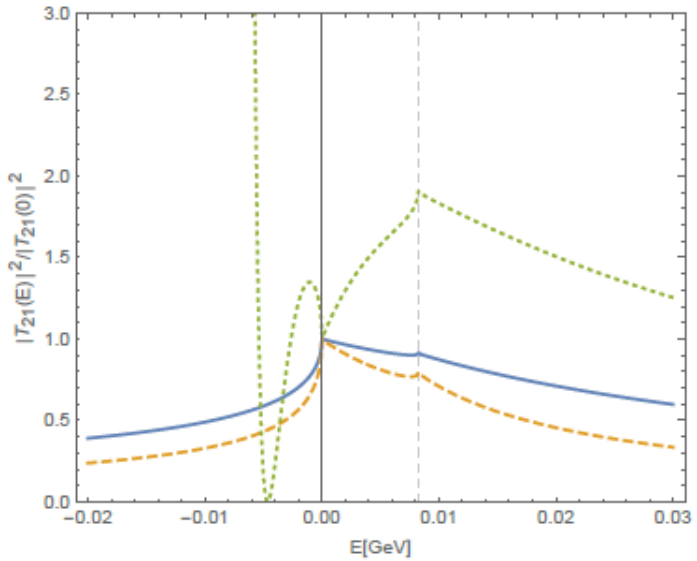
$a_{22} = -1$ fm, $a_{23} = 5$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)(the first row for $a_{22} < 0$ and the second row for $a_{22} > 0$) 2nd RS	3rd RS	6th RS
				8th RS	5th RS	4th RS
0.6	5	0.6	5	$-0.0666 \pm 0.0000i, -0.0380 \pm 0.0011$	$-0.0464 \pm 0.0007i$	$-0.0548 \pm 0.0007i$
0.6	5	1	5	$-0.0258 \pm 0.0000i, -0.0075 \pm 0.0006i$	$-0.0107 \pm 0.0005i$	$-0.0192 \pm 0.0004i$
0.6	5	-1	5	$-0.0262 \pm 0.0009i, -0.0078 \pm 0.0001i$	$-0.0111 \pm 0.0003i$	$-0.0196 \pm 0.0003i$

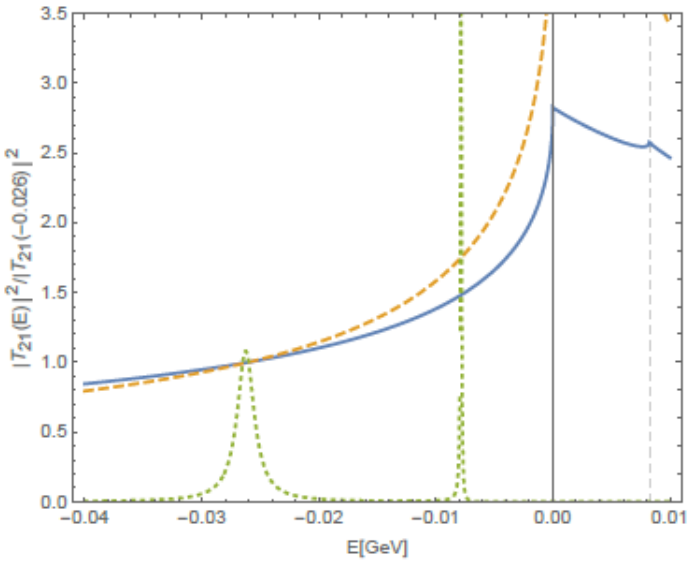
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(-0.026)|^2$

Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

Case 1:

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

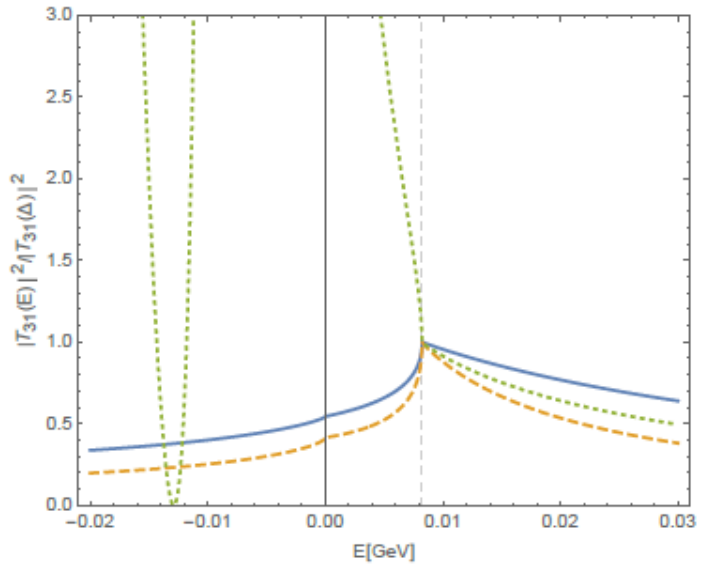
$a_{22} = 0.6$ fm, $a_{23} = 5$ fm

$a_{22} = 1$ fm, $a_{23} = 5$ fm

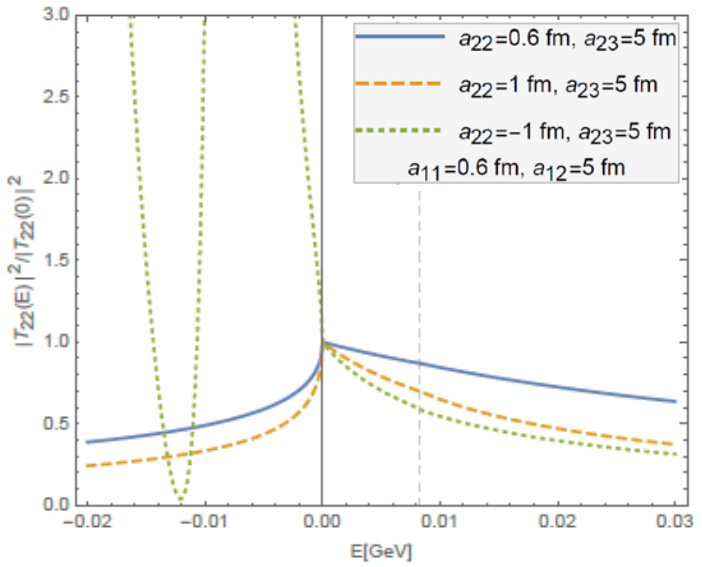
$a_{22} = -1$ fm, $a_{23} = 5$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)(the first row for $a_{22} < 0$ and the second row for $a_{22} > 0$) 2nd RS / 8th RS	3rd RS / 5th RS	6th RS / 4th RS
0.6	5	0.6	5	$-0.0666 \pm 0.0000i, -0.0380 \pm 0.0011$	$-0.0464 \pm 0.0007i$	$-0.0548 \pm 0.0007i$
0.6	5	1	5	$-0.0258 \pm 0.0000i, -0.0075 \pm 0.0006i$	$-0.0107 \pm 0.0005i$	$-0.0192 \pm 0.0004i$
0.6	5	-1	5	$-0.0262 \pm 0.0009i, -0.0078 \pm 0.0001i$	$-0.0111 \pm 0.0003i$	$-0.0196 \pm 0.0003i$

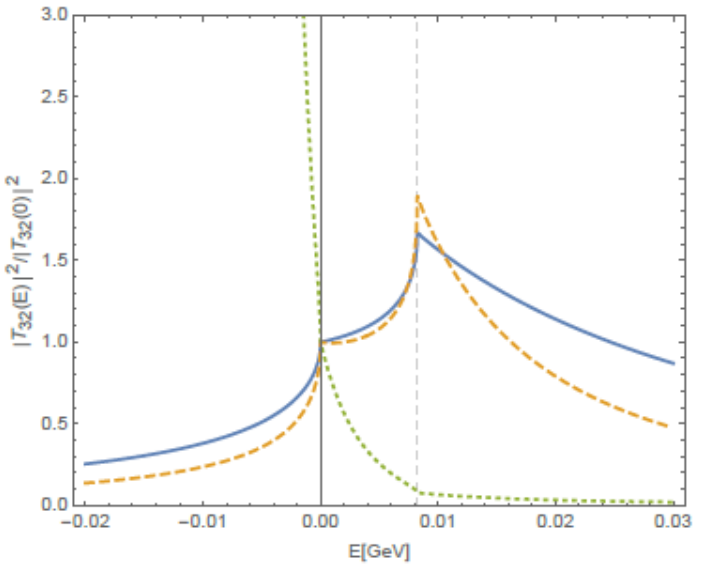
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$



(e)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



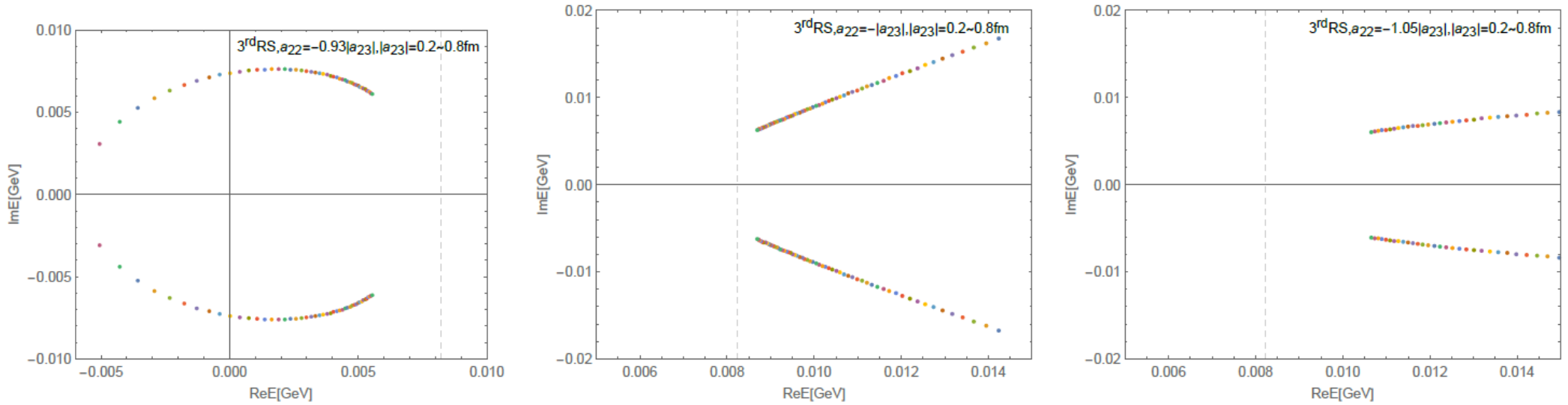
(f)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$

Threshold Behaviors Case 2: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2: Couplings in CH2 &CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 3rd Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

FIG. 13. Poles on the 3rdRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 2a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

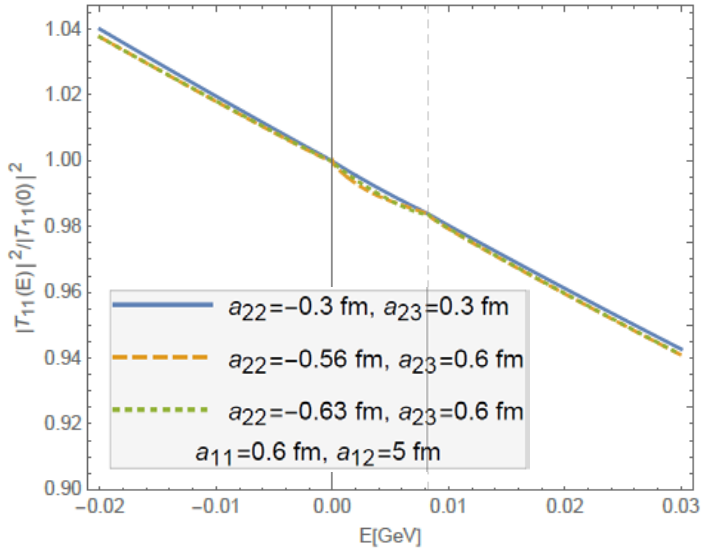
$a_{22} = -0.3$ fm, $a_{23} = 0.3$ fm

$a_{22} = -0.56$ fm, $a_{23} = 0.6$ fm

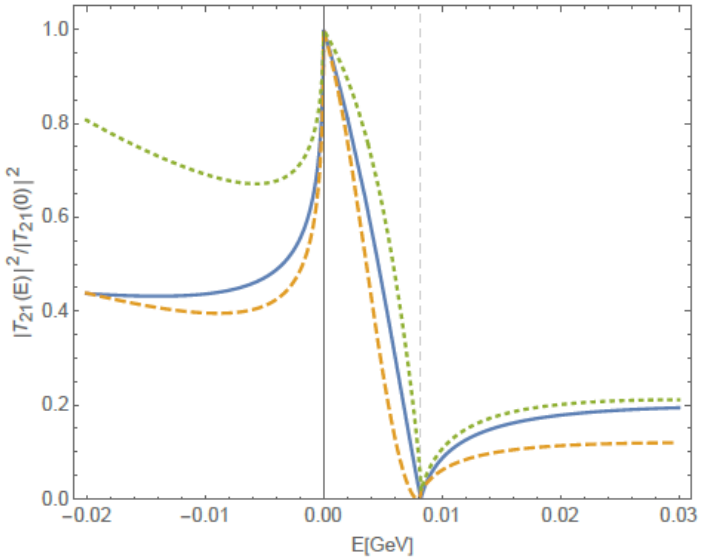
$a_{22} = -0.63$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				3rd RS	4th RS	6th RS
0.6	5	−0.3	0.3	$0.0122 \pm 0.0131i$	$0.0121 \pm 0.0131i$	$-0.0120 \pm 0.0000i$
0.6	5	−0.56	0.6	$0.0042 \pm 0.0073i$	$0.0041 \pm 0.0073i$	$-0.0128 \pm 0.0001i$
0.6	5	−0.63	0.6	$0.0130 \pm 0.0075i$	$0.0129 \pm 0.0076i$	$-0.0033 \pm 0.0000i$

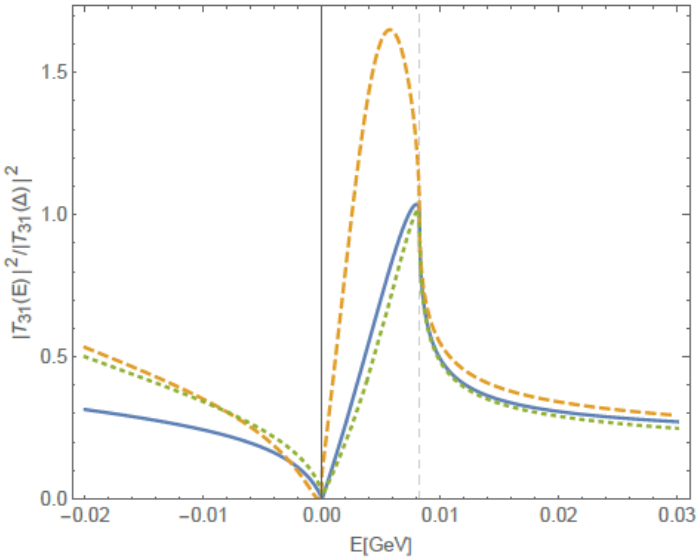
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Threshold Behaviors Case 2a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

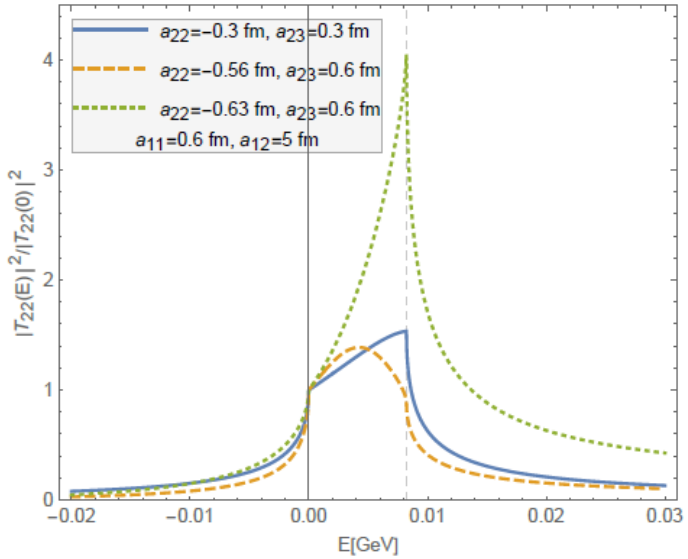
$a_{22} = -0.3$ fm, $a_{23} = 0.3$ fm

$a_{22} = -0.56$ fm, $a_{23} = 0.6$ fm

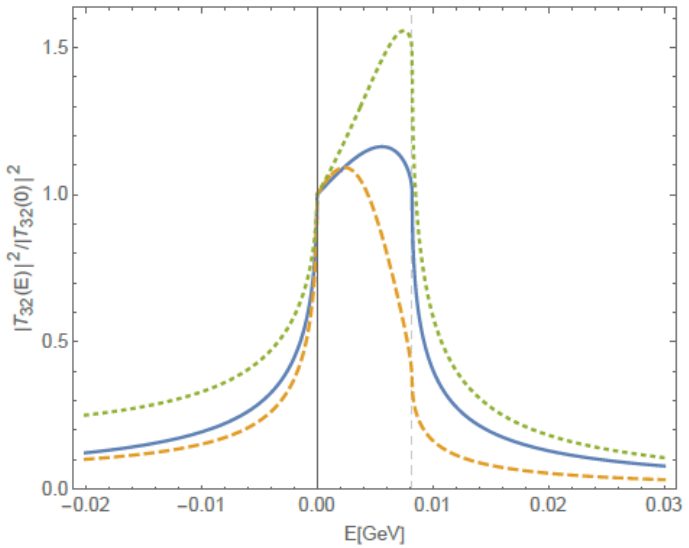
$a_{22} = -0.63$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV) 3rd RS	4th RS	6th RS
0.6	5	−0.3	0.3	$0.0122 \pm 0.0131i$	$0.0121 \pm 0.0131i$	$-0.0120 \pm 0.0000i$
0.6	5	−0.56	0.6	$0.0042 \pm 0.0073i$	$0.0041 \pm 0.0073i$	$-0.0128 \pm 0.0001i$
0.6	5	−0.63	0.6	$0.0130 \pm 0.0075i$	$0.0129 \pm 0.0076i$	$-0.0033 \pm 0.0000i$

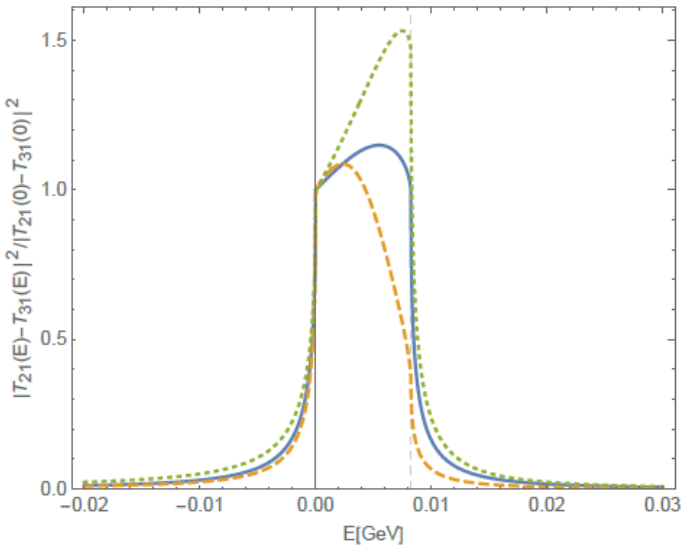
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 2b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

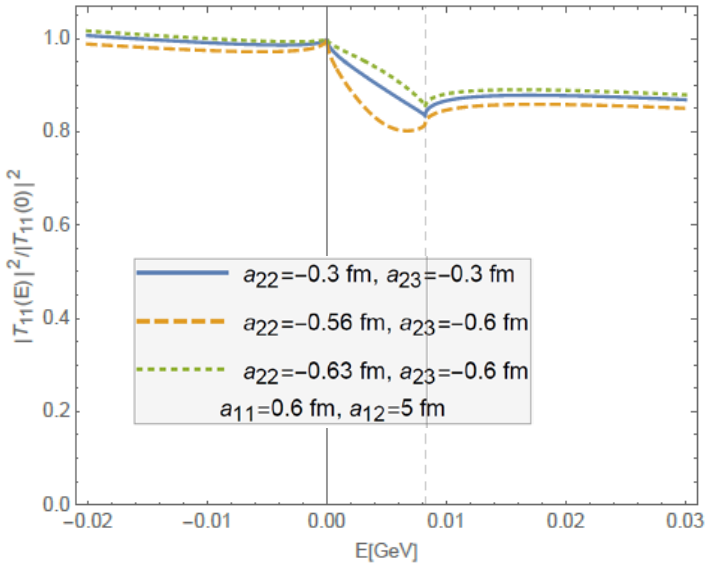
$a_{22} = -0.3$ fm, $a_{23} = -0.3$ fm

$a_{22} = -0.56$ fm, $a_{23} = -0.6$ fm

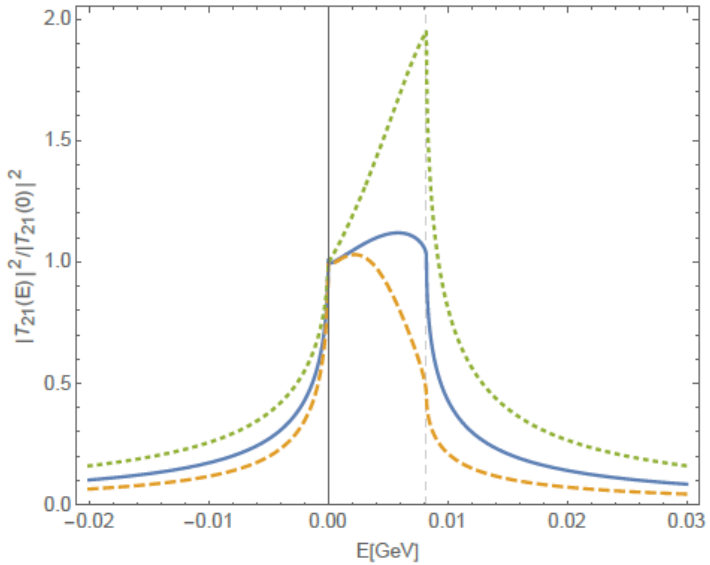
$a_{22} = -0.63$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				3rd RS	4th RS	6th RS
0.6	5	−0.3	0.3	$0.0122 \pm 0.0131i$	$0.0121 \pm 0.0131i$	$-0.0120 \pm 0.0000i$
0.6	5	−0.56	0.6	$0.0042 \pm 0.0073i$	$0.0041 \pm 0.0073i$	$-0.0128 \pm 0.0001i$
0.6	5	−0.63	0.6	$0.0130 \pm 0.0075i$	$0.0129 \pm 0.0076i$	$-0.0033 \pm 0.0000i$

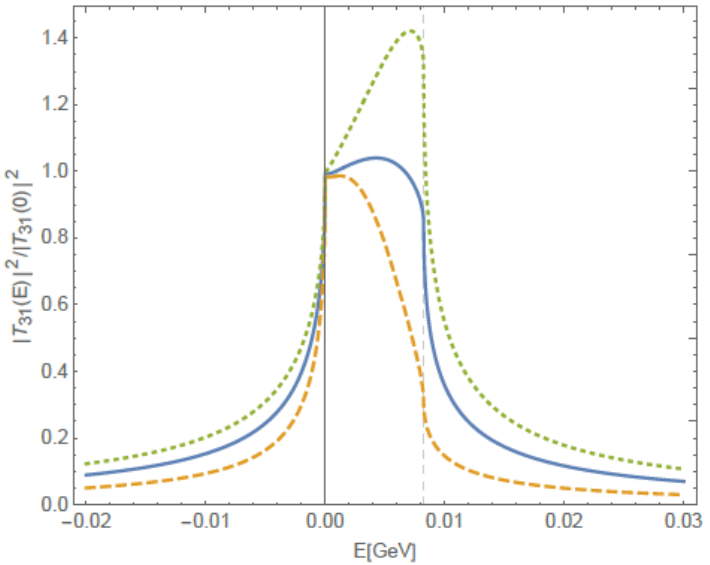
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



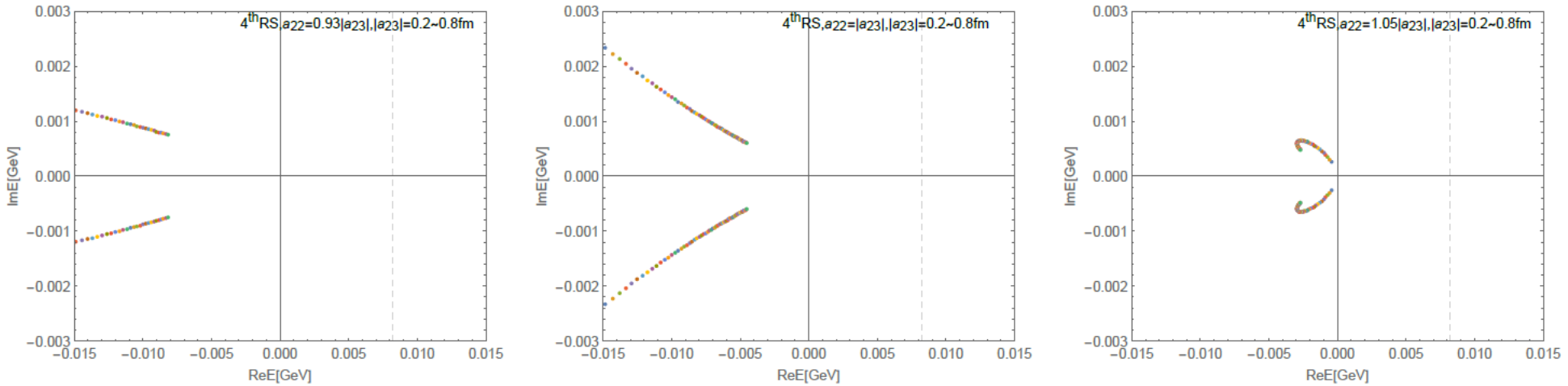
(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 3: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

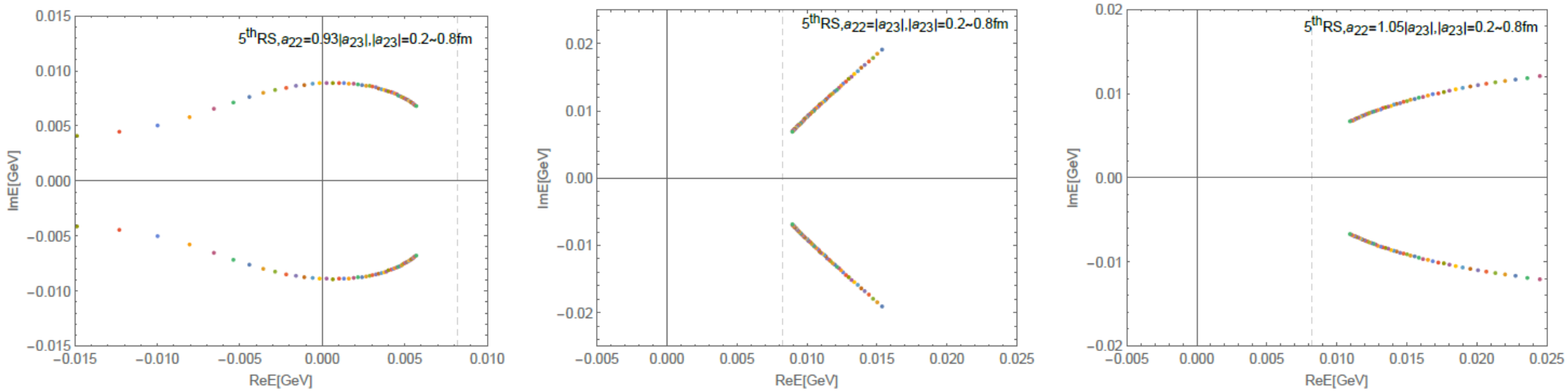
FIG. 18. Poles on the 4thRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 3: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 5th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 19. Poles on the 5thRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 3a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

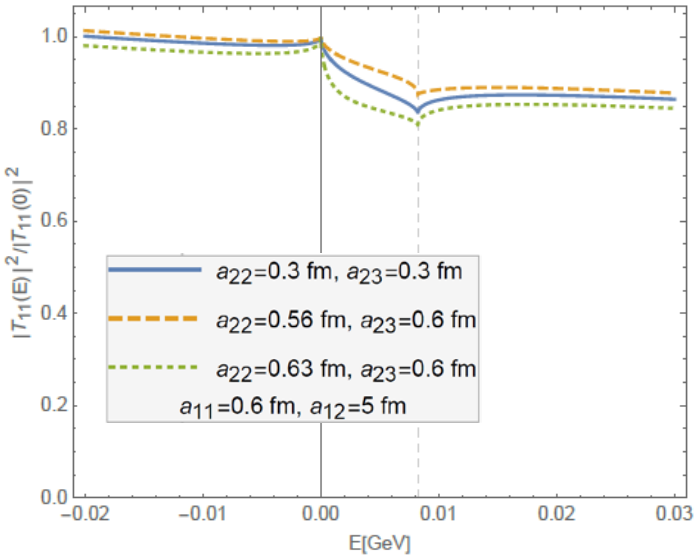
$a_{22} = 0.3$ fm, $a_{23} = 0.3$ fm

$a_{22} = 0.56$ fm, $a_{23} = 0.6$ fm

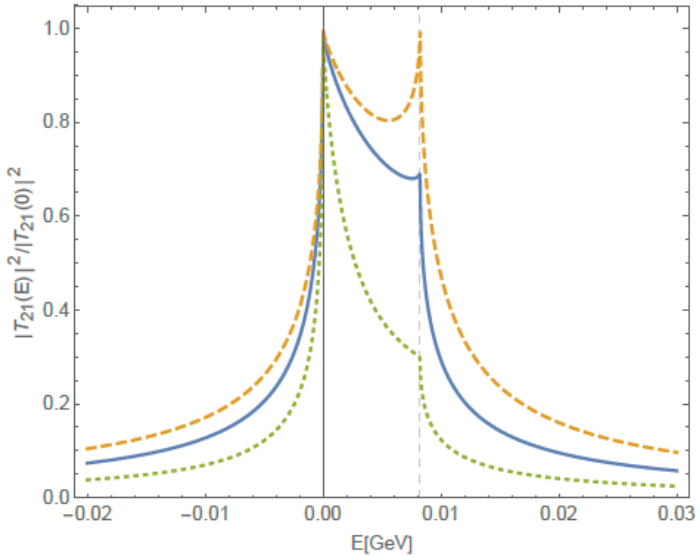
$a_{22} = 0.63$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				4^{th} RS	5^{th} RS	6^{th} RS
0.6	5	0.3	0.3	$-0.0111 \pm 0.0016i$	$0.0130 \pm 0.0147i$	$0.0129 \pm 0.0114i$
0.6	5	0.56	0.6	$-0.0123 \pm 0.0010i$	$0.0043 \pm 0.0082i$	$0.0048 \pm 0.0066i$
0.6	5	0.63	0.6	$-0.0029 \pm 0.0006i$	$0.0135 \pm 0.0084i$	$0.0133 \pm 0.0066i$

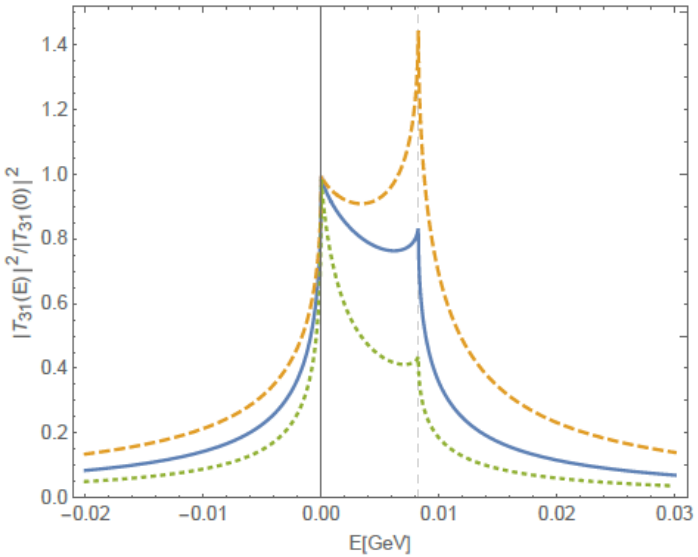
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 3a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

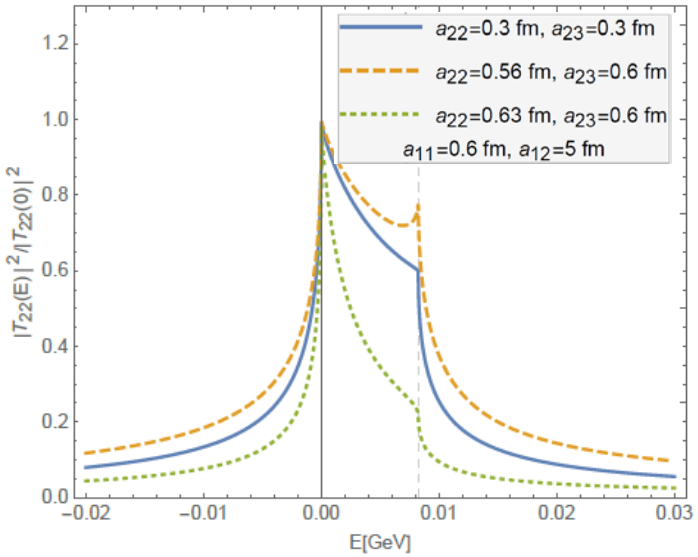
$a_{22} = 0.3$ fm, $a_{23} = 0.3$ fm

$a_{22} = 0.56$ fm, $a_{23} = 0.6$ fm

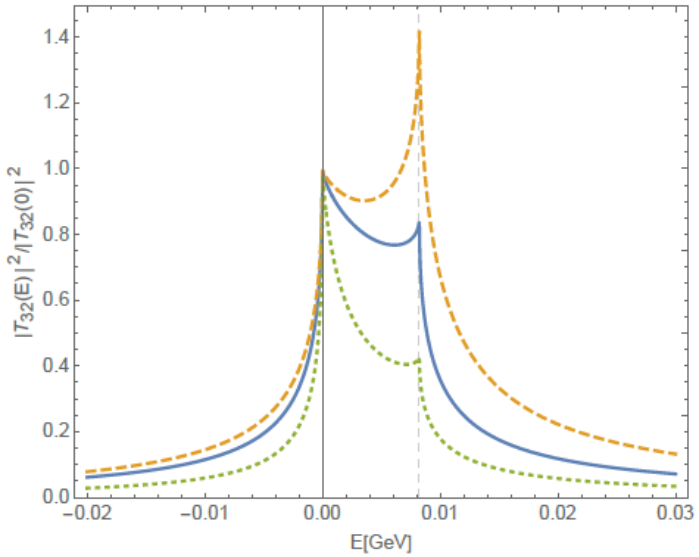
$a_{22} = 0.63$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				4^{th} RS	5^{th} RS	6^{th} RS
0.6	5	0.3	0.3	$-0.0111 \pm 0.0016i$	$0.0130 \pm 0.0147i$	$0.0129 \pm 0.0114i$
0.6	5	0.56	0.6	$-0.0123 \pm 0.0010i$	$0.0043 \pm 0.0082i$	$0.0048 \pm 0.0066i$
0.6	5	0.63	0.6	$-0.0029 \pm 0.0006i$	$0.0135 \pm 0.0084i$	$0.0133 \pm 0.0066i$

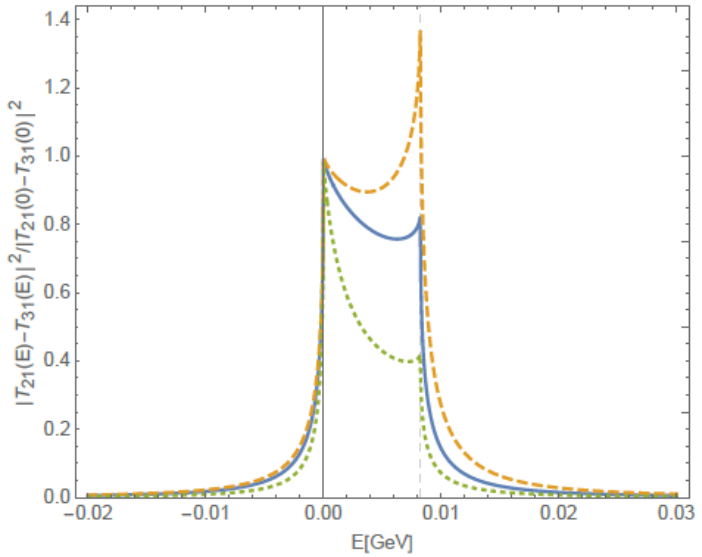
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 3b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

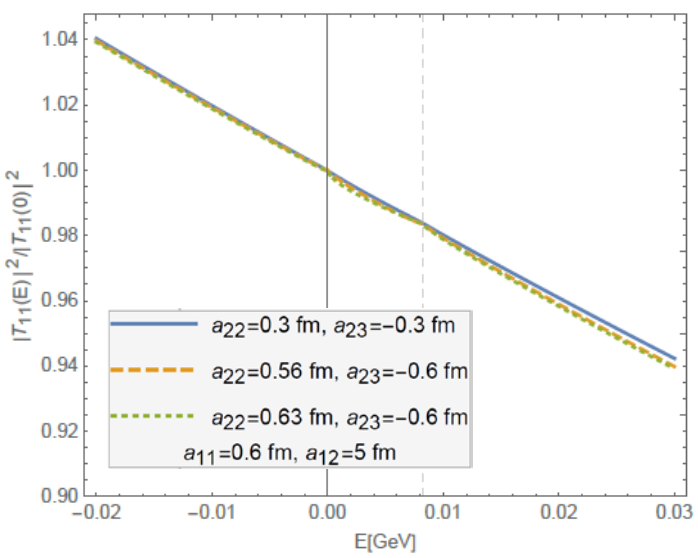
$a_{22} = 0.3$ fm, $a_{23} = -0.3$ fm

$a_{22} = 0.56$ fm, $a_{23} = -0.6$ fm

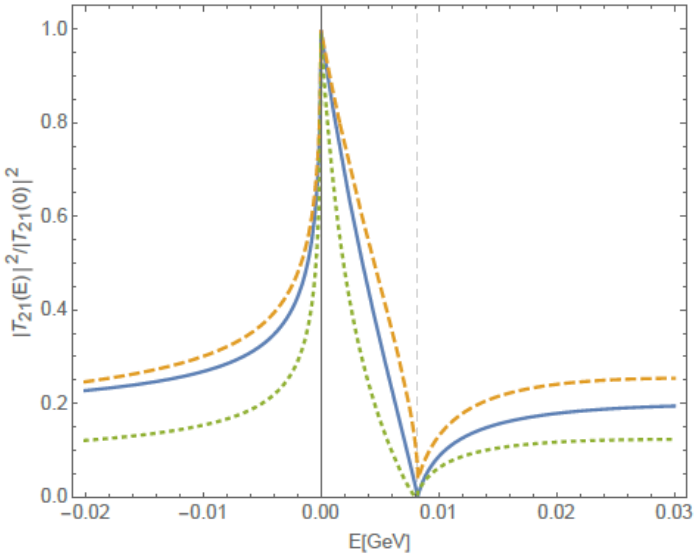
$a_{22} = 0.63$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				4^{th} RS	5^{th} RS	6^{th} RS
0.6	5	0.3	0.3	$-0.0111 \pm 0.0016i$	$0.0130 \pm 0.0147i$	$0.0129 \pm 0.0114i$
0.6	5	0.56	0.6	$-0.0123 \pm 0.0010i$	$0.0043 \pm 0.0082i$	$0.0048 \pm 0.0066i$
0.6	5	0.63	0.6	$-0.0029 \pm 0.0006i$	$0.0135 \pm 0.0084i$	$0.0133 \pm 0.0066i$

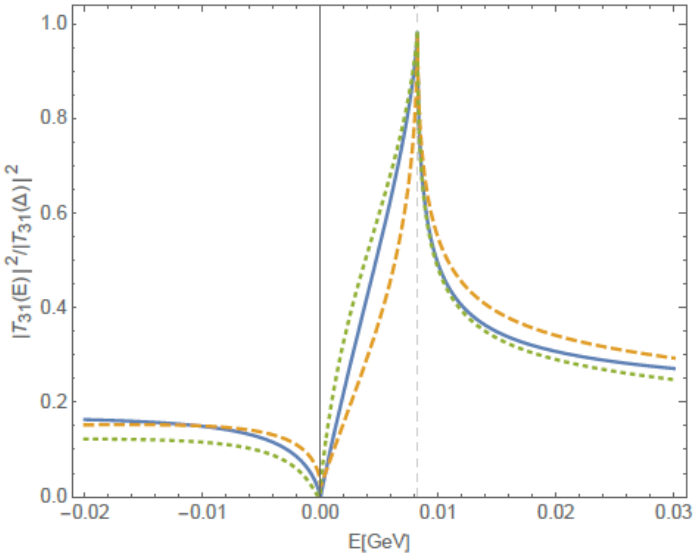
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Threshold Behaviors Case 4: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \gg 1$

Case 4: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are strong.

Take three groups of a_{22}&$|a_{23}|$ and there are no pole near the thresholds:

TABLE IX. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 4. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\|a_{23}\|$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

Threshold Behaviors Case 4a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \gg 1$

Case 4a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

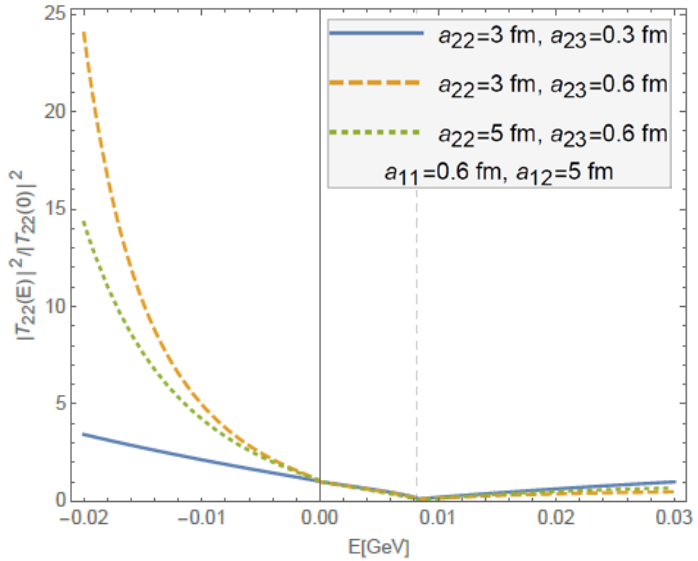
$a_{22} = 3$ fm, $a_{23} = 0.3$ fm

$a_{22} = 3$ fm, $a_{23} = 0.6$ fm

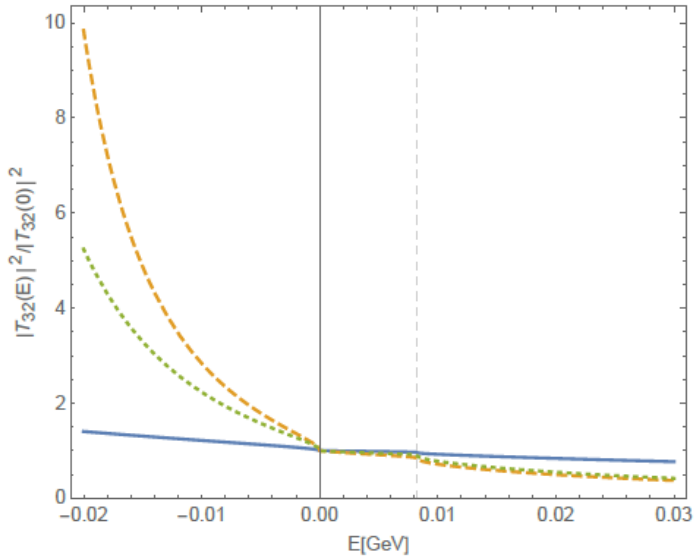
$a_{22} = 5$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

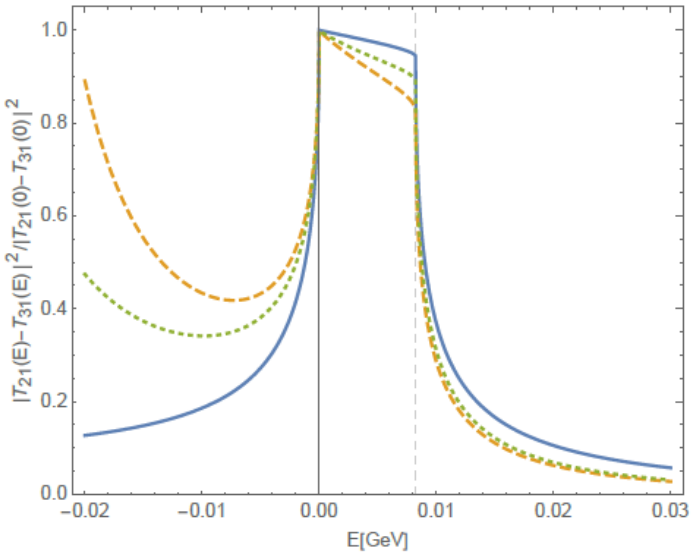
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Summary

- **Classification of the near threshold behaviors of the line shapes in a kind of symmetrical-coupled-channel system ($a_{22} = a_{23}, a_{12} = \pm a_{13}$) in a weak coupling limit $|a_{12}|\sqrt{2M_3\Delta} \gg 1$ in the ZREFT framework.**
- **$|T_{21} - T_{31}|$ have peaks in their line shapes even there is no near threshold pole.**
- **The poles on the 4th and 5th Riemann sheets also have observable effect on the line shapes.**

Thank you for your attention!

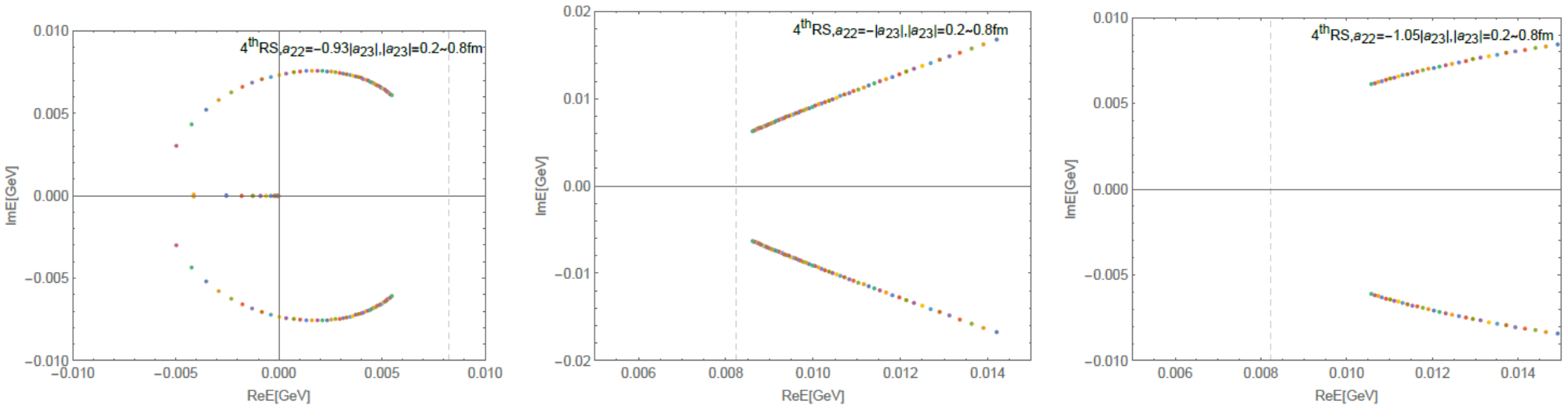
Back up

Threshold Behaviors Case 2: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2: Couplings in CH2 &CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

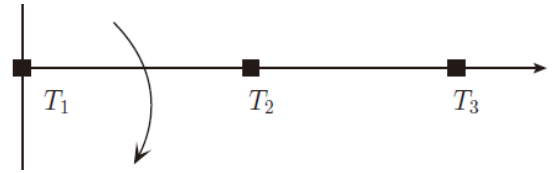
The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

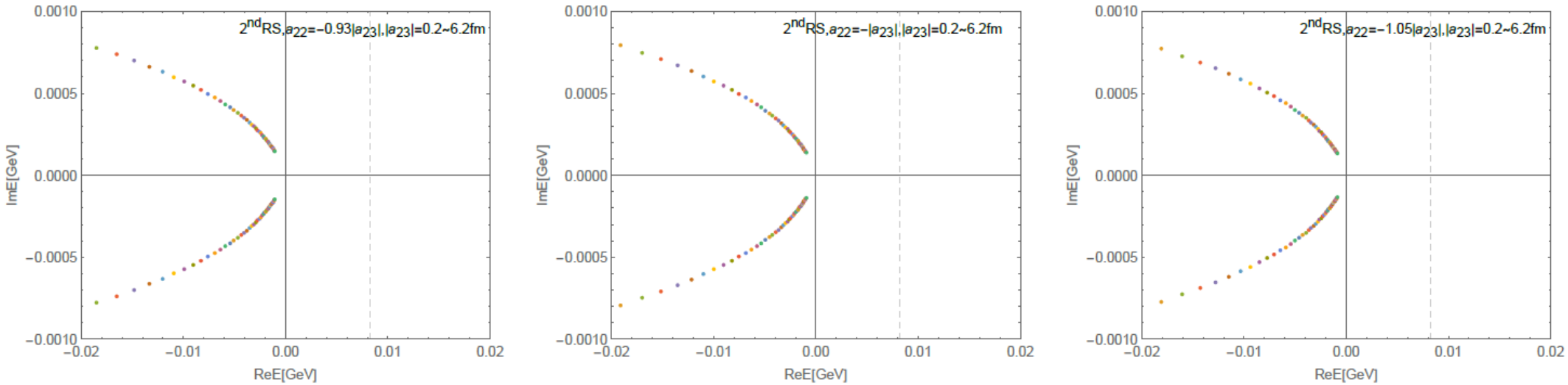
FIG. 14. Poles on the 4thRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2 \sim 6.2$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

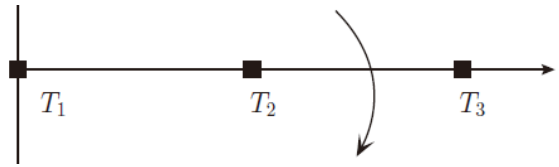
The pole trajectories on the 2nd Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

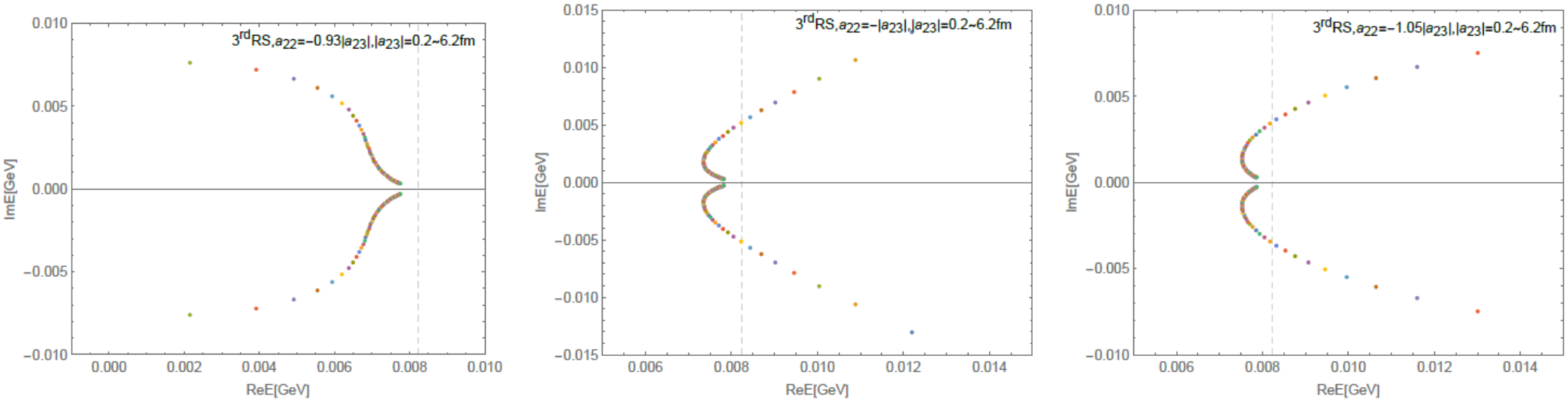
FIG. 3. Poles on the 2ndRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2 \sim 6.2$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

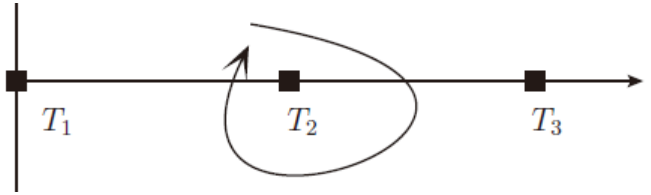
The pole trajectories on the 3rd Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

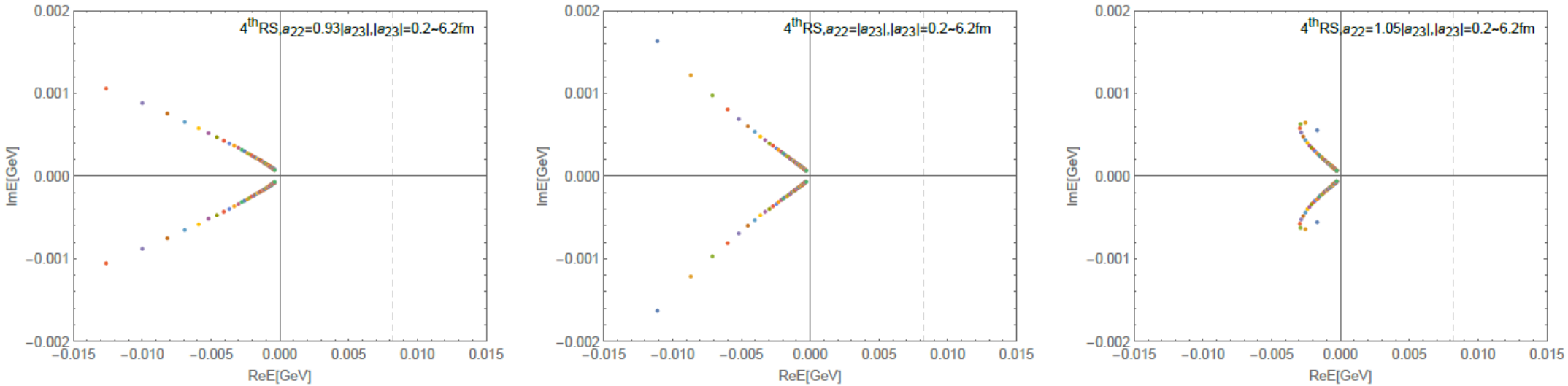
FIG. 4. Poles on the 3rdRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3. The transition of the pole positions as $|a_{22}|/|a_{23}|$ become larger is continuous.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2 \sim 6.2$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



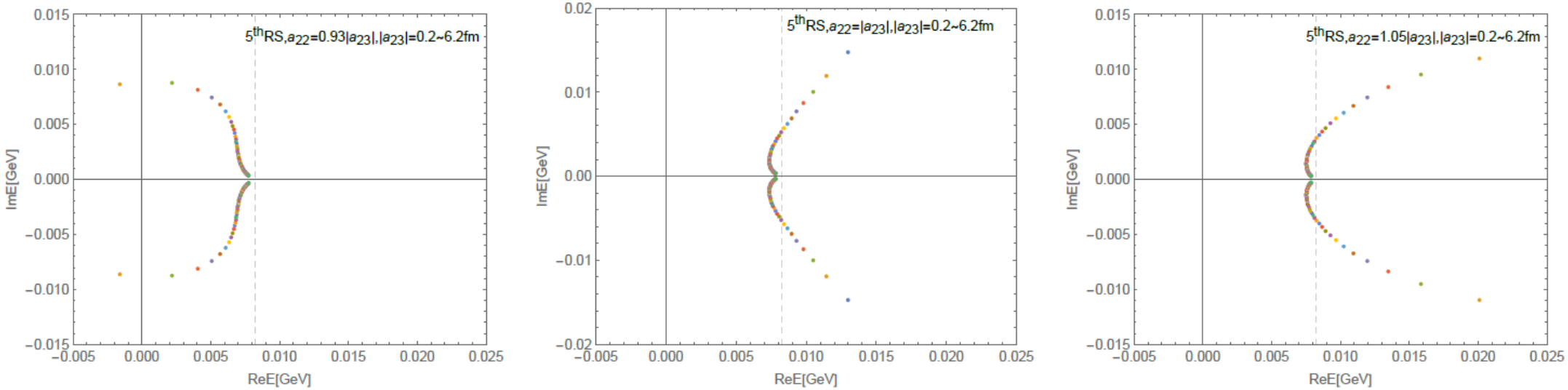
(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 7. Poles on the 4thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Pole Positions for Different a_{22} and $|a_{23}|$

The pole trajectories for $|a_{23}| = 0.2\sim6.2$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 5th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 8. Poles on the 5thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3. The transition of the pole positions as $|a_{22}|/|a_{23}|$ become larger is continuous.

Threshold Behaviors Case 4a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \gg 1$

Case 4a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

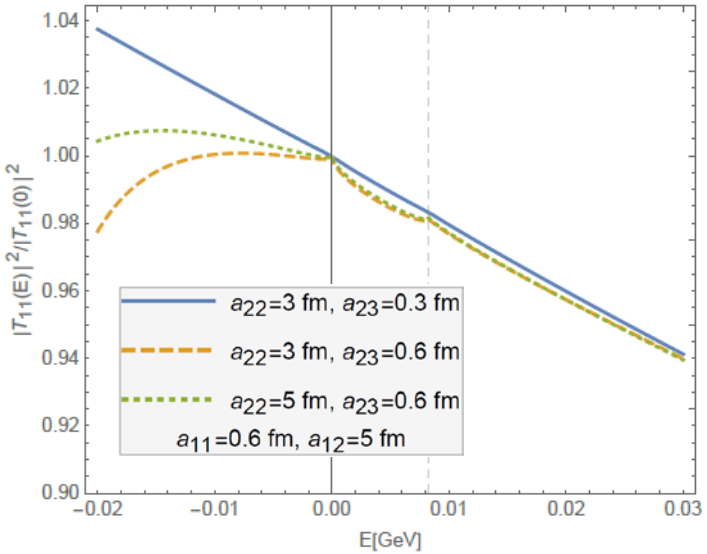
$a_{22} = 3$ fm, $a_{23} = 0.3$ fm

$a_{22} = 3$ fm, $a_{23} = 0.6$ fm

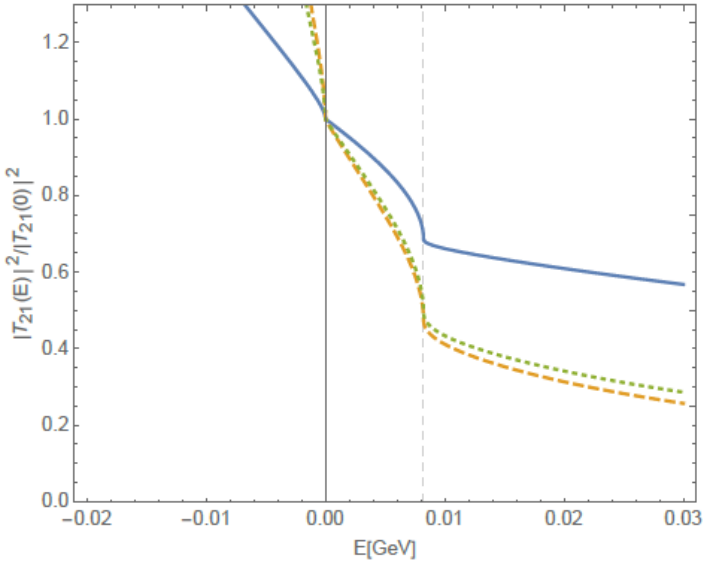
$a_{22} = 5$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

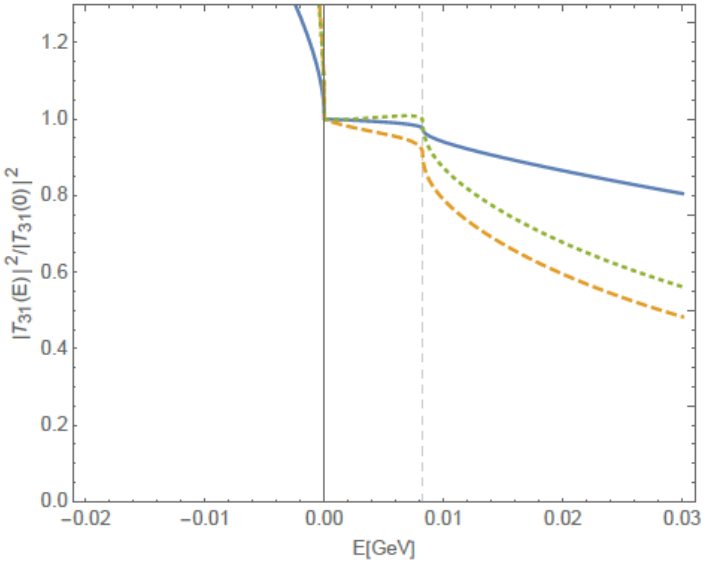
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 4b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \gg 1$

Case 4b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

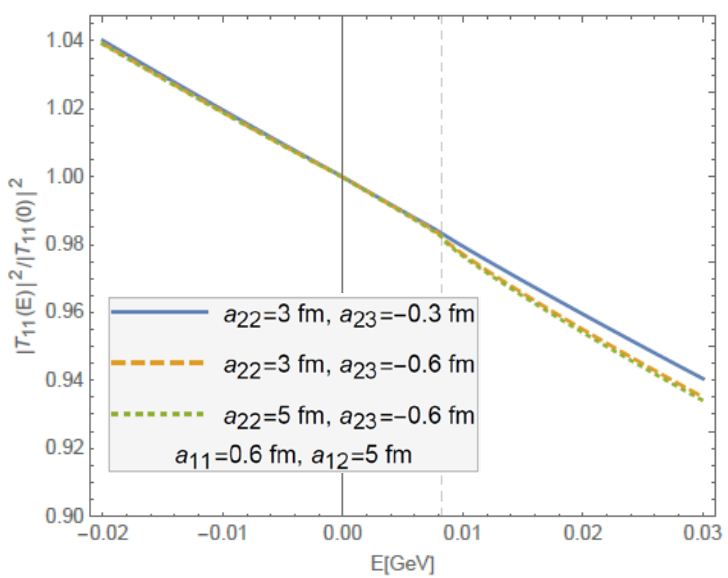
$a_{22} = 3$ fm, $a_{23} = -0.3$ fm

$a_{22} = 3$ fm, $a_{23} = -0.6$ fm

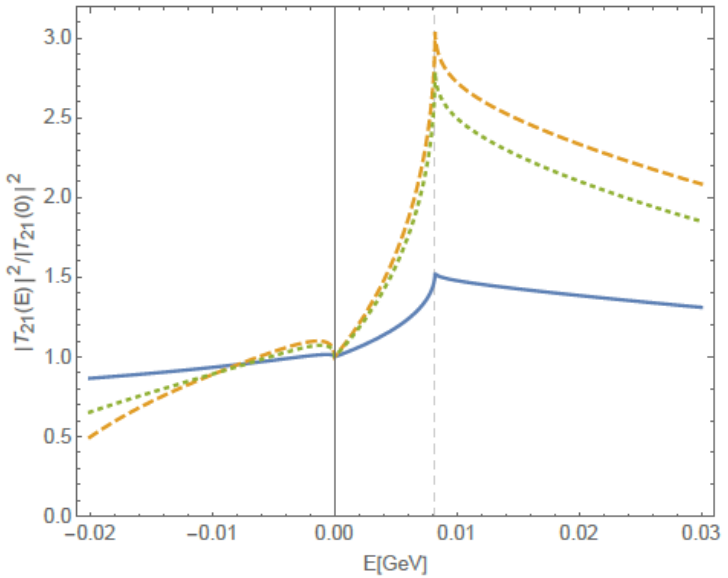
$a_{22} = 5$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

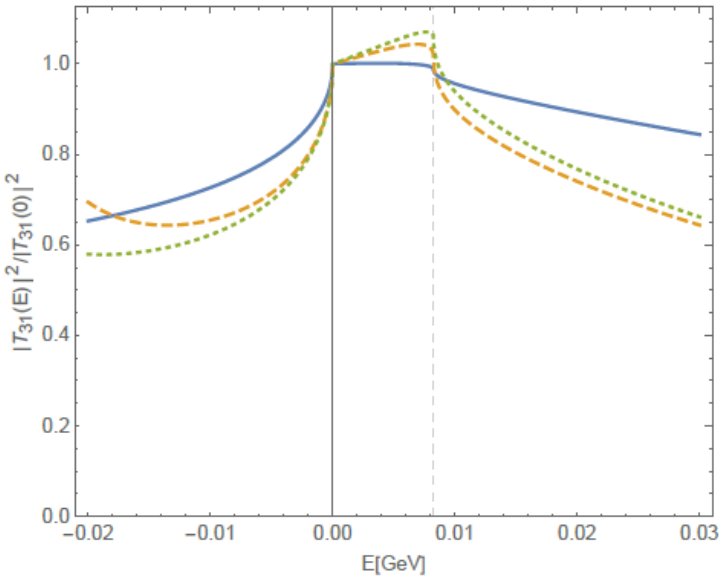
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

Case 1:

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

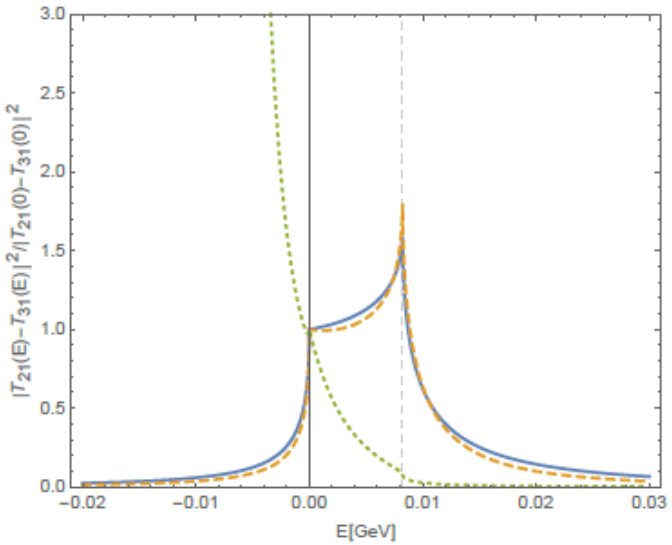
$a_{22} = 0.6$ fm, $a_{23} = 5$ fm

$a_{22} = 1$ fm, $a_{23} = 5$ fm

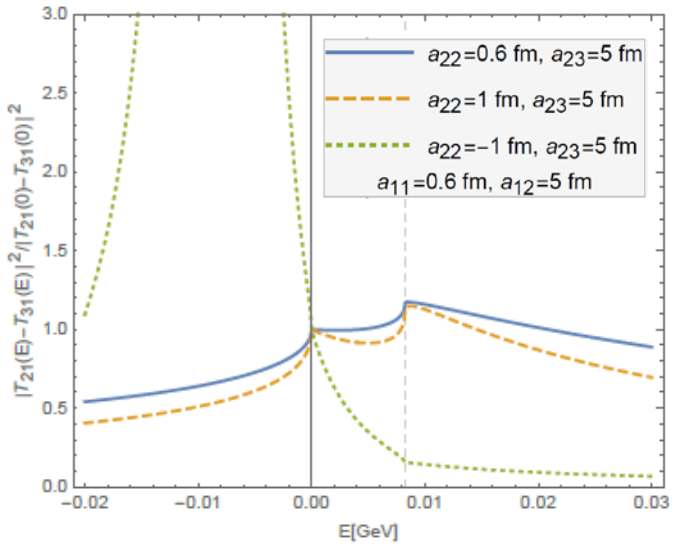
$a_{22} = -1$ fm, $a_{23} = 5$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)(the first row for $a_{22} < 0$ and the second row for $a_{22} > 0$) 2^{nd} RS / 8^{th} RS	3^{rd} RS / 5^{th} RS	6^{th} RS / 4^{th} RS
0.6	5	0.6	5	$-0.0666 \pm 0.0000i, -0.0380 \pm 0.0011$	$-0.0464 \pm 0.0007i$	$-0.0548 \pm 0.0007i$
0.6	5	1	5	$-0.0258 \pm 0.0000i, -0.0075 \pm 0.0006i$	$-0.0107 \pm 0.0005i$	$-0.0192 \pm 0.0004i$
0.6	5	−1	5	$-0.0262 \pm 0.0009i, -0.0078 \pm 0.0001i$	$-0.0111 \pm 0.0003i$	$-0.0196 \pm 0.0003i$

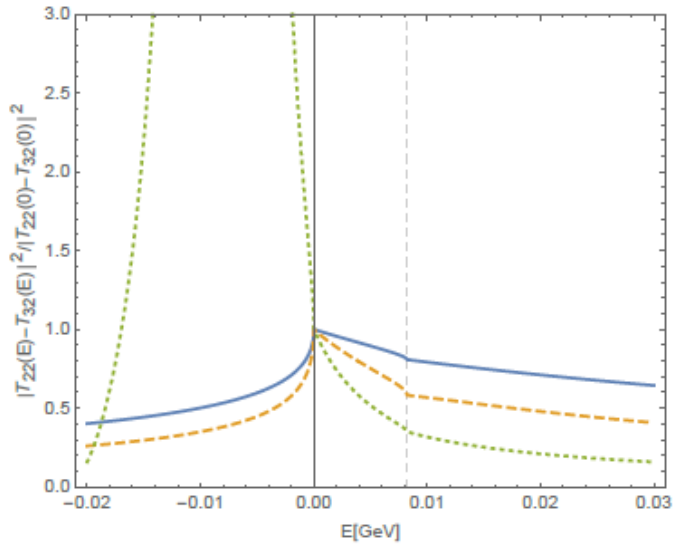
The line shapes for three groups of a_{22}&$|a_{23}|$:



(g)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$, $a_{12} = a_{13}$



(h)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$, $a_{12} = -a_{13}$



(i)$|\mathbf{T}_{22}(E) - \mathbf{T}_{32}(E)|^2/|\mathbf{T}_{22}(0) - \mathbf{T}_{32}(0)|^2$

Pole Positions for Different a_{22} and $|a_{23}|$

TABLE II. Numerical results of pole positions for different a_{22} and $|a_{23}|$ near $E = 0$ and $E = \Delta$.

a_{22}	$\lvert a_{23}\rvert$	Pole position		
		$\mathrm{Re}E < 0$	$0 < \mathrm{Re}E < \Delta$	$\mathrm{Re}E > \Delta$
$a_{22}/\lvert a_{23}\rvert = 0.93, 1, 1.05$	$\lvert a_{23}\rvert = 0.2 \sim 6.2$ fm $\lvert a_{23}\rvert\sqrt{2M_3\Delta} = 0.13 \sim 3.97$	$4^{\text{th}}\mathrm{RS}(+-+)$ $8^{\text{th}}\mathrm{RS}(+--)$	$5^{\text{th}}\mathrm{RS}(++-)$ $6^{\text{th}}\mathrm{RS}(-+-)$	$5^{\text{th}}\mathrm{RS}(++-)$ $6^{\text{th}}\mathrm{RS}(-+-)$
$a_{22}/\lvert a_{23}\rvert = -0.93, -1, -1.05$	$\lvert a_{23}\rvert = 0.2 \sim 6.2$ fm $\lvert a_{23}\rvert\sqrt{2M_3\Delta} = 0.13 \sim 3.97$	$2^{\text{nd}}\mathrm{RS}(-++)$ $6^{\text{th}}\mathrm{RS}(-+-)$	$3^{\text{rd}}\mathrm{RS}(--+)$ $4^{\text{th}}\mathrm{RS}(+-+)$	$3^{\text{rd}}\mathrm{RS}(--+)$ $4^{\text{th}}\mathrm{RS}(+-+)$
$\lvert a_{22}\rvert\sqrt{2M_3\Delta} \gg 1$	$\lvert a_{23}\rvert\sqrt{2M_3\Delta} < 1$	no pole	no pole	no pole
$a_{22} = 0.2 \sim 1.2$ fm $0.13 < \lvert a_{22}\rvert\sqrt{2M_3\Delta} < 0.76$	$\lvert a_{23}\rvert = 5$ fm	$4^{\text{th}}\mathrm{RS}(+-+)$ $5^{\text{th}}\mathrm{RS}(++-)$ $8^{\text{th}}\mathrm{RS}(+--)$(two poles)	no pole	no pole
$-a_{22} = 0.2 \sim 1.2$ fm $0.13 < \lvert a_{22}\rvert\sqrt{2M_3\Delta} < 0.76$	$\lvert a_{23}\rvert = 5$ fm	$2^{\text{nd}}\mathrm{RS}(-++)$(two poles) $3^{\text{rd}}\mathrm{RS}(--+)$ $6^{\text{th}}\mathrm{RS}(-+-)$	no pole	no pole

$a_{11} = 0.6$ fm,
$a_{12} = 5$ fm

$T_1 = -0.64$ GeV,
$T_2 = 0$ GeV,
$T_3 = \Delta = 0.0082$ GeV

Threshold Behaviors Case 5: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \ll -1$

Case 5: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are strong.

Take three groups of a_{22}&$|a_{23}|$ and there are no pole near the thresholds:

TABLE X. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 5. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\|a_{23}\|$(fm)	Pole position (GeV)
0.6	5	−3	0.3	no pole
0.6	5	−3	0.6	no pole
0.6	5	−5	0.6	no pole

Threshold Behaviors Case 5a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \ll -1$

Case 5a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

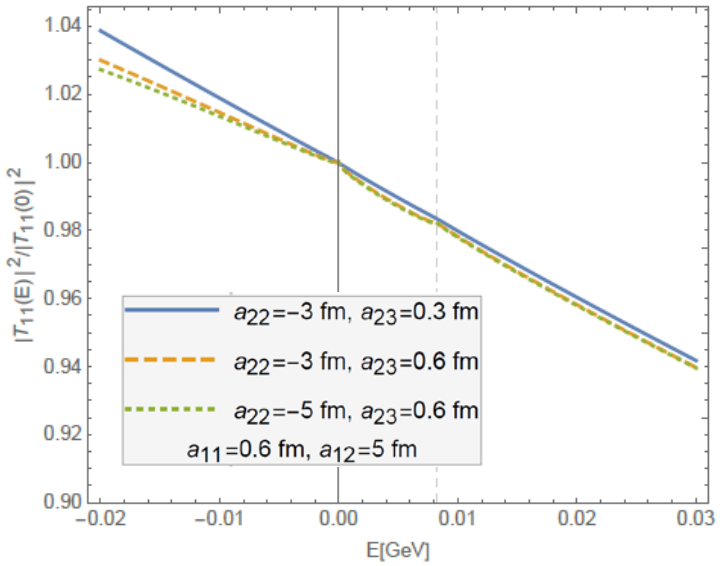
$a_{22} = -3$ fm, $a_{23} = 0.3$ fm

$a_{22} = -3$ fm, $a_{23} = 0.6$ fm

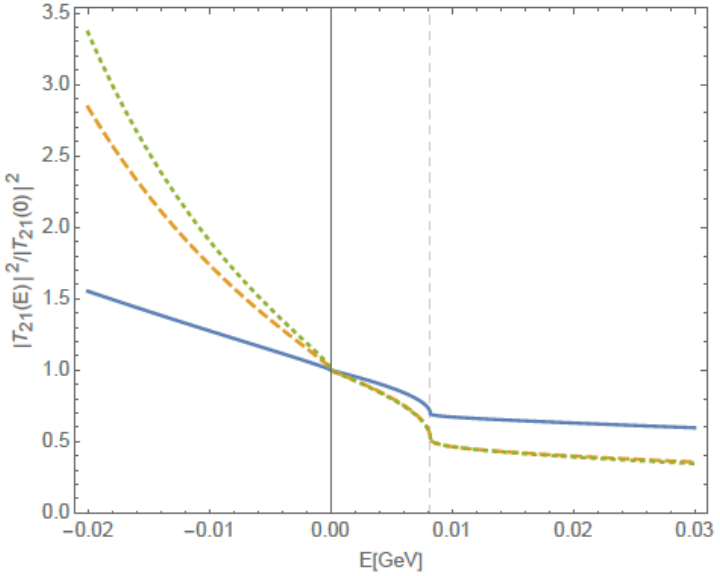
$a_{22} = -5$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

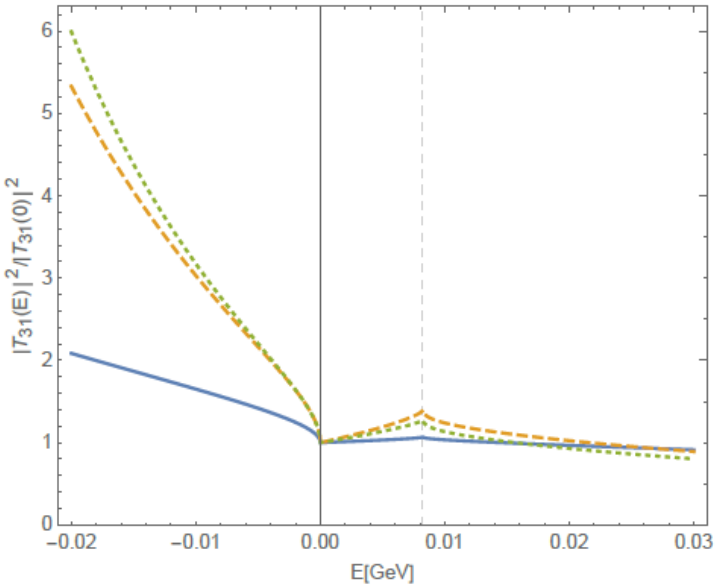
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 5a: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \ll -1$

Case 5a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

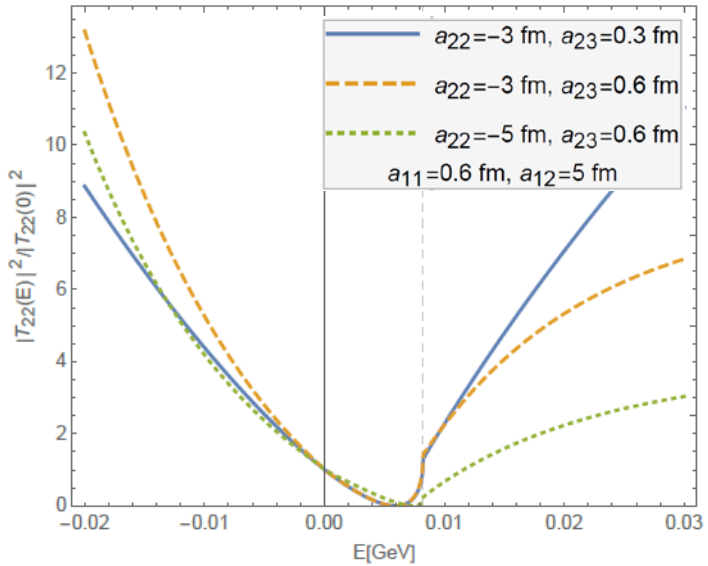
$a_{22} = -3$ fm, $a_{23} = 0.3$ fm

$a_{22} = -3$ fm, $a_{23} = 0.6$ fm

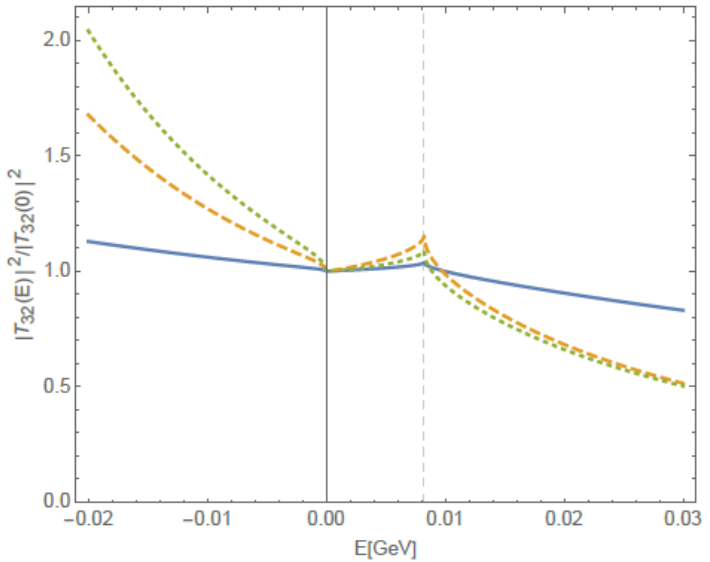
$a_{22} = -5$ fm, $a_{23} = 0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

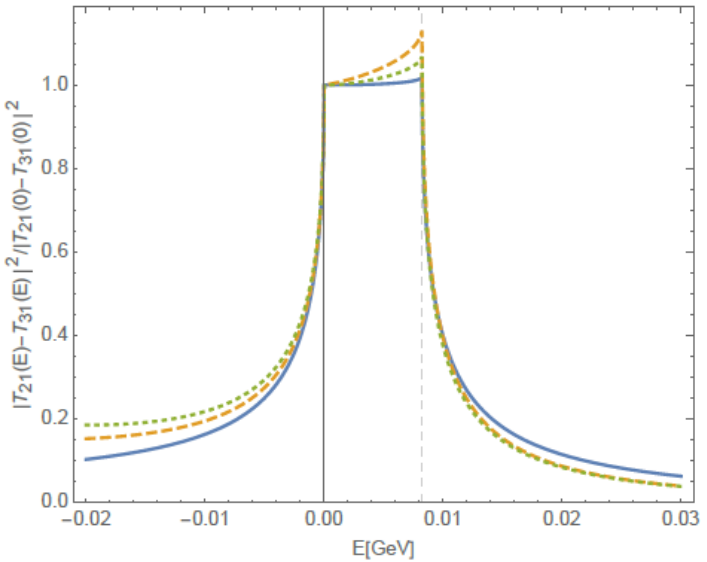
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 5b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \ll -1$

Case 5b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

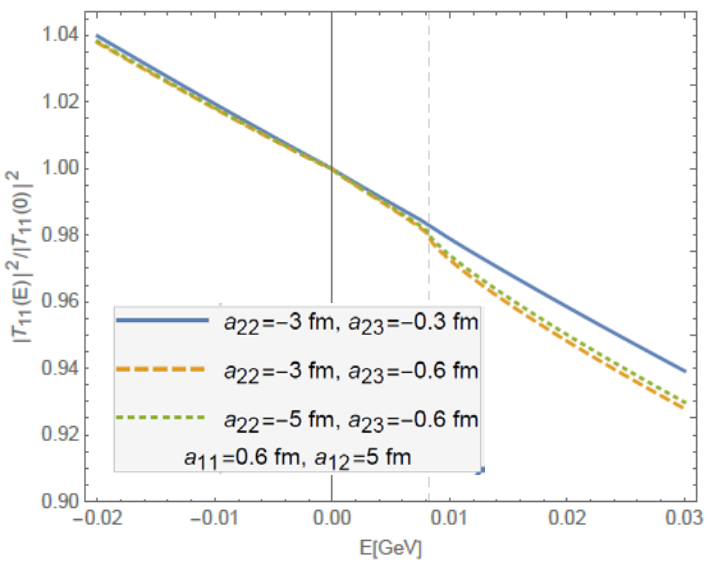
$a_{22} = -3$ fm, $a_{23} = -0.3$ fm

$a_{22} = -3$ fm, $a_{23} = -0.6$ fm

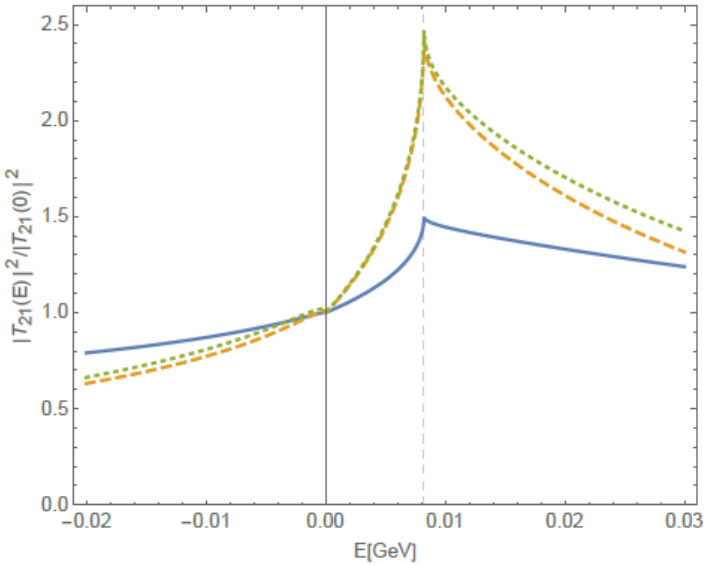
$a_{22} = -5$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

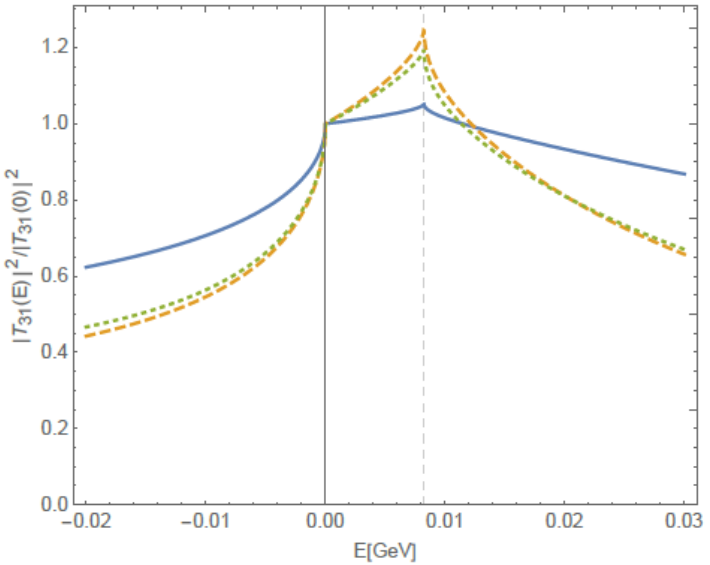
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



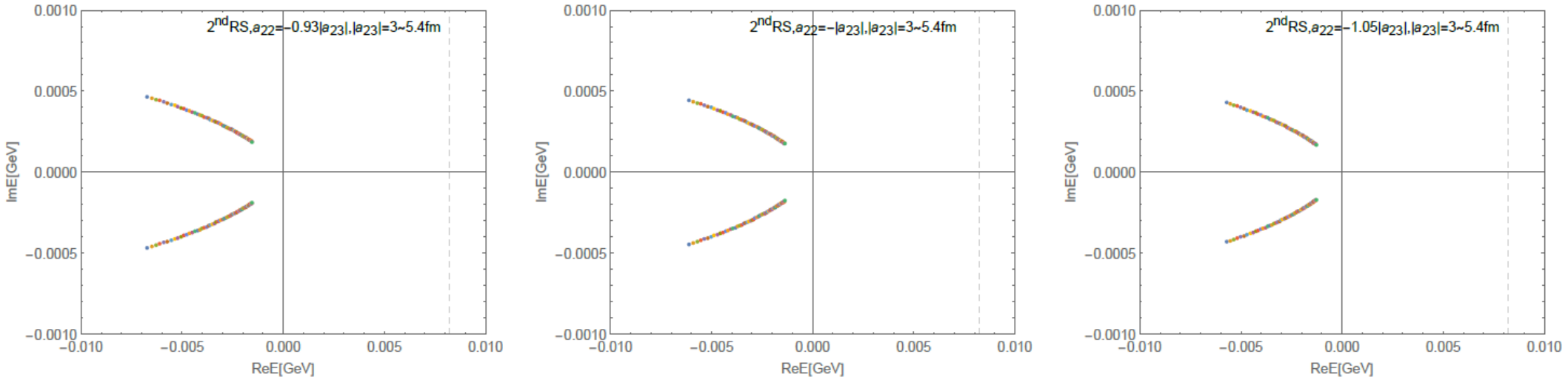
(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 6: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 2nd Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

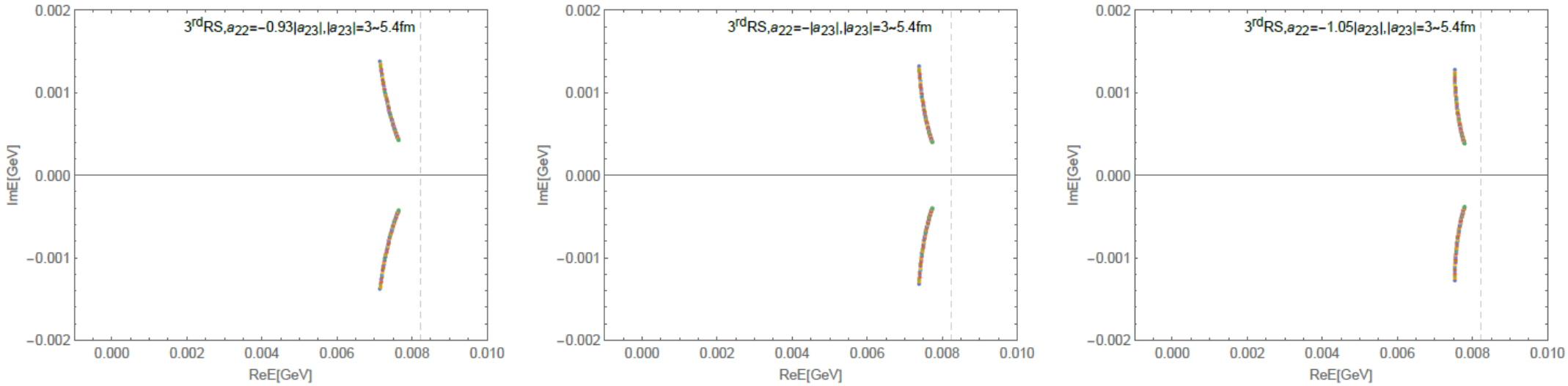
FIG. 27. Poles on the 2ndRS with $|a_{23}| = 3 \sim 5.4$fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 6: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 3rd Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

FIG. 28. Poles on the 3rdRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 6a: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

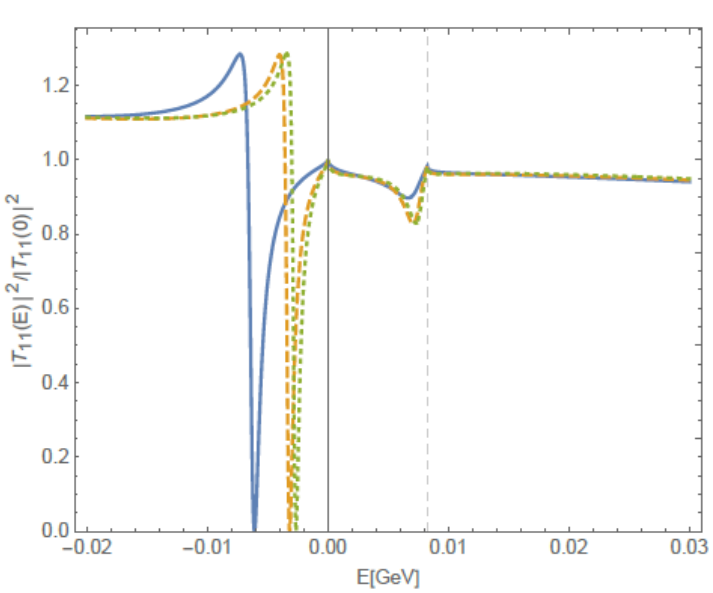
$a_{22} = -3$ fm, $a_{23} = 3$ fm

$a_{22} = -3.72$ fm, $a_{23} = 4$ fm

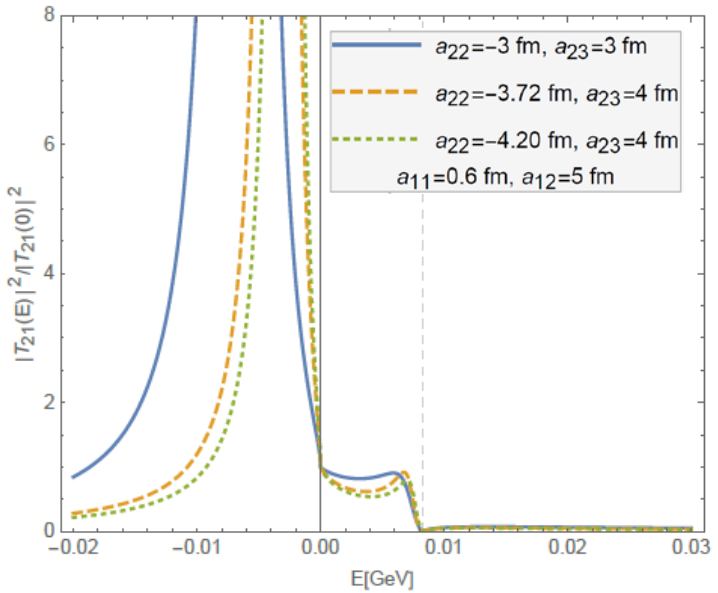
$a_{22} = -4.2$ fm, $a_{23} = 4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				2^{nd} RS	3^{rd} RS	4^{th} RS	6^{th} RS
0.6	5	-3	3	$-0.0063 \pm 0.0005i$	$0.0074 \pm 0.0013i$	$0.0072 \pm 0.0013i$	$-0.0010 \pm 0.0000i$
0.6	5	-3.72	4	$-0.0033 \pm 0.0003i$	$0.0074 \pm 0.0008i$	$0.0072 \pm 0.0008i$	$-0.0008 \pm 0.0000i$
0.6	5	-4.2	4	$-0.0028 \pm 0.0003i$	$0.0076 \pm 0.0007i$	$0.0075 \pm 0.0007i$	$-0.0006 \pm 0.0000i$

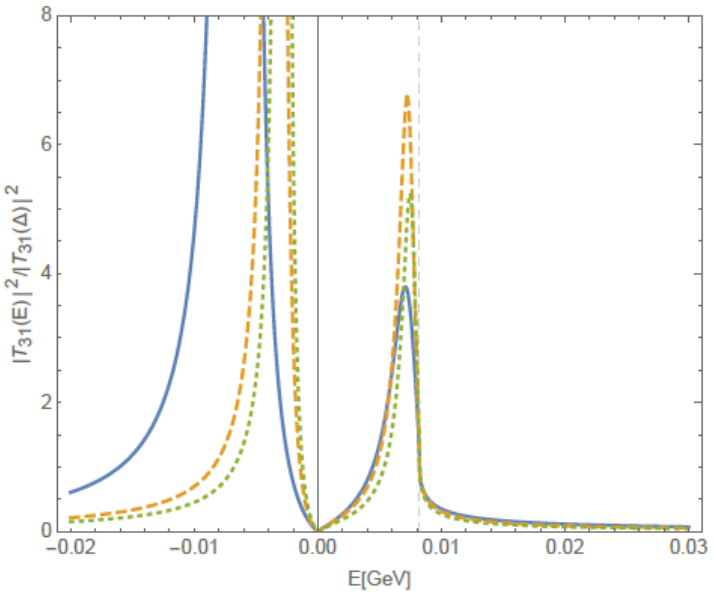
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Threshold Behaviors Case 6a: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

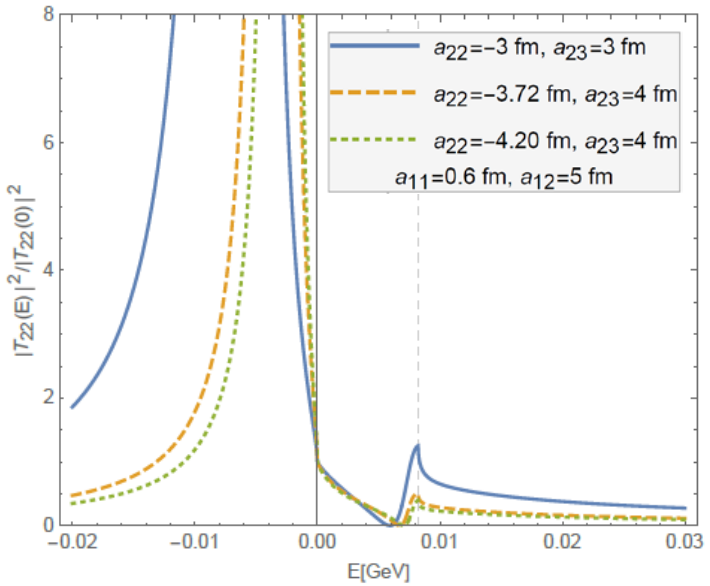
$a_{22} = -3$ fm, $a_{23} = 3$ fm

$a_{22} = -3.72$ fm, $a_{23} = 4$ fm

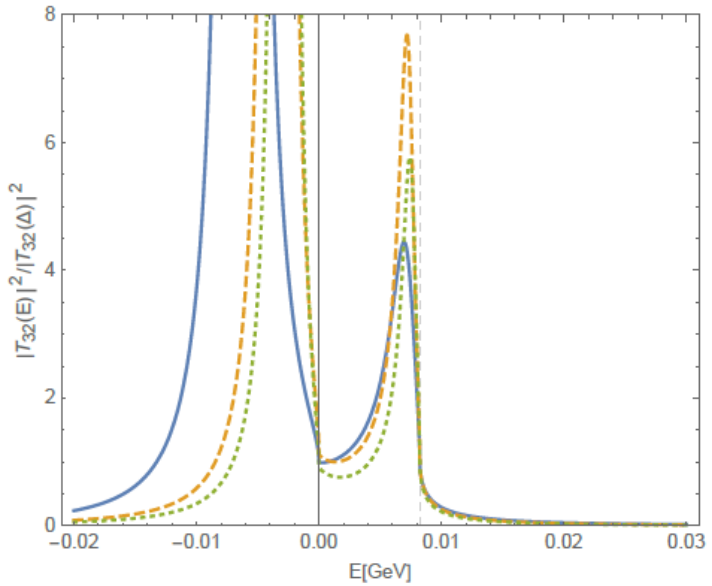
$a_{22} = -4.2$ fm, $a_{23} = 4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				2nd RS	3rd RS	4th RS	6th RS
0.6	5	−3	3	$-0.0063 \pm 0.0005i$	$0.0074 \pm 0.0013i$	$0.0072 \pm 0.0013i$	$-0.0010 \pm 0.0000i$
0.6	5	−3.72	4	$-0.0033 \pm 0.0003i$	$0.0074 \pm 0.0008i$	$0.0072 \pm 0.0008i$	$-0.0008 \pm 0.0000i$
0.6	5	−4.2	4	$-0.0028 \pm 0.0003i$	$0.0076 \pm 0.0007i$	$0.0075 \pm 0.0007i$	$-0.0006 \pm 0.0000i$

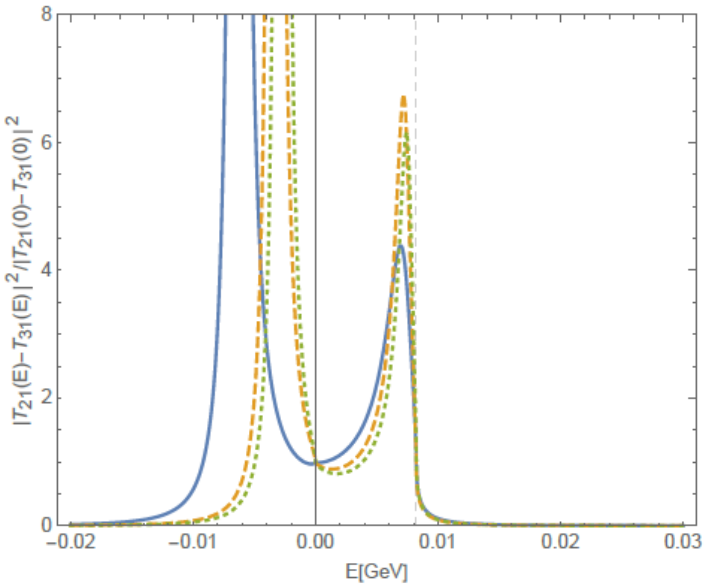
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 6b: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

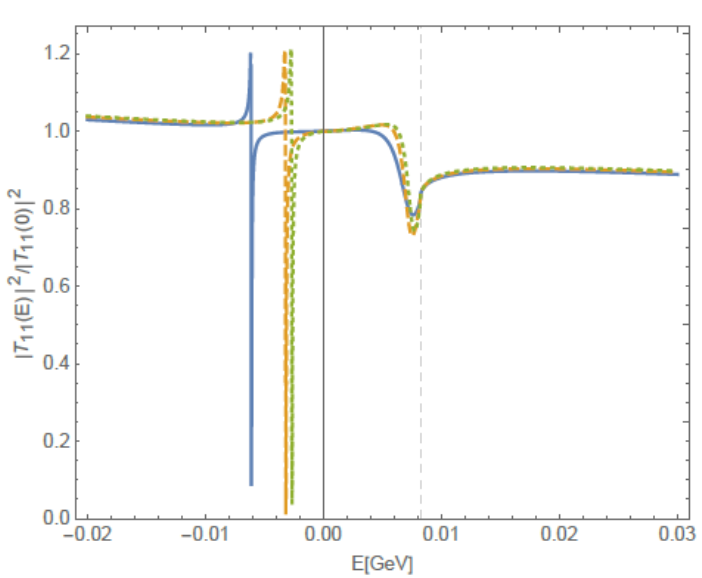
$a_{22} = -3$ fm, $a_{23} = -3$ fm

$a_{22} = -3.72$ fm, $a_{23} = -4$ fm

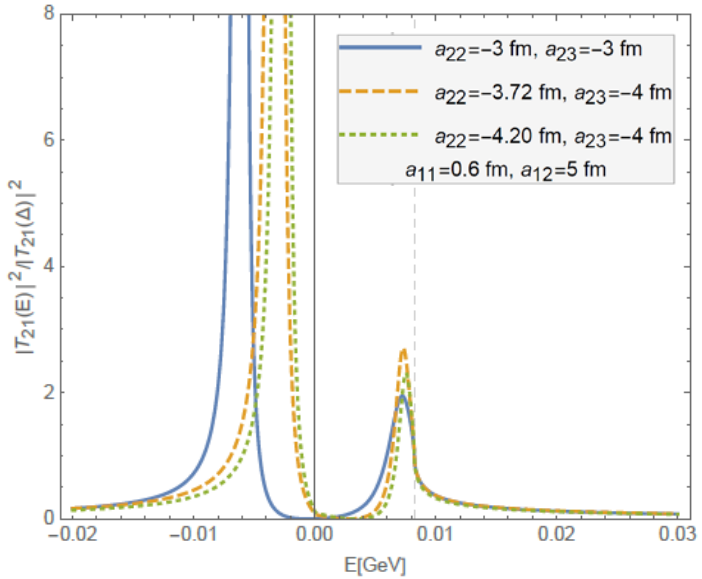
$a_{22} = -4.2$ fm, $a_{23} = -4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				2^{nd} RS	3^{rd} RS	4^{th} RS	6^{th} RS
0.6	5	-3	3	$-0.0063 \pm 0.0005i$	$0.0074 \pm 0.0013i$	$0.0072 \pm 0.0013i$	$-0.0010 \pm 0.0000i$
0.6	5	-3.72	4	$-0.0033 \pm 0.0003i$	$0.0074 \pm 0.0008i$	$0.0072 \pm 0.0008i$	$-0.0008 \pm 0.0000i$
0.6	5	-4.2	4	$-0.0028 \pm 0.0003i$	$0.0076 \pm 0.0007i$	$0.0075 \pm 0.0007i$	$-0.0006 \pm 0.0000i$

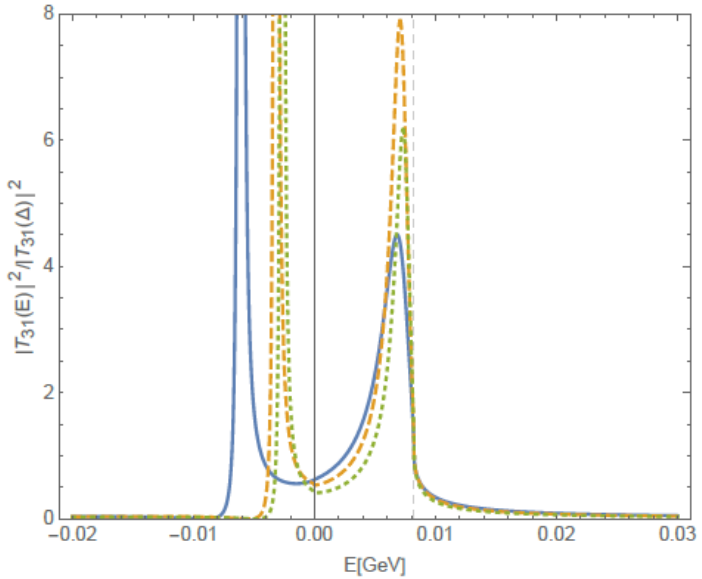
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(\Delta)|^2$



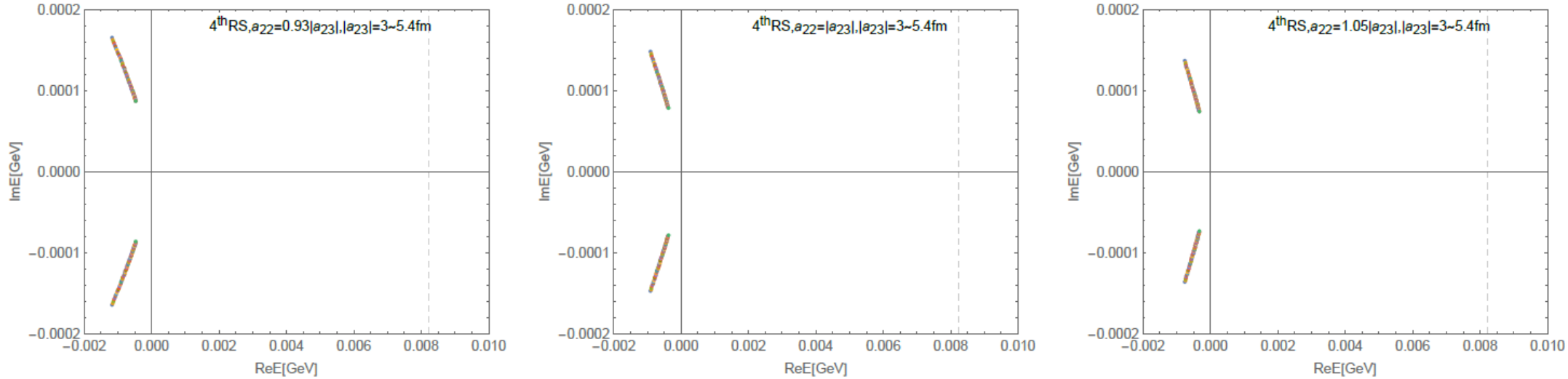
(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Threshold Behaviors Case 7: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

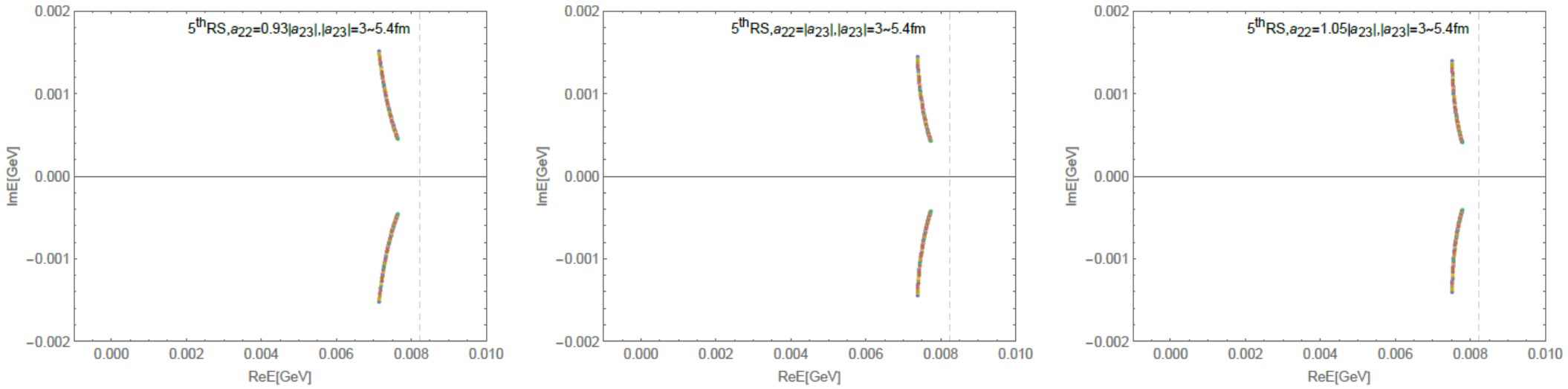
FIG. 33. Poles on the 4thRS with $|a_{23}| = 3 \sim 5.4$fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 7: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 5th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 34. Poles on the 5thRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 7a: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

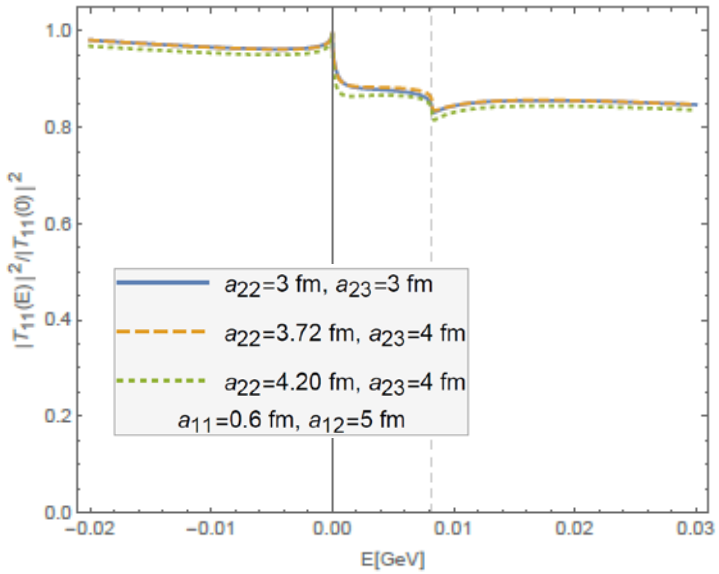
$a_{22} = 3$ fm, $a_{23} = 3$ fm

$a_{22} = 3.72$ fm, $a_{23} = 4$ fm

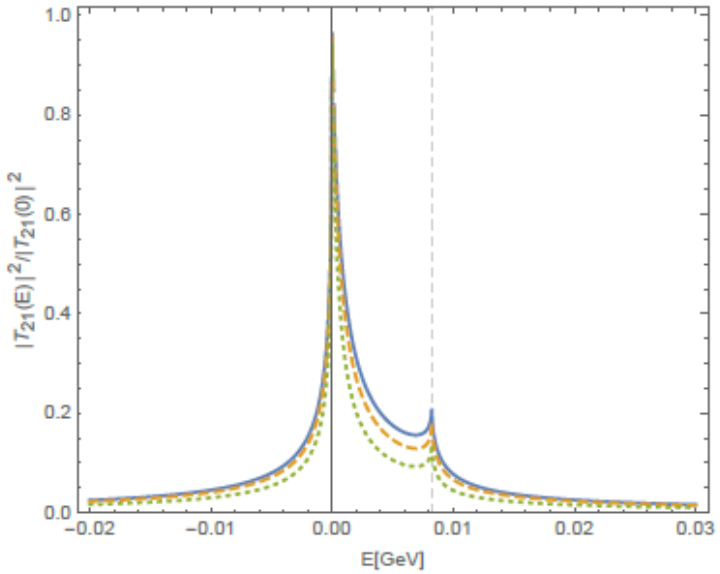
$a_{22} = 4.2$ fm, $a_{23} = 4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				4^{th} RS	5^{th} RS	6^{th} RS	8^{th} RS
0.6	5	3	3	$-0.0009 \pm 0.0001i$	$0.0074 \pm 0.0015i$	$0.0074 \pm 0.0012i$	$-0.0061 \pm 0.0000i$
0.6	5	3.72	4	$-0.0008 \pm 0.0001i$	$0.0074 \pm 0.0009i$	$0.0074 \pm 0.0006i$	$-0.0032 \pm 0.0000i$
0.6	5	4.2	4	$-0.0005 \pm 0.0001i$	$0.0076 \pm 0.0008i$	$0.0076 \pm 0.0006i$	$-0.0026 \pm 0.0000i$

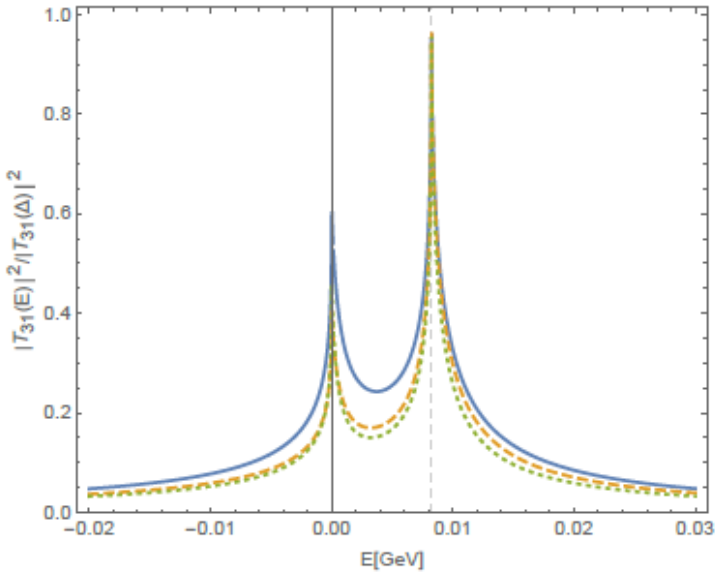
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$



(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(0)|^2$



(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Threshold Behaviors Case 7a: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7a: $a_{23} > 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

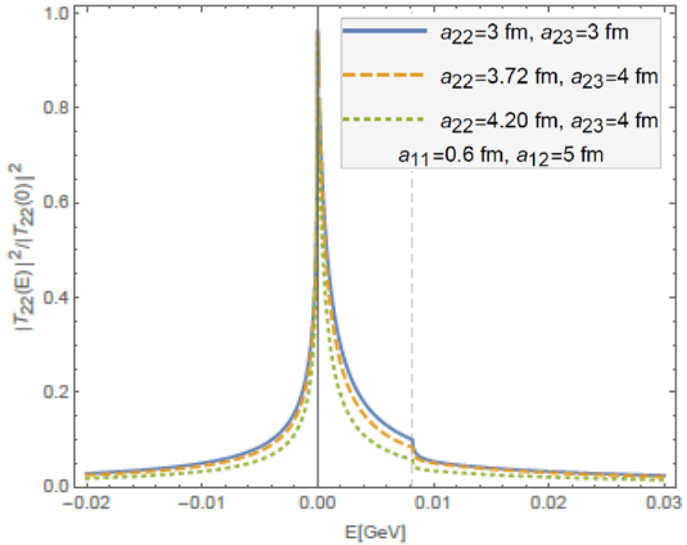
$a_{22} = 3$ fm, $a_{23} = 3$ fm

$a_{22} = 3.72$ fm, $a_{23} = 4$ fm

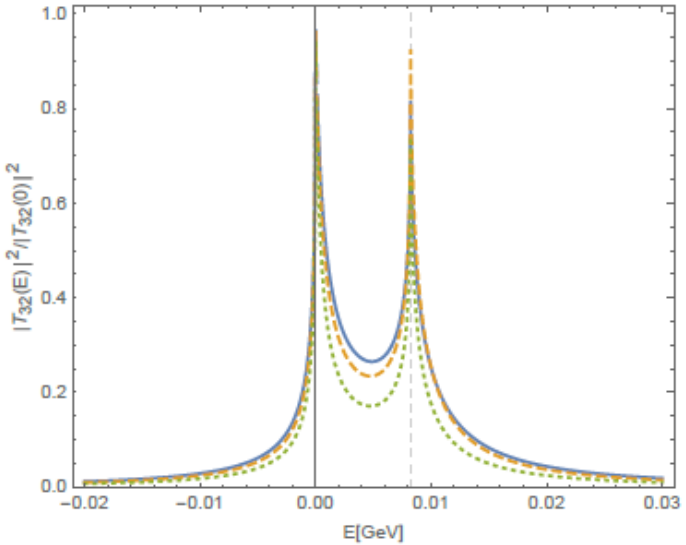
$a_{22} = 4.2$ fm, $a_{23} = 4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				4^{th} RS	5^{th} RS	6^{th} RS	8^{th} RS
0.6	5	3	3	$-0.0009 \pm 0.0001i$	$0.0074 \pm 0.0015i$	$0.0074 \pm 0.0012i$	$-0.0061 \pm 0.0000i$
0.6	5	3.72	4	$-0.0008 \pm 0.0001i$	$0.0074 \pm 0.0009i$	$0.0074 \pm 0.0006i$	$-0.0032 \pm 0.0000i$
0.6	5	4.2	4	$-0.0005 \pm 0.0001i$	$0.0076 \pm 0.0008i$	$0.0076 \pm 0.0006i$	$-0.0026 \pm 0.0000i$

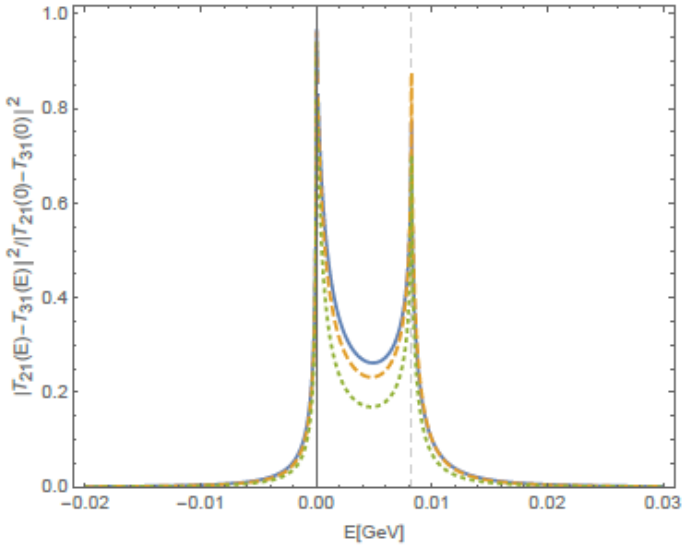
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 7b: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

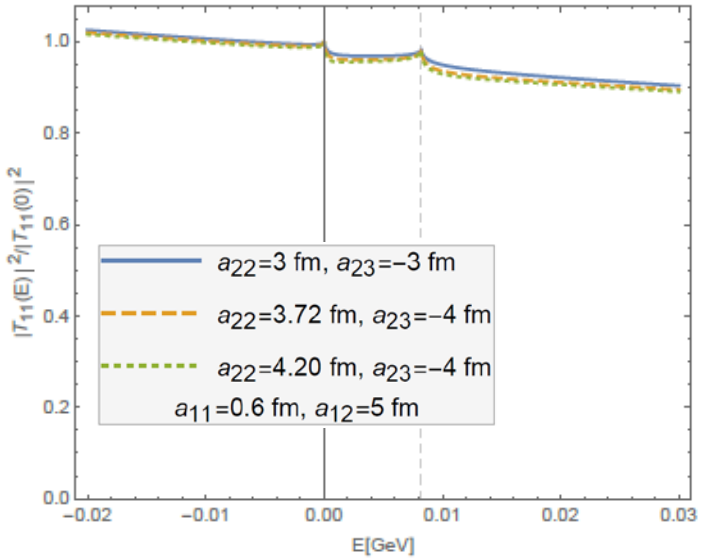
$a_{22} = 3$ fm, $a_{23} = -3$ fm

$a_{22} = 3.72$ fm, $a_{23} = -4$ fm

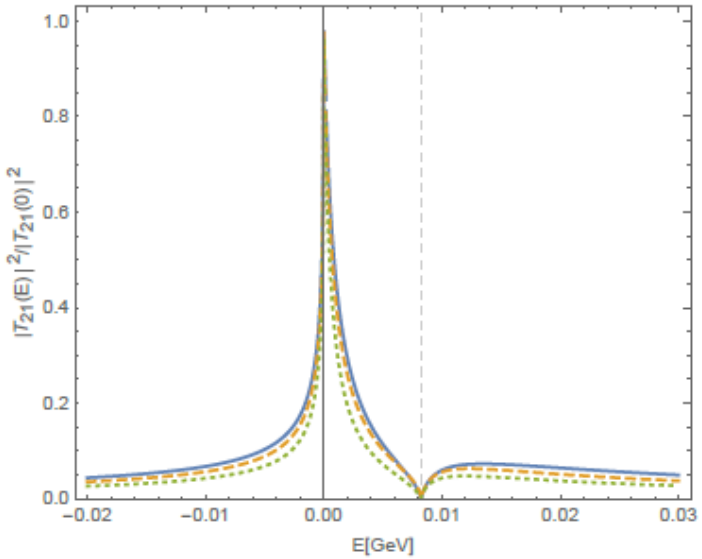
$a_{22} = 4.2$ fm, $a_{23} = -4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				4^{th} RS	5^{th} RS	6^{th} RS	8^{th} RS
0.6	5	3	3	$-0.0009 \pm 0.0001i$	$0.0074 \pm 0.0015i$	$0.0074 \pm 0.0012i$	$-0.0061 \pm 0.0000i$
0.6	5	3.72	4	$-0.0008 \pm 0.0001i$	$0.0074 \pm 0.0009i$	$0.0074 \pm 0.0006i$	$-0.0032 \pm 0.0000i$
0.6	5	4.2	4	$-0.0005 \pm 0.0001i$	$0.0076 \pm 0.0008i$	$0.0076 \pm 0.0006i$	$-0.0026 \pm 0.0000i$

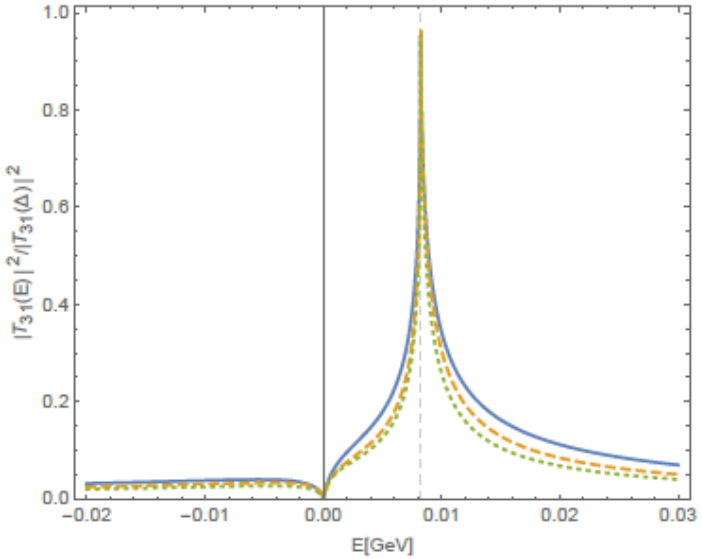
The line shapes for three groups of a_{22}&$|a_{23}|$:



(a)$|\mathbf{T}_{11}(E)|^2/|\mathbf{T}_{11}(0)|^2$

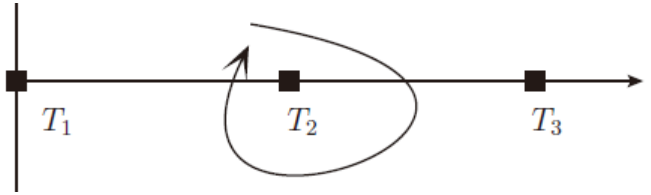


(b)$|\mathbf{T}_{21}(E)|^2/|\mathbf{T}_{21}(\Delta)|^2$



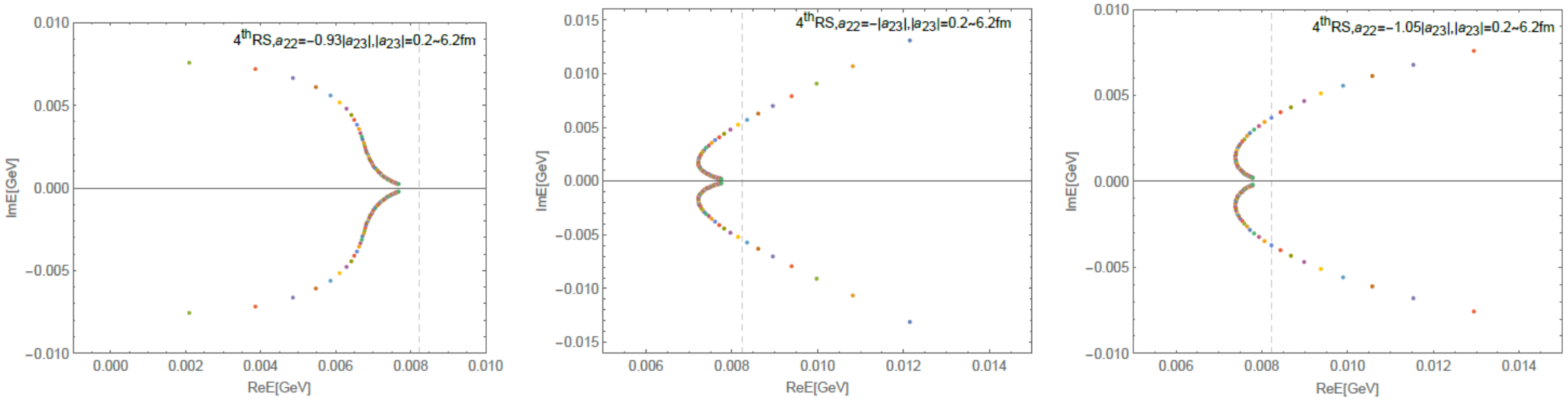
(c)$|\mathbf{T}_{31}(E)|^2/|\mathbf{T}_{31}(\Delta)|^2$

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2\sim6.2$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

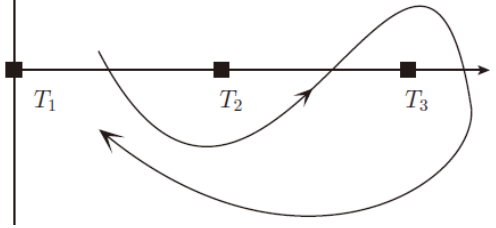
The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

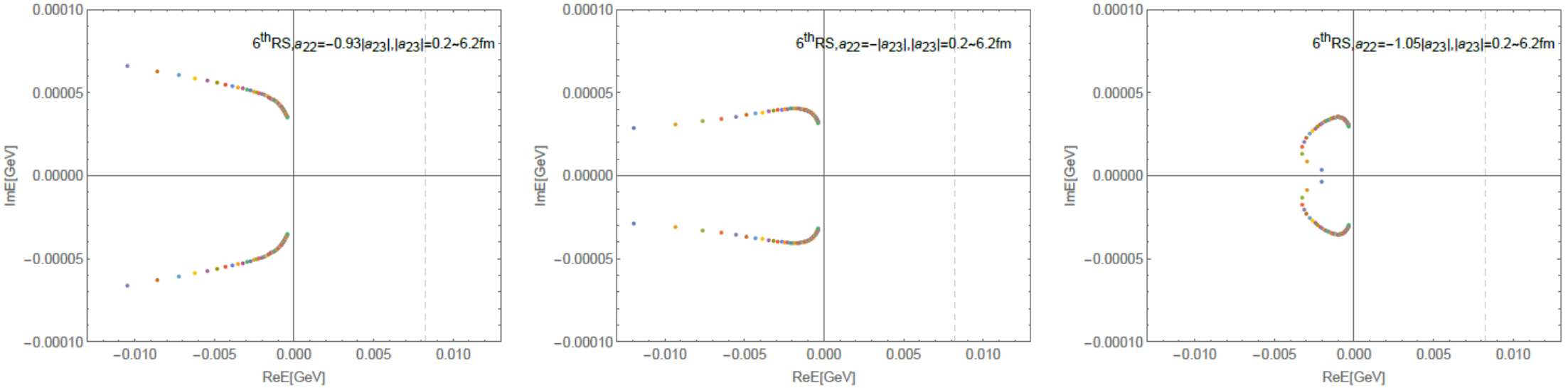
FIG. 5. Poles on the 4thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2 \sim 6.2$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

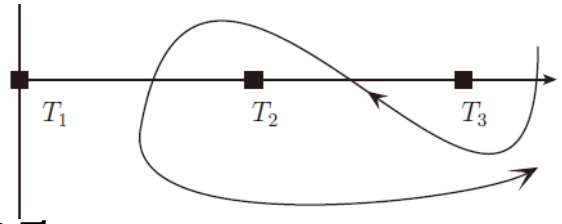
The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

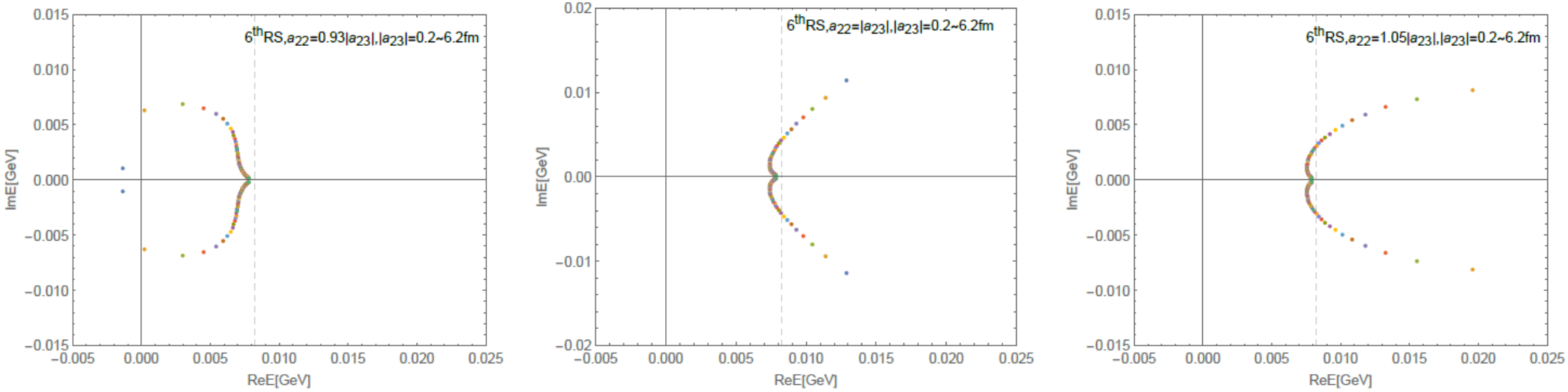
FIG. 6. Poles on the 6thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2 \sim 6.2$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

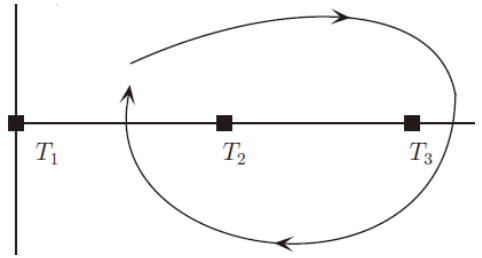
The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

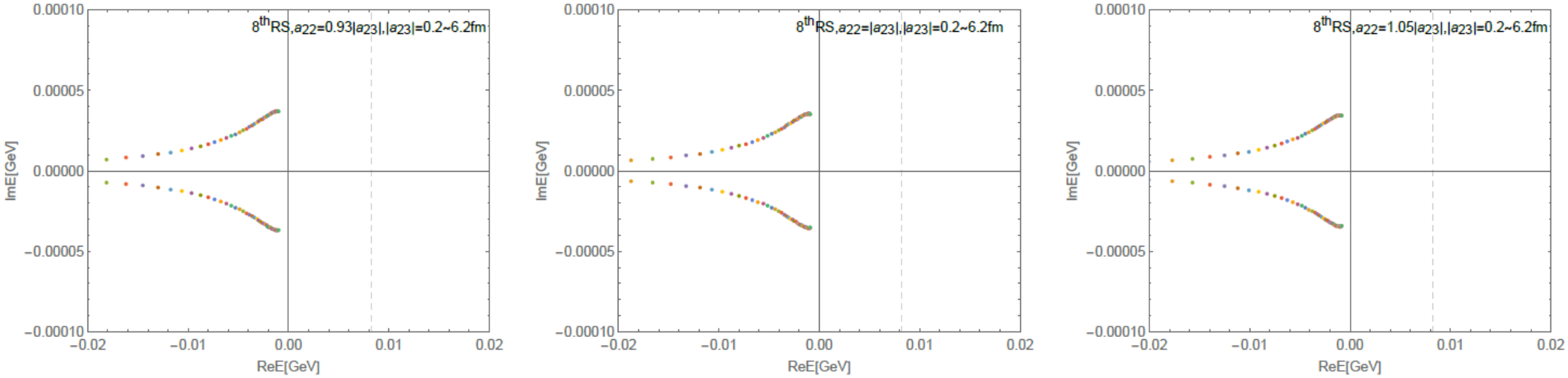
FIG. 9. Poles on the 6thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3. The transition of the pole positions as $|a_{22}|/|a_{23}|$ become larger is continuous.

Pole Positions for Different a_{22} and $|a_{23}|$



The pole trajectories for $|a_{23}| = 0.2\sim6.2$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 8th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm $T_1 = -0.64$ GeV, $T_2 = 0$ GeV, $T_3 = \Delta = 0.0082$ GeV



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

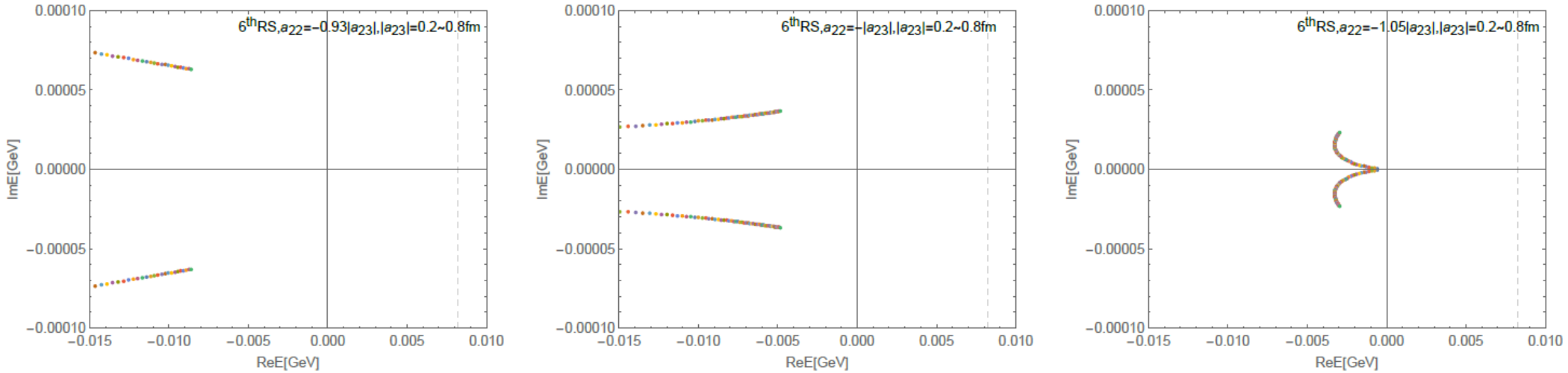
FIG. 10. Poles on the 8thRS with $|a_{23}| = 0.2 \sim 6.2$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 2: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2: Couplings in CH2 &CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

FIG. 15. Poles on the 6thRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 2b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

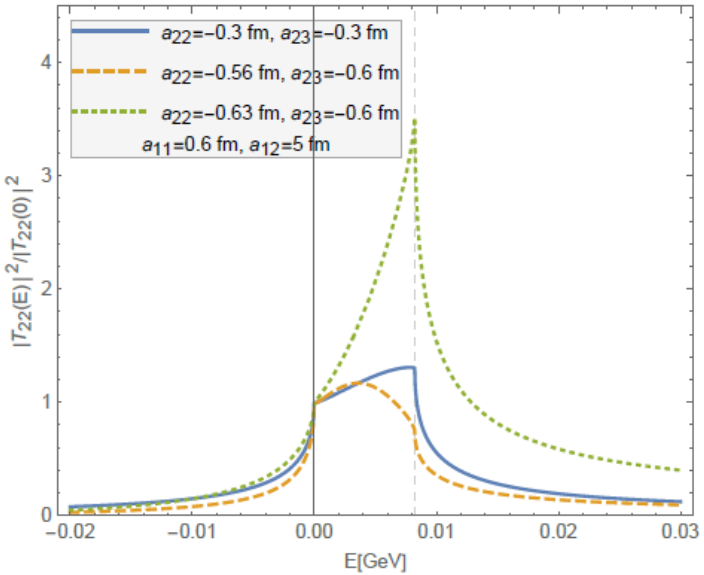
$a_{22} = -0.3$ fm, $a_{23} = -0.3$ fm

$a_{22} = -0.56$ fm, $a_{23} = -0.6$ fm

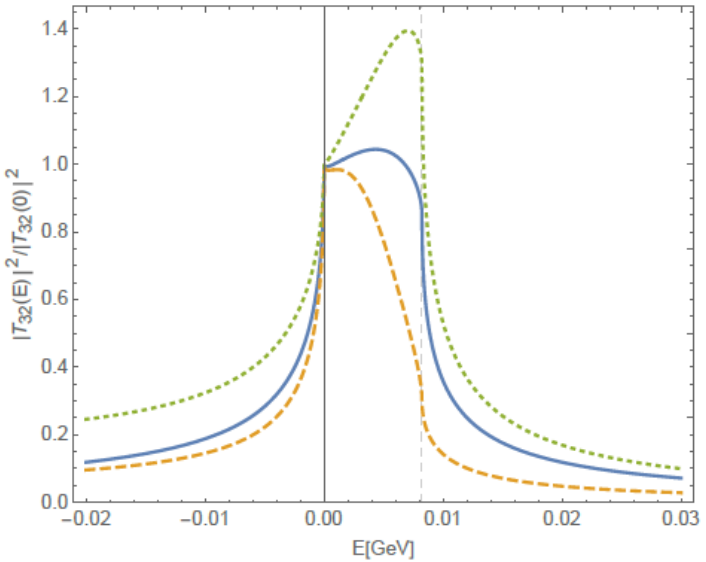
$a_{22} = -0.63$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				3^{rd} RS	4^{th} RS	6^{th} RS
0.6	5	−0.3	0.3	$0.0122 \pm 0.0131i$	$0.0121 \pm 0.0131i$	$-0.0120 \pm 0.0000i$
0.6	5	−0.56	0.6	$0.0042 \pm 0.0073i$	$0.0041 \pm 0.0073i$	$-0.0128 \pm 0.0001i$
0.6	5	−0.63	0.6	$0.0130 \pm 0.0075i$	$0.0129 \pm 0.0076i$	$-0.0033 \pm 0.0000i$

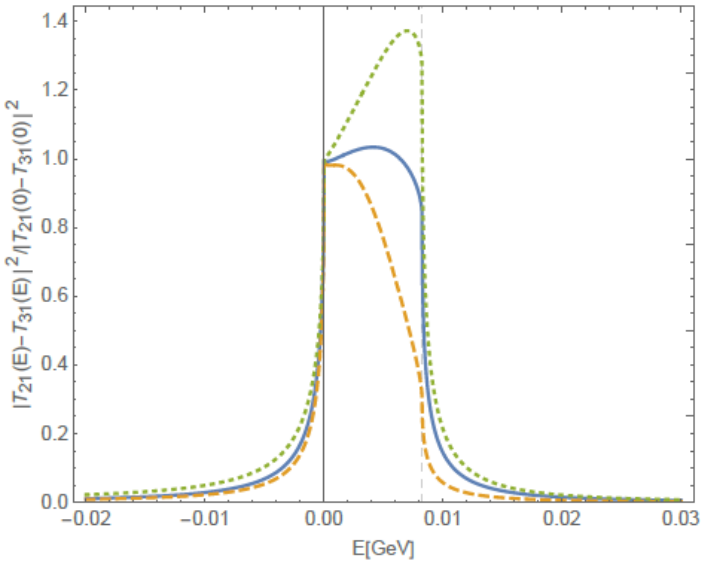
The lineshapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



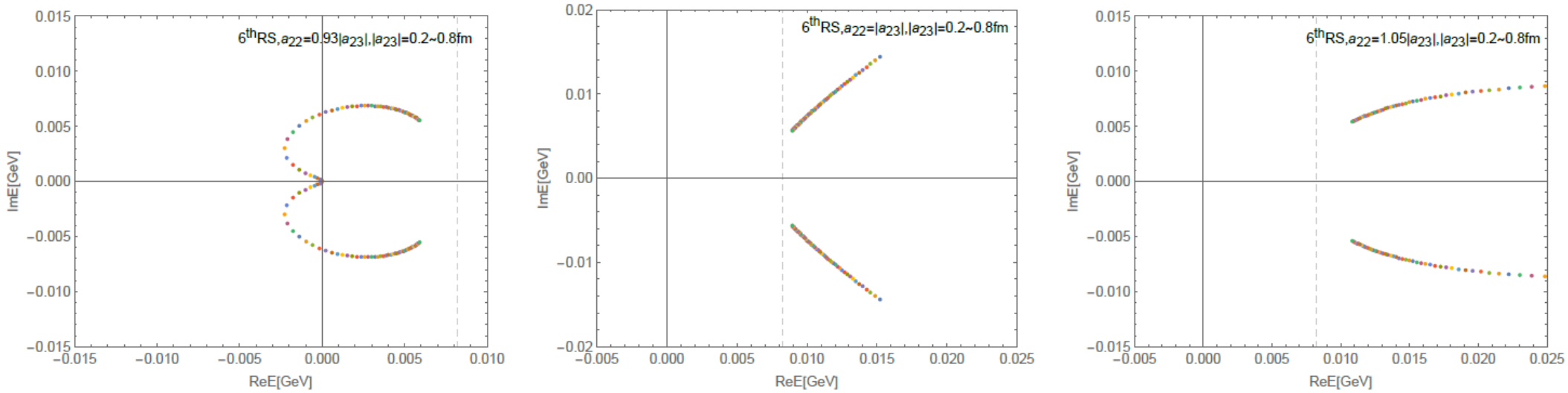
(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 3: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are strong.

The pole trajectories for $|a_{23}| = 0.2\sim0.8$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 20. Poles on the 6thRS with $|a_{23}| = 0.2 \sim 0.8$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 3b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

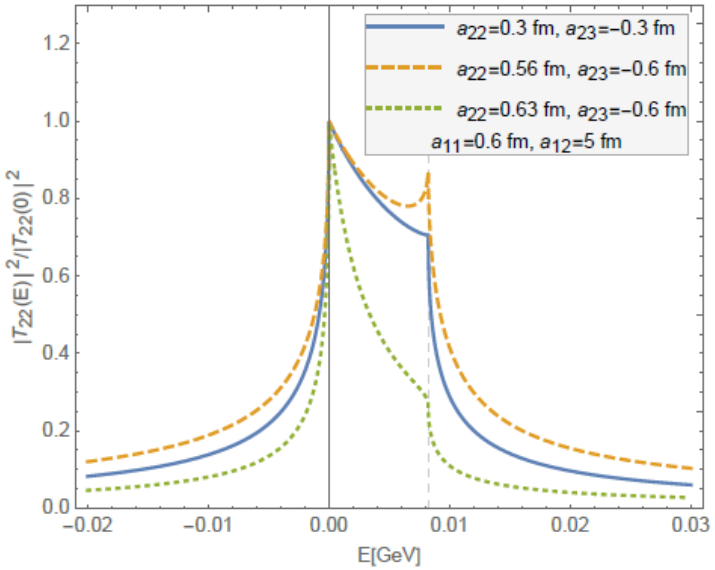
$a_{22} = 0.3$ fm, $a_{23} = -0.3$ fm

$a_{22} = 0.56$ fm, $a_{23} = -0.6$ fm

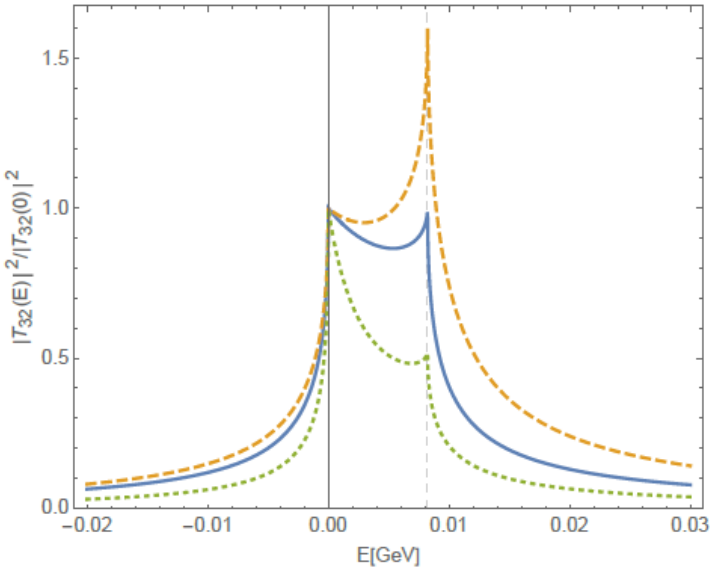
$a_{22} = 0.63$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				4^{th} RS	5^{th} RS	6^{th} RS
0.6	5	0.3	0.3	$-0.0111 \pm 0.0016i$	$0.0130 \pm 0.0147i$	$0.0129 \pm 0.0114i$
0.6	5	0.56	0.6	$-0.0123 \pm 0.0010i$	$0.0043 \pm 0.0082i$	$0.0048 \pm 0.0066i$
0.6	5	0.63	0.6	$-0.0029 \pm 0.0006i$	$0.0135 \pm 0.0084i$	$0.0133 \pm 0.0066i$

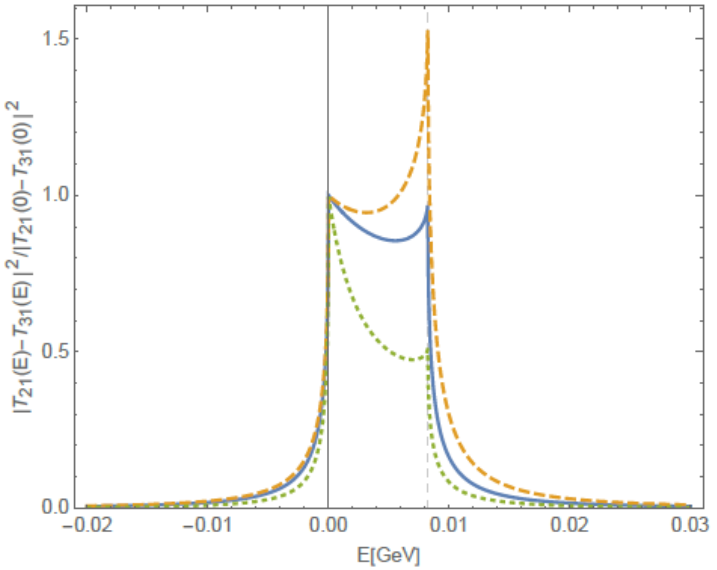
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 4b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \gg 1$

Case 4b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

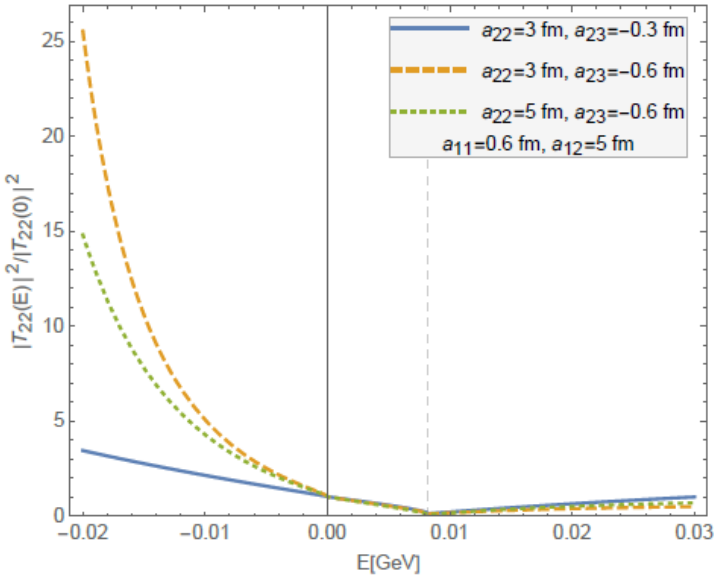
$a_{22} = 3$ fm, $a_{23} = -0.3$ fm

$a_{22} = 3$ fm, $a_{23} = -0.6$ fm

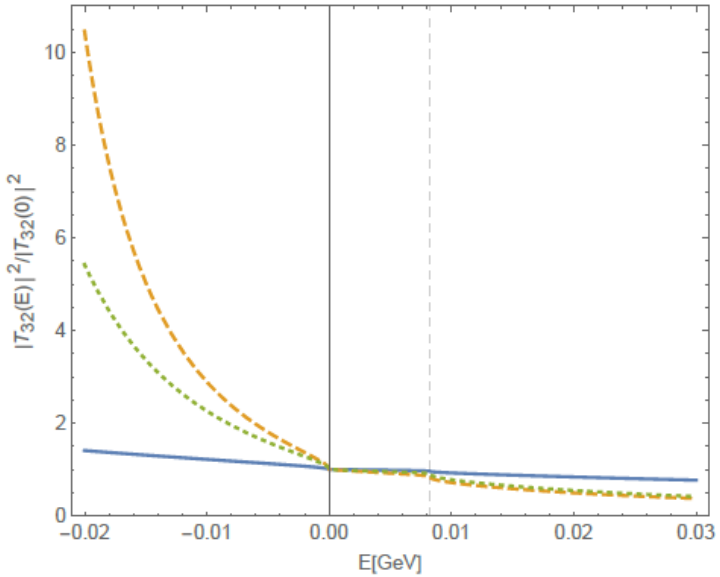
$a_{22} = 5$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

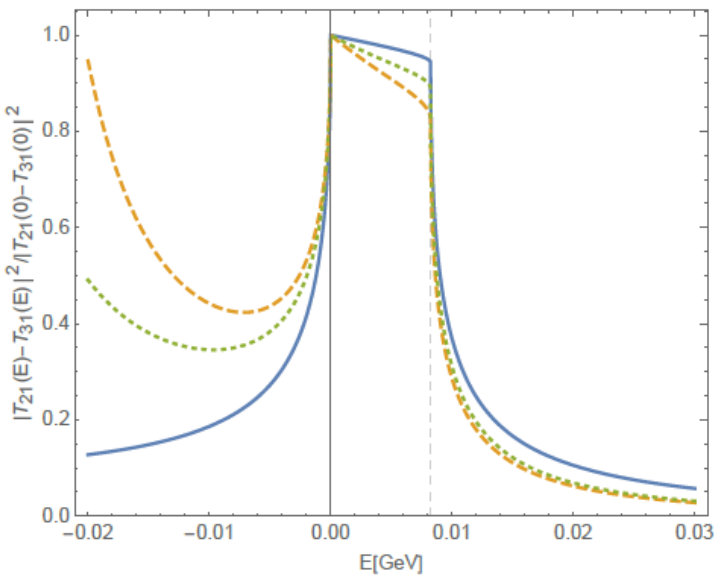
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 5b: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22}\sqrt{2M_3\Delta} \ll -1$

Case 5b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

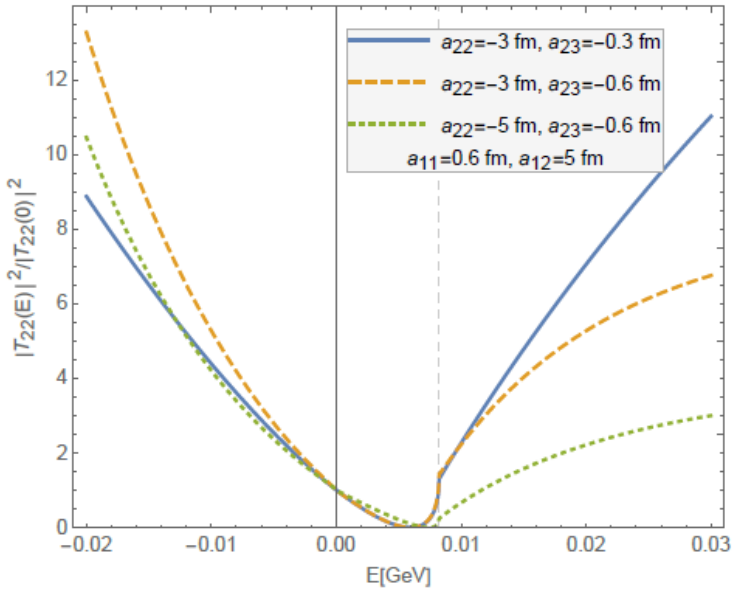
$a_{22} = -3$ fm, $a_{23} = -0.3$ fm

$a_{22} = -3$ fm, $a_{23} = -0.6$ fm

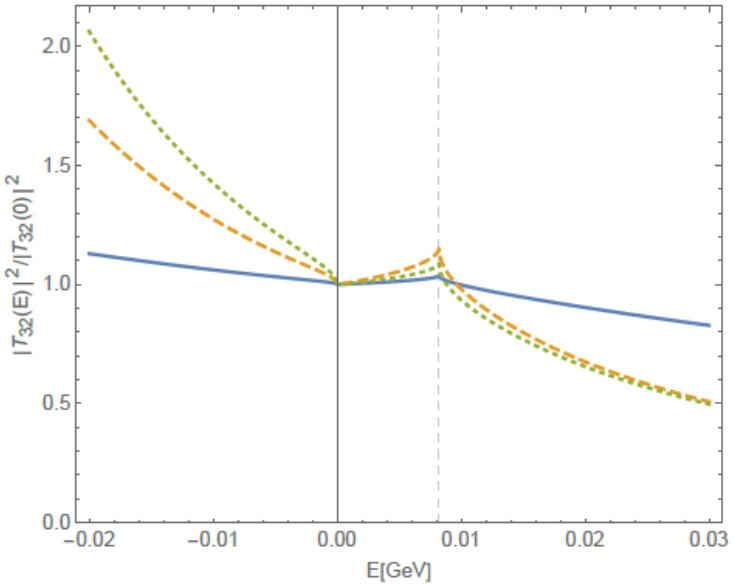
$a_{22} = -5$ fm, $a_{23} = -0.6$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)
0.6	5	3	0.3	no pole
0.6	5	3	0.6	no pole
0.6	5	5	0.6	no pole

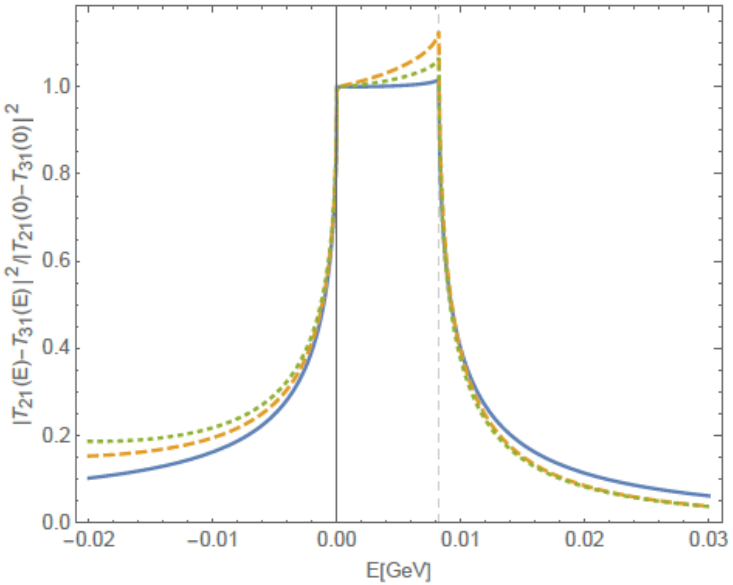
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



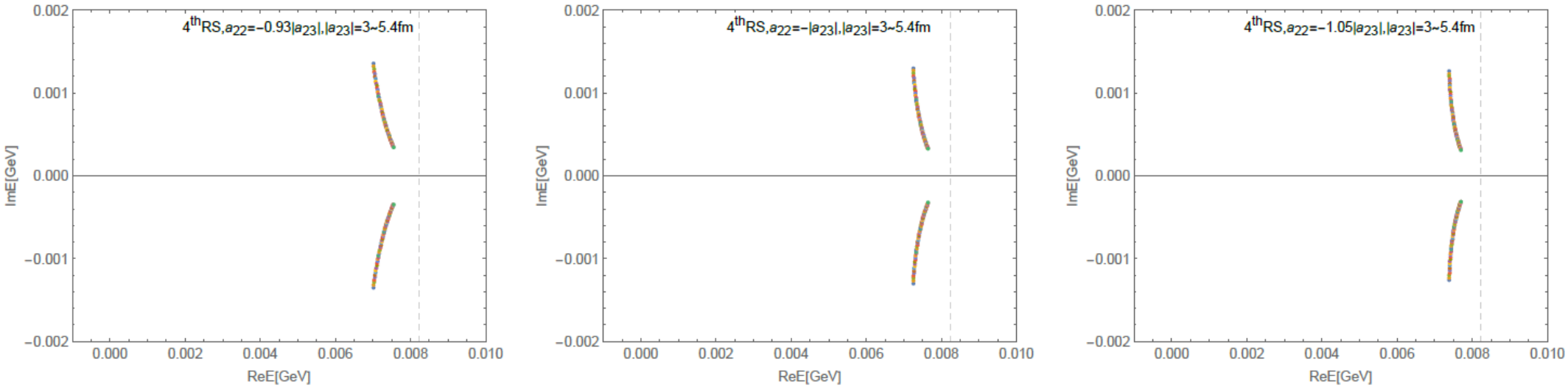
(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 6: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 4th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

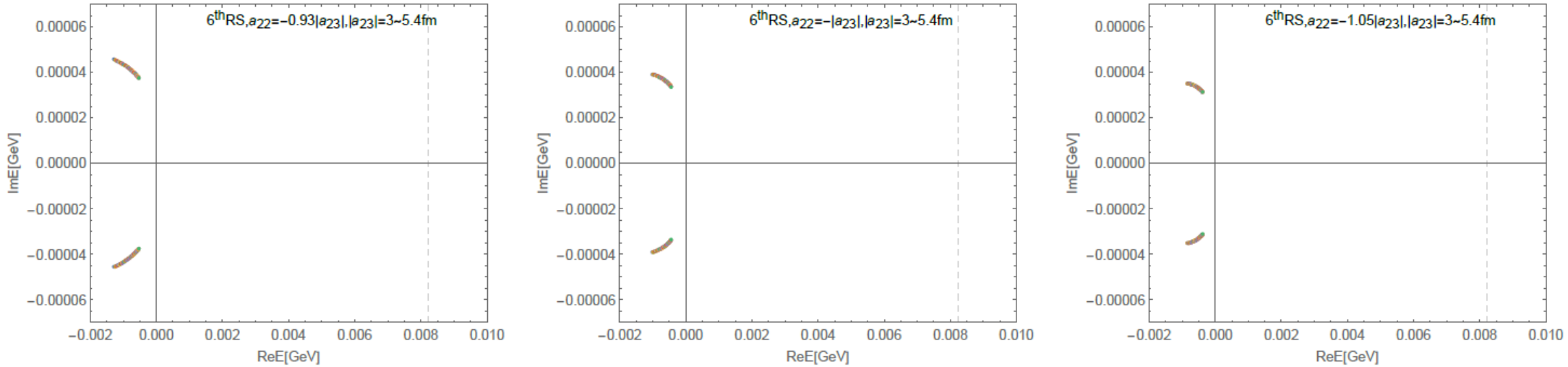
FIG. 29. Poles on the 4thRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 6: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = -0.93, -1, -1.05$:

The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = -0.93|a_{23}|$ (b)$a_{22} = -|a_{23}|$ (c)$a_{22} = -1.05|a_{23}|$

FIG. 30. Poles on the 6thRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx -|a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 6b: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

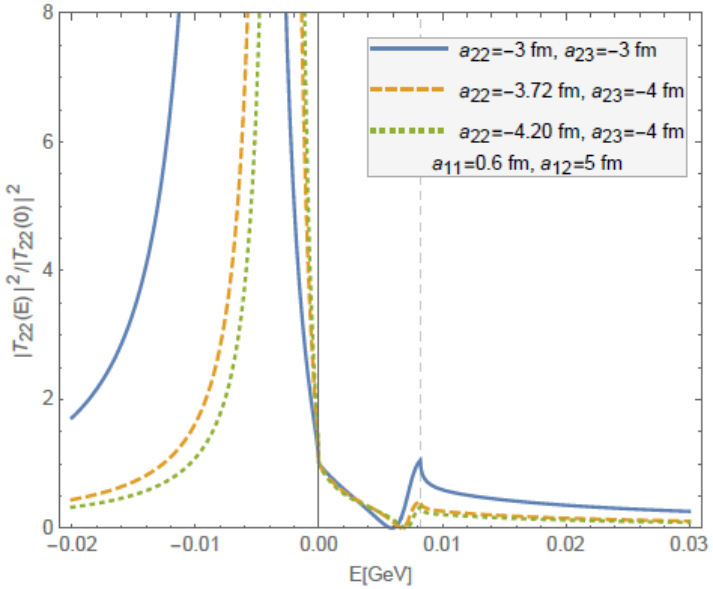
$a_{22} = -3$ fm, $a_{23} = -3$ fm

$a_{22} = -3.72$ fm, $a_{23} = -4$ fm

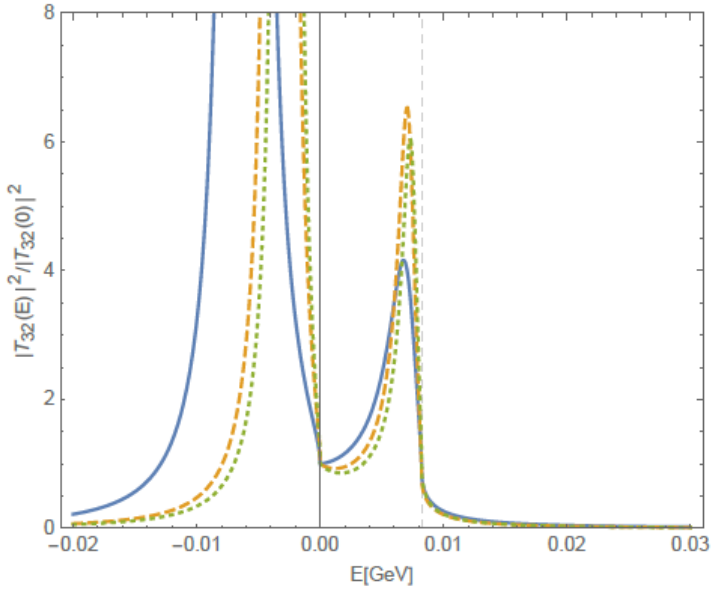
$a_{22} = -4.2$ fm, $a_{23} = -4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				2nd RS	3rd RS	4th RS	6th RS
0.6	5	−3	3	$-0.0063 \pm 0.0005i$	$0.0074 \pm 0.0013i$	$0.0072 \pm 0.0013i$	$-0.0010 \pm 0.0000i$
0.6	5	−3.72	4	$-0.0033 \pm 0.0003i$	$0.0074 \pm 0.0008i$	$0.0072 \pm 0.0008i$	$-0.0008 \pm 0.0000i$
0.6	5	−4.2	4	$-0.0028 \pm 0.0003i$	$0.0076 \pm 0.0007i$	$0.0075 \pm 0.0007i$	$-0.0006 \pm 0.0000i$

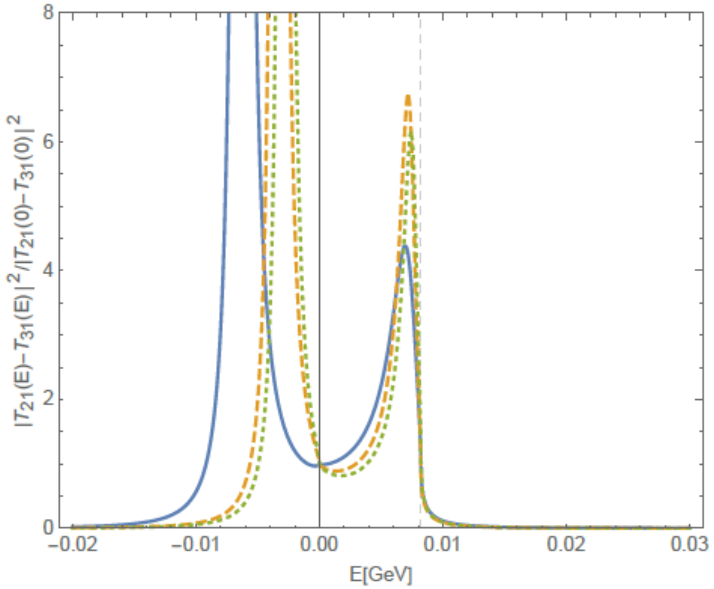
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



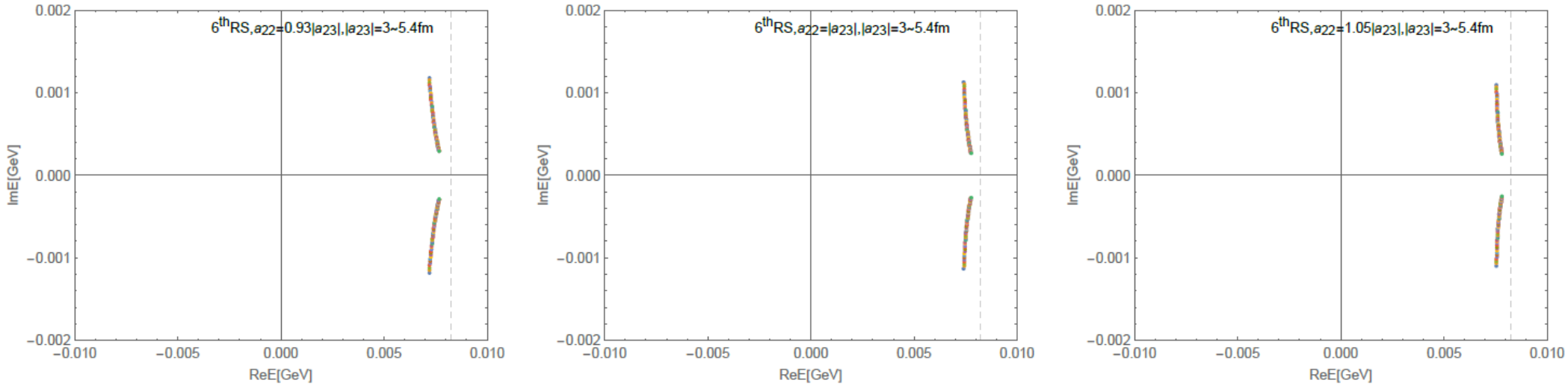
(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Threshold Behaviors Case 7: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 6th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

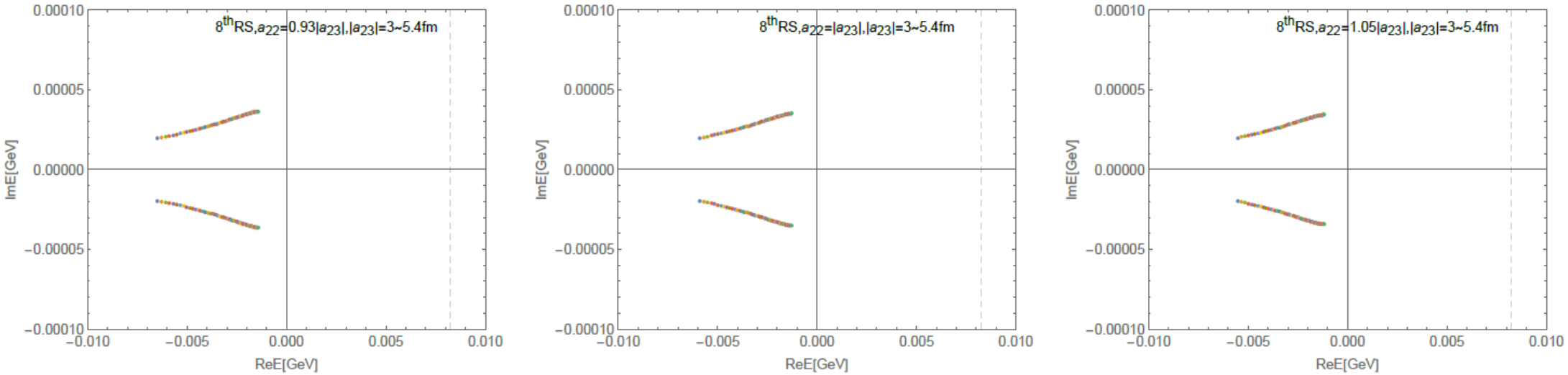
FIG. 35. Poles on the 6thRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 7: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole trajectories for $|a_{23}| = 3\sim5.4$ fm & $\frac{a_{22}}{|a_{23}|} = 0.93, 1, 1.05$:

The pole trajectories on the 8th Riemann sheet $a_{11} = 0.6$ fm, $a_{12} = 5$ fm



(a)$a_{22} = 0.93|a_{23}|$ (b)$a_{22} = |a_{23}|$ (c)$a_{22} = 1.05|a_{23}|$

FIG. 36. Poles on the 8thRS with $|a_{23}| = 3 \sim 5.4$ fm, $a_{22} \approx |a_{23}|$. Here we use the masses of $J/\psi\pi_0$, $D^0\bar{D}^{*0}$, and D^+D^{*-} for channel-1, channel-2, and channel-3.

Threshold Behaviors Case 7b: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7b: $a_{23} < 0$

$a_{11} = 0.6$ fm, $a_{12} = 5$ fm

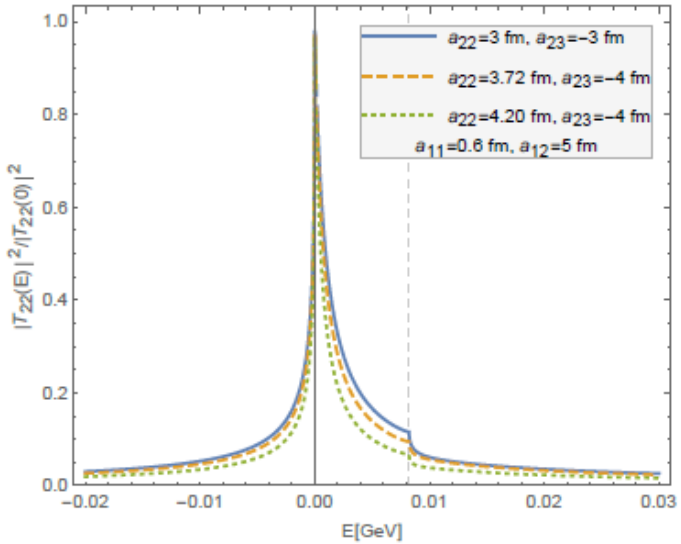
$a_{22} = 3$ fm, $a_{23} = -3$ fm

$a_{22} = 3.72$ fm, $a_{23} = -4$ fm

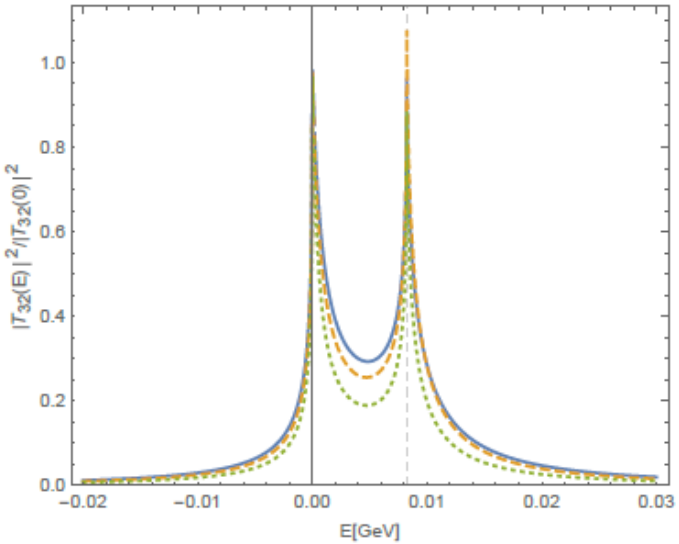
$a_{22} = 4.2$ fm, $a_{23} = -4$ fm

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)			
				4^{th} RS	5^{th} RS	6^{th} RS	8^{th} RS
0.6	5	3	3	$-0.0009 \pm 0.0001i$	$0.0074 \pm 0.0015i$	$0.0074 \pm 0.0012i$	$-0.0061 \pm 0.0000i$
0.6	5	3.72	4	$-0.0008 \pm 0.0001i$	$0.0074 \pm 0.0009i$	$0.0074 \pm 0.0006i$	$-0.0032 \pm 0.0000i$
0.6	5	4.2	4	$-0.0005 \pm 0.0001i$	$0.0076 \pm 0.0008i$	$0.0076 \pm 0.0006i$	$-0.0026 \pm 0.0000i$

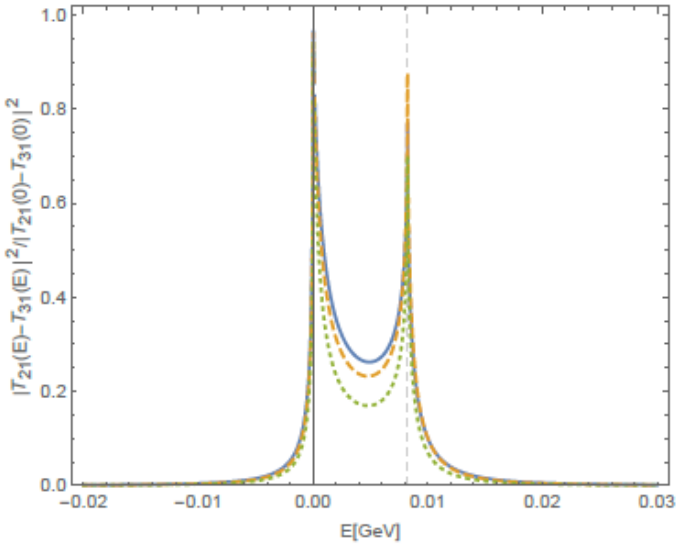
The line shapes for three groups of a_{22}&$|a_{23}|$:



(d)$|\mathbf{T}_{22}(E)|^2/|\mathbf{T}_{22}(0)|^2$



(e)$|\mathbf{T}_{32}(E)|^2/|\mathbf{T}_{32}(0)|^2$



(f)$|\mathbf{T}_{21}(E) - \mathbf{T}_{31}(E)|^2/|\mathbf{T}_{21}(0) - \mathbf{T}_{31}(0)|^2$

Duality Between the Poles for $a_{22} < 0$ and $a_{22} > 0$

The poles of the scattering amplitude are the zeros of det

$$\det[r_1, r_2, r_3, a_{11}, a_{22}, a_{33}, a_{23}]$$
$$= -\left[\frac{1}{a_{11}} - ir_1k_1\right]\left\{\left[\frac{1}{a_{22}} - ir_2k_2\right]\left[\frac{1}{a_{33}} - ir_3k_3\right] - \frac{1}{a_{23}^2}\right\} + \frac{1}{a_{12}^2}\left[\frac{1}{a_{33}} - ir_3k_3\right] + \frac{1}{a_{13}^2}\left[\frac{1}{a_{22}} - ir_2k_2\right] + \frac{2}{a_{12}a_{13}a_{23}} = 0$$

Reflection of parameters: $\mathcal{R} : a_{ii} \to -a_{ii}, a_{23} \to -a_{23}, r_i \to -r_i, i = 1, 2, 3$

Reflection of determinant:

$$\mathcal{R}\{\det[r_1, r_2, r_3, a_{11}, a_{22}, a_{33}, a_{23}]\} = \det[-r_1, -r_2, -r_3, -a_{11}, -a_{22}, -a_{33}, -a_{23}] = -\det[r_1, r_2, r_3, a_{11}, a_{22}, a_{33}, a_{23}]$$

Duality of the poles:

$$\det[r_1, r_2, r_3, a_{11}, a_{22}, a_{33}, a_{23}](E) = 0 \Longrightarrow \det[-r_1, -r_2, -r_3, -a_{11}, -a_{22}, -a_{33}, -a_{23}](E) = 0$$

$$1^{\text{st}}\text{RS}(+++) \overset{\mathcal{R}}{\longleftrightarrow} 7^{\text{th}}\text{RS}(---), 2^{\text{nd}}\text{RS}(-++) \overset{\mathcal{R}}{\longleftrightarrow} 8^{\text{th}}\text{RS}(+--),$$
$$3^{\text{rd}}\text{RS}(--+) \overset{\mathcal{R}}{\longleftrightarrow} 5^{\text{th}}\text{RS}(++-), 4^{\text{th}}\text{RS}(+-+) \overset{\mathcal{R}}{\longleftrightarrow} 6^{\text{th}}\text{RS}(-+-).$$

Duality Between the Poles for $a_{22} < 0$ and $a_{22} > 0$

The symmetry of the last two channels: $a_{22} = a_{33}, a_{12} = \pm a_{13}$

In the weak coupling limit $|a_{12}|\sqrt{2M_3\Delta} \gg 1$, the det reads

$$\det[r_1, r_2, r_3, a_{11}, a_{22}] \simeq -[\frac{1}{a_{11}} - ir_1k_1]\left\{[\frac{1}{a_{22}} - ir_2k_2][\frac{1}{a_{22}} - ir_3k_3] - \frac{1}{a_{23}^2}\right\} + O\left(\frac{\sqrt{2M_3\Delta}}{a_{12}^2}\right)$$

If $|k_1| \gg 1/|a_{11}|$, we have approximate reflection of parameters:

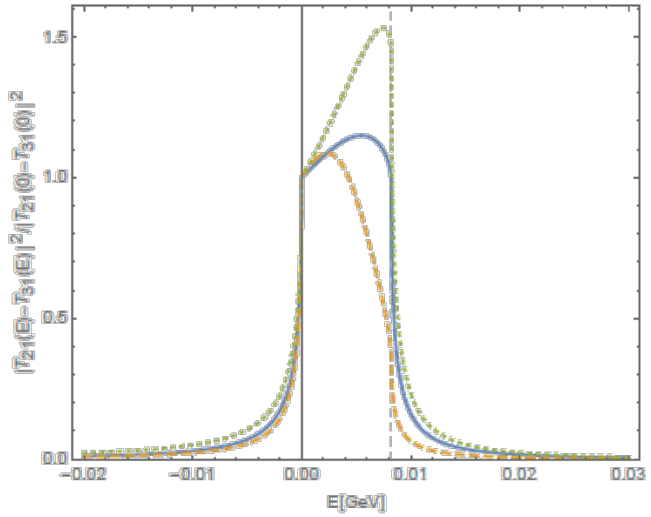
$$\mathcal{AR} : a_{22} \to -a_{22}, r_i \to -r_i, i = 1, 2, 3.,$$

Approximate reflection of determinant:

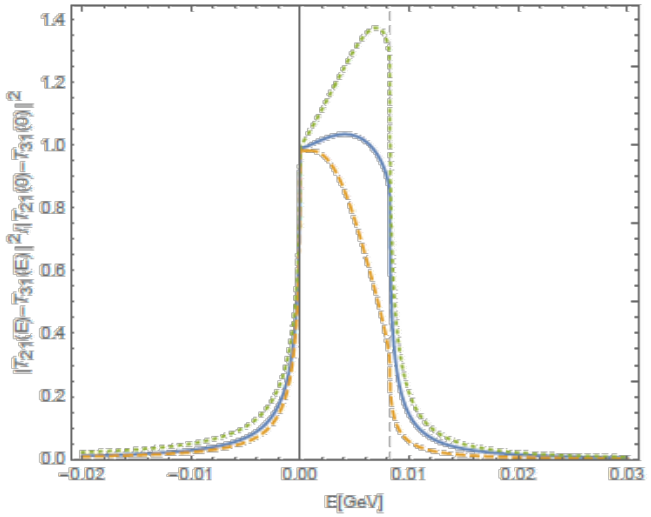
$$\mathcal{AR}\{\det[r_1, r_2, r_3, a_{22}]\} = \det[-r_1, -r_2, -r_3, -a_{22}] = -\det[r_1, r_2, r_3, a_{22}]$$

Approximate duality of the poles: $\mathrm{RS} : (r_1, r_2, r_3) \overset{\mathcal{AR}}{\longleftrightarrow} \mathrm{RS} : (-r_1, -r_2, -r_3).$

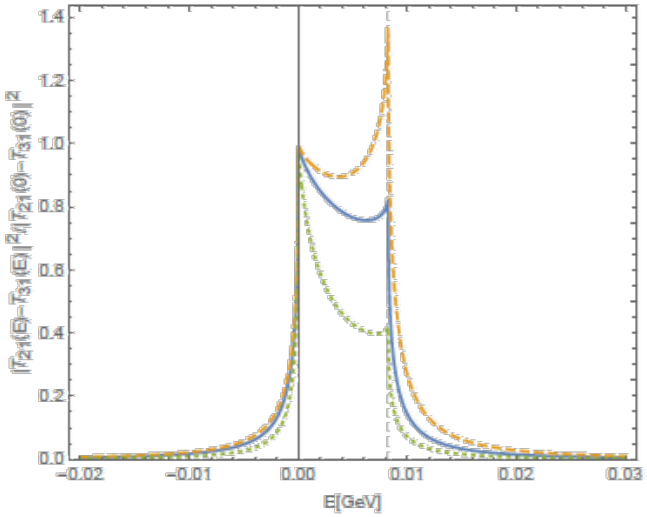
Line shapes of $|T_{21} - T_{31}|^2, a_{12} = a_{13}$



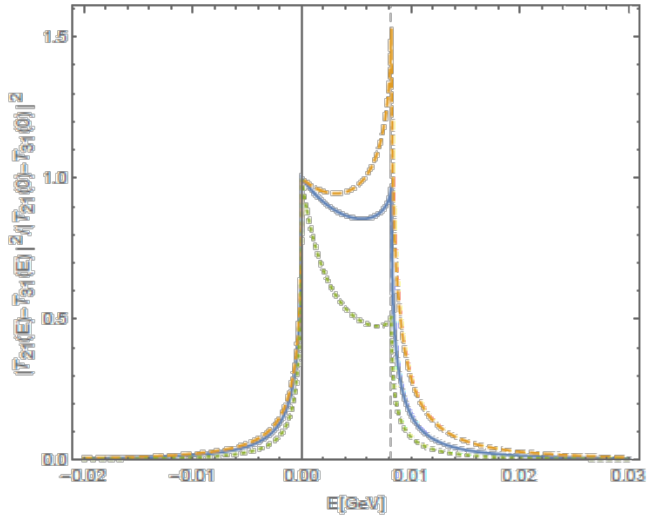
(a)Case 2a



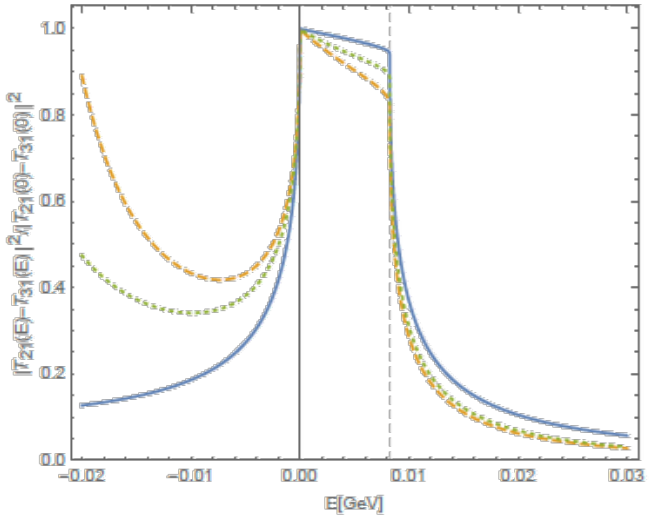
(b)Case 2b



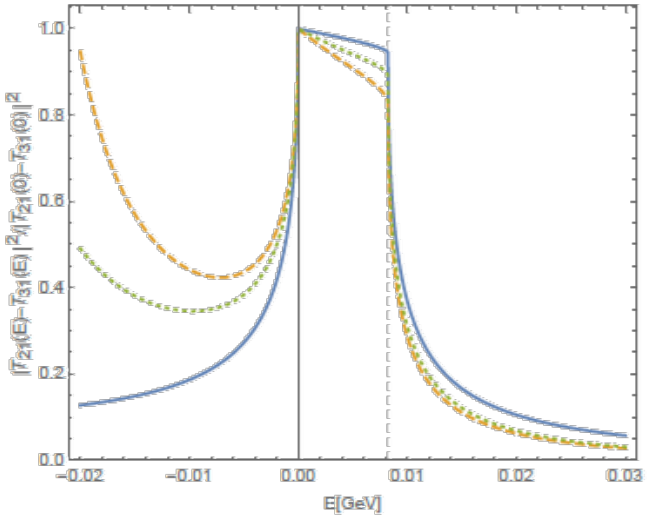
(c)Case 3a



(d)Case 3b

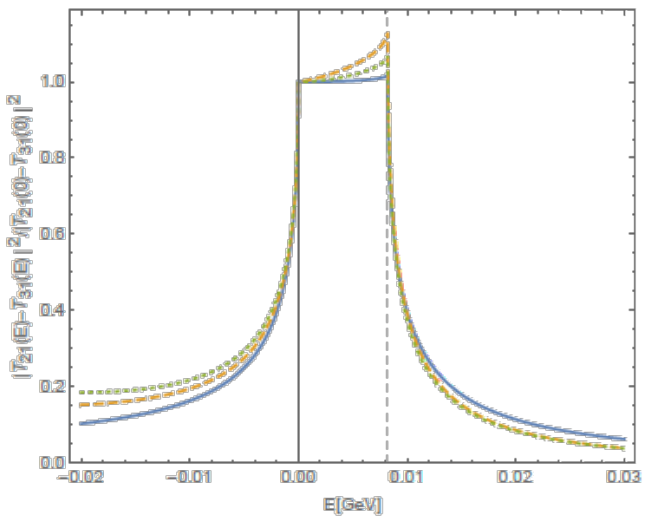


(e)Case 4a

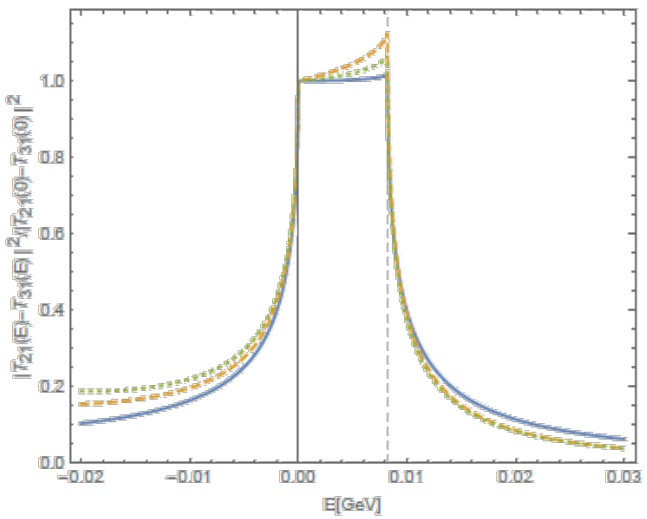


(f)Case 4b

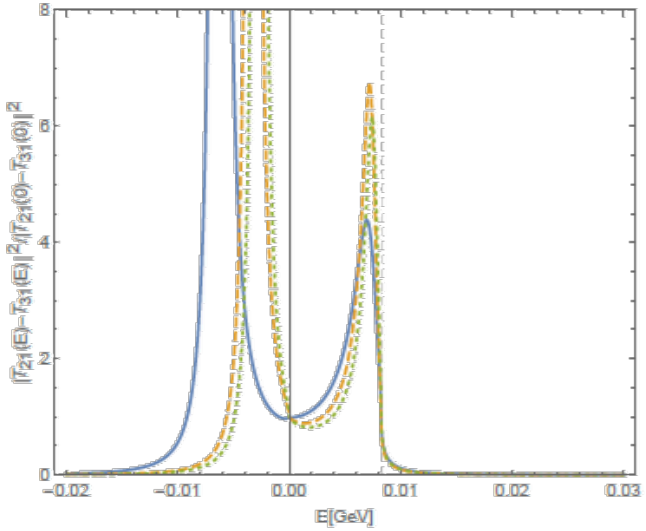
Line shapes of $|T_{21} - T_{31}|^2, a_{12} = a_{13}$



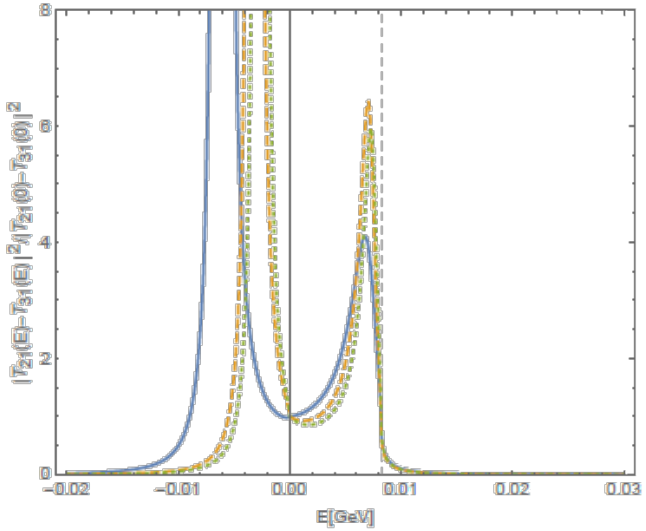
(g)Case 5a



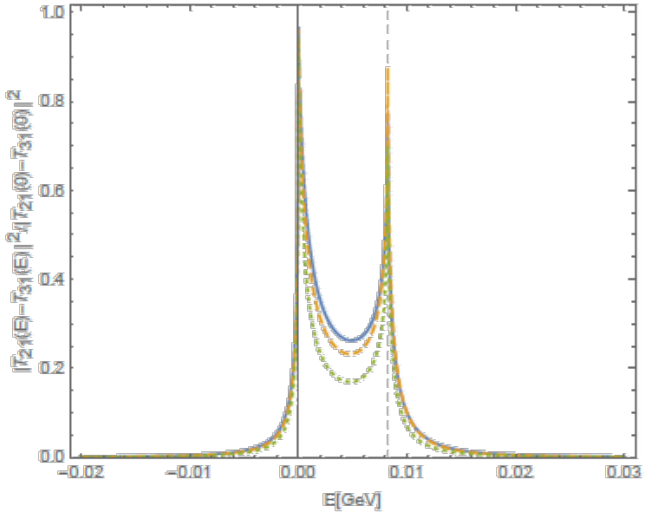
(h)Case 5b



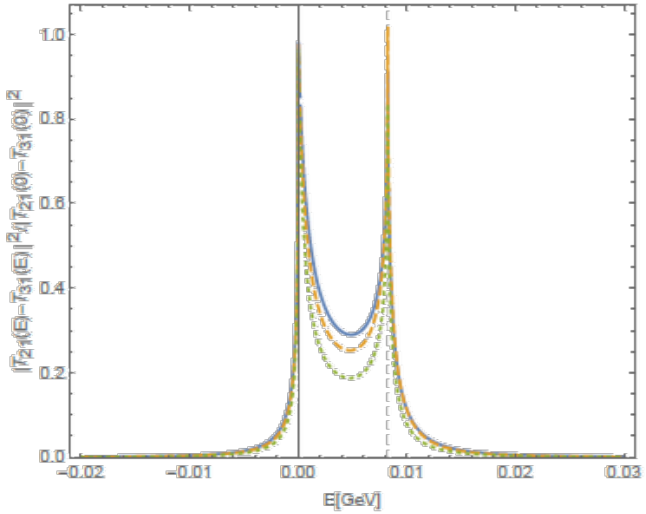
(i)Case 6a



(j)Case 6b

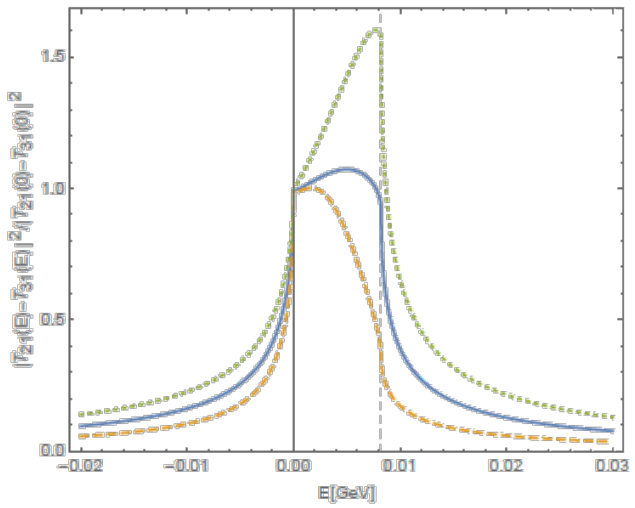


(k)Case 7a

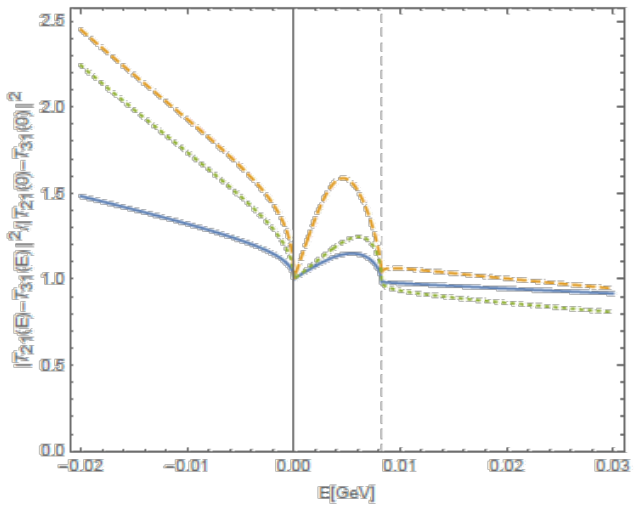


(l)Case 7b

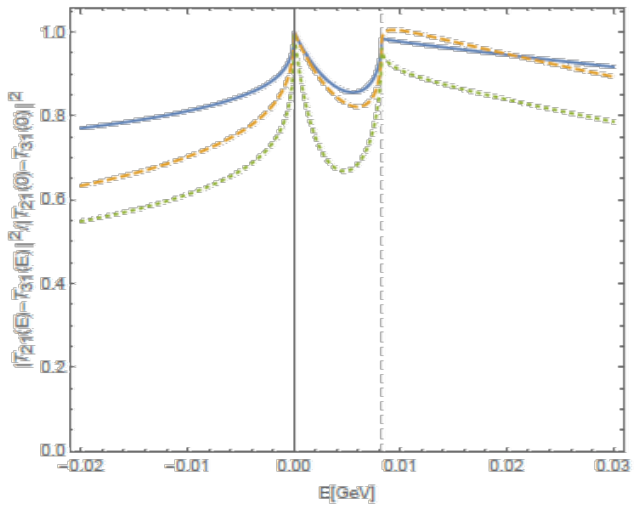
Line Shapes of $|T_{21} - T_{31}|^2, a_{12} = -a_{13}$



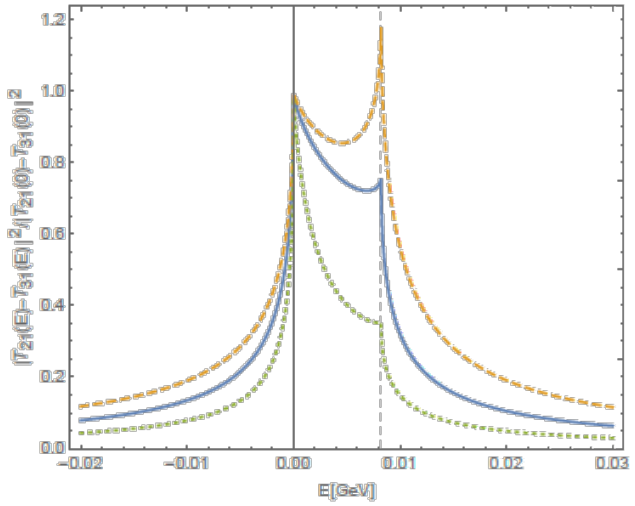
(a)Case 2a



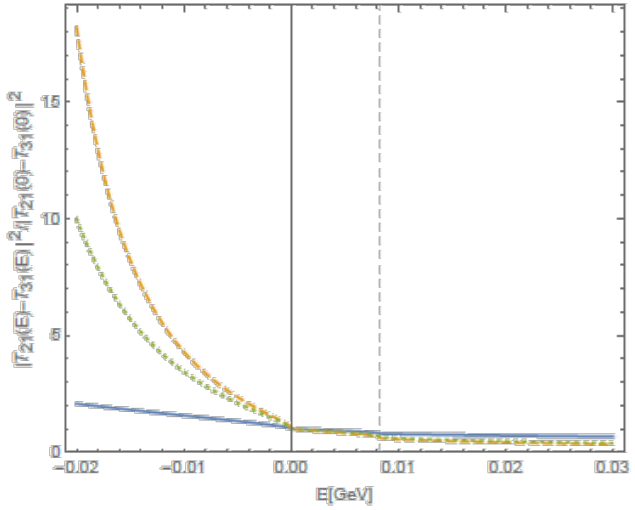
(b)Case 2b



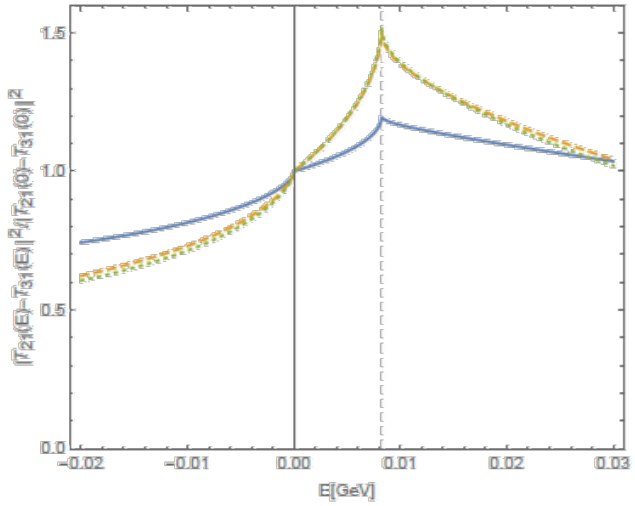
(c)Case 3a



(d)Case 3b

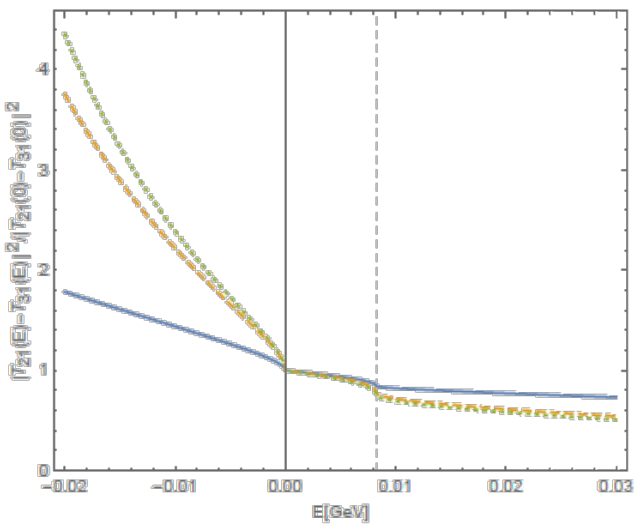


(e)Case 4a

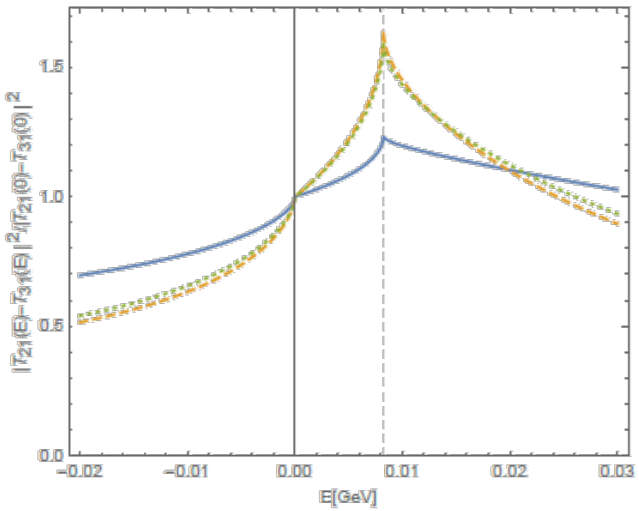


(f)Case 4b

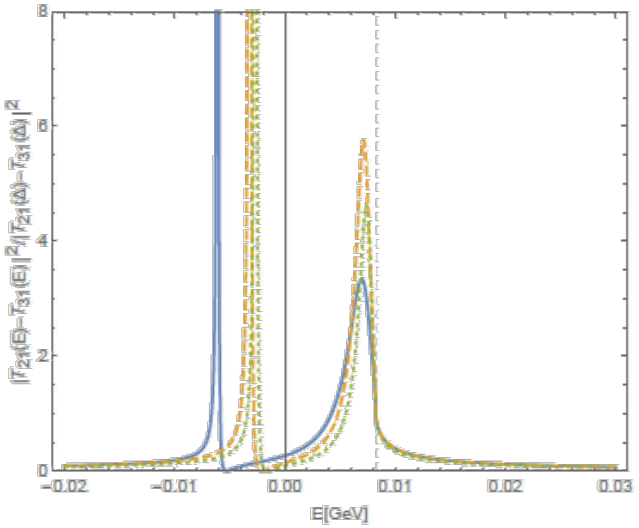
Line Shapes of $|T_{21} - T_{31}|^2, a_{12} = -a_{13}$



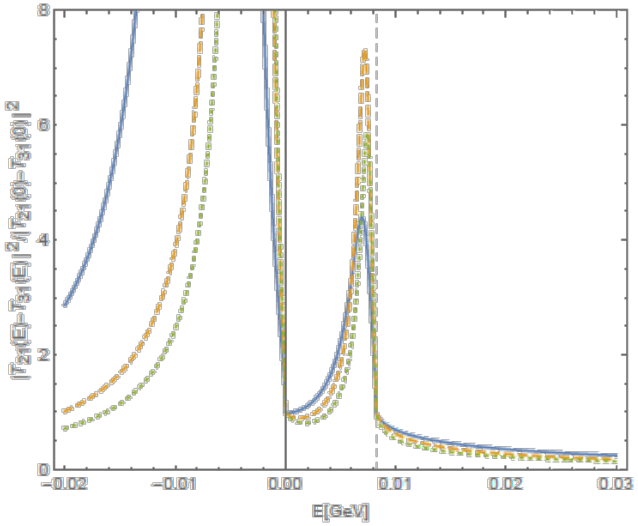
(g)Case 5a



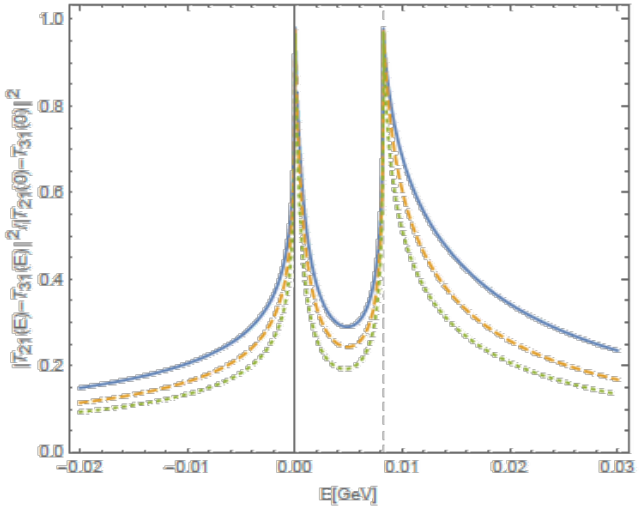
(h)Case 5b



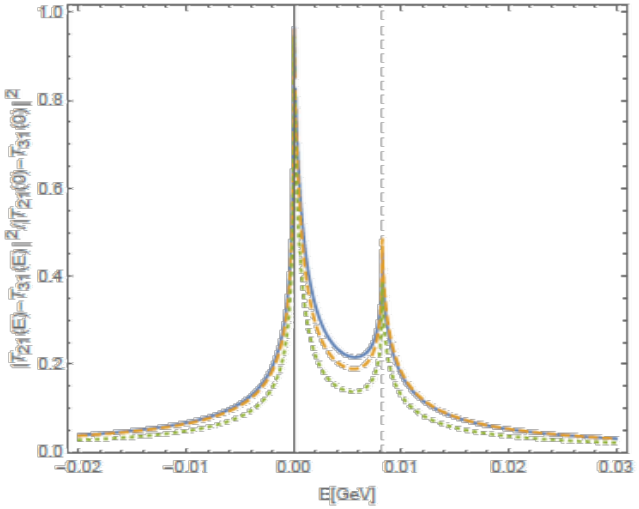
(i)Case 6a



(j)Case 6b

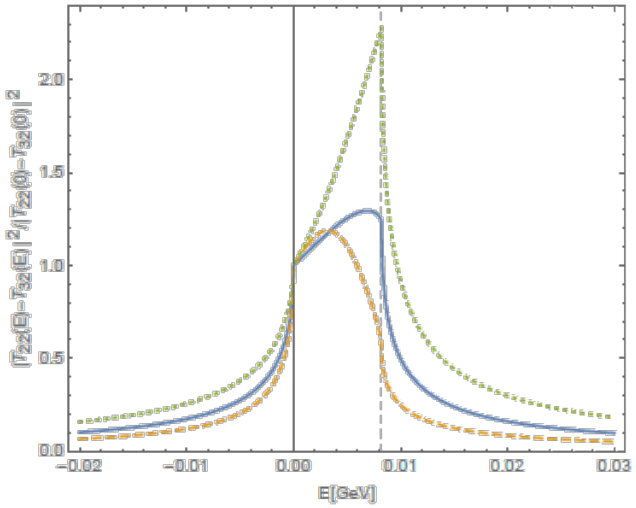


(k)Case 7a

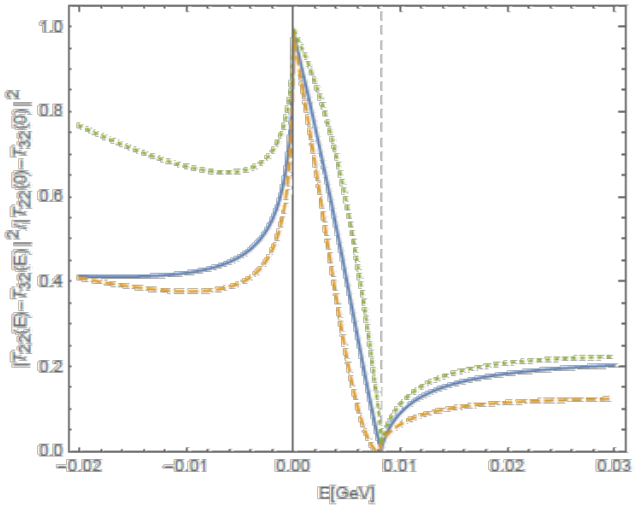


(l)Case 7b

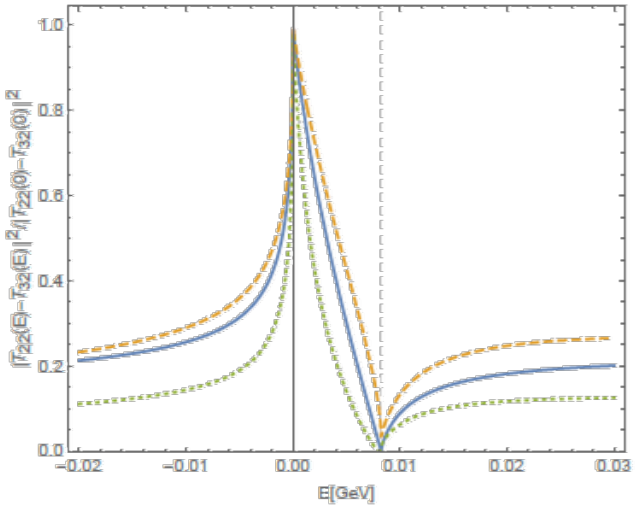
Line Shapes of $|T_{22} - T_{32}|^2$



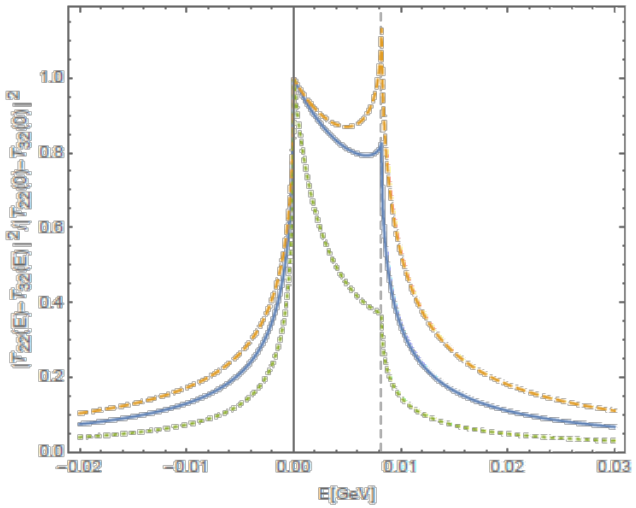
(a)Case 2a



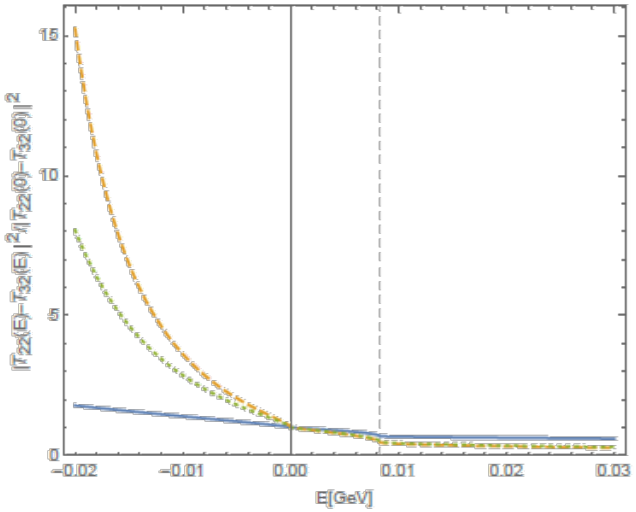
(b)Case 2b



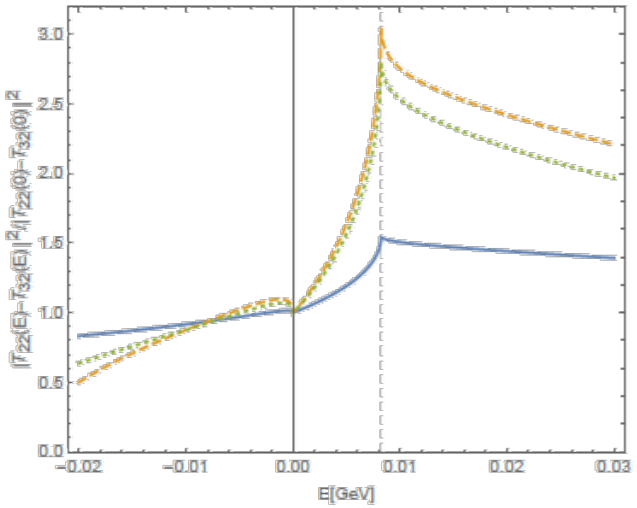
(c)Case 3a



(d)Case 3b

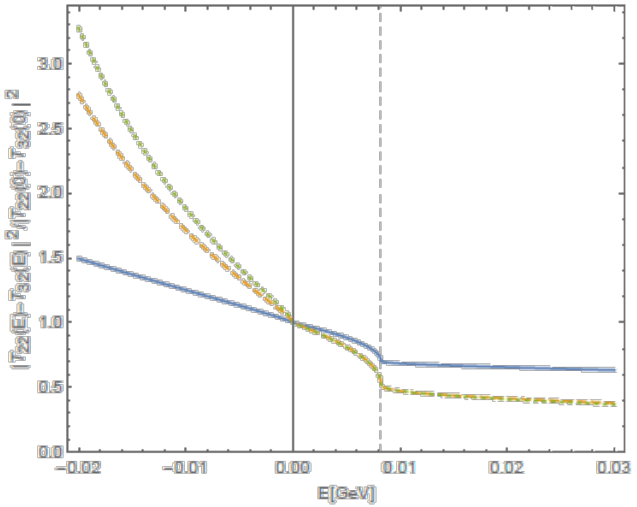


(e)Case 4a

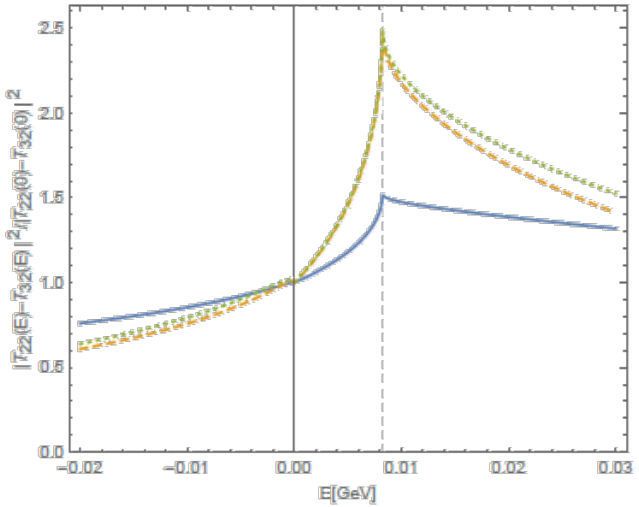


(f)Case 4b

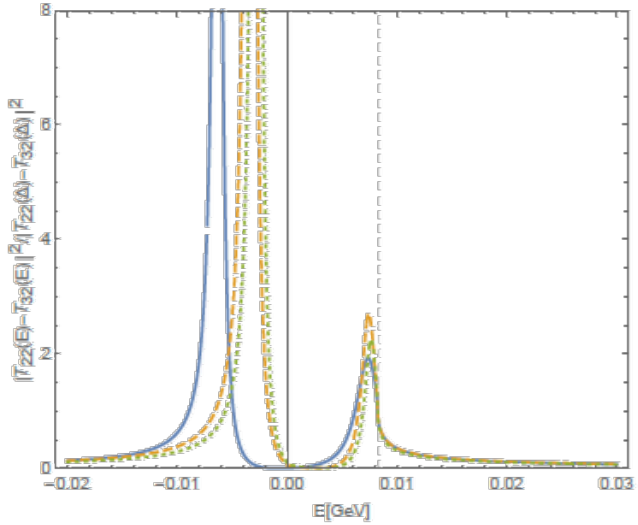
Line Shapes of $|T_{22} - T_{32}|^2$



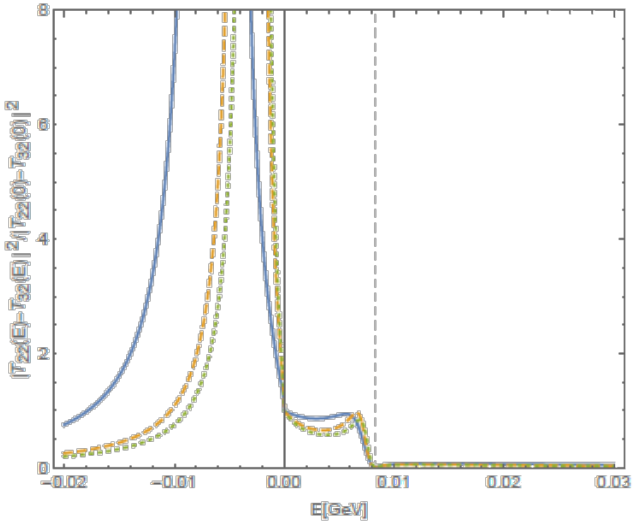
(g)Case 5a



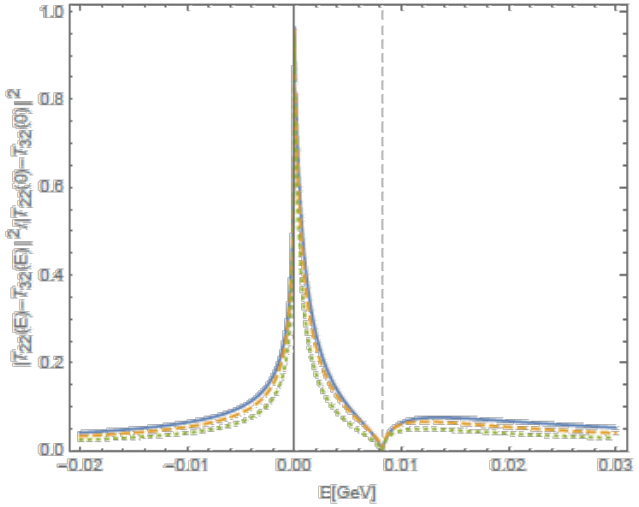
(h)Case 5b



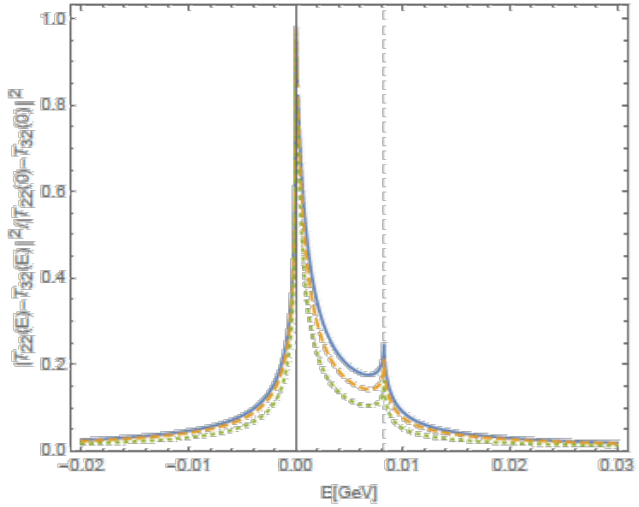
(i)Case 6a



(j)Case 6b



(k)Case 7a



(l)Case 7b

Threshold Behaviors Case 1: $|a_{22}|\sqrt{2M_3\Delta} \lesssim 1, |a_{23}|\sqrt{2M_3\Delta} \gg 1$

Case 1: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are weak.

The pole positions for three groups of a_{22}&$|a_{23}|$:

TABLE IV. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 1. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\|a_{23}\|$(fm)	Pole position (GeV)(the first row for $a_{22} < 0$ and the second row for $a_{22} > 0$)		
				2^{nd} RS	3^{rd} RS	6^{th} RS
				8^{th} RS	5^{th} RS	4^{th} RS
0.6	5	0.6	5	$-0.0666 \pm 0.0000i, -0.0380 \pm 0.0011$	$-0.0464 \pm 0.0007i$	$-0.0548 \pm 0.0007i$
0.6	5	1	5	$-0.0258 \pm 0.0000i, -0.0075 \pm 0.0006i$	$-0.0107 \pm 0.0005i$	$-0.0192 \pm 0.0004i$
0.6	5	-1	5	$-0.0262 \pm 0.0009i, -0.0078 \pm 0.0001i$	$-0.0111 \pm 0.0003i$	$-0.0196 \pm 0.0003i$

Threshold Behaviors Case 2: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx -|a_{23}|$

Case 2: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are strong.

The pole positions for three groups of a_{22}&$|a_{23}|$:

TABLE VI. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 2. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				3^{rd} RS	4^{th} RS	6^{th} RS
0.6	5	-0.3	0.3	$0.0122 \pm 0.0131i$	$0.0121 \pm 0.0131i$	$-0.0120 \pm 0.0000i$
0.6	5	-0.56	0.6	$0.0042 \pm 0.0073i$	$0.0041 \pm 0.0073i$	$-0.0128 \pm 0.0001i$
0.6	5	-0.63	0.6	$0.0130 \pm 0.0075i$	$0.0129 \pm 0.0076i$	$-0.0033 \pm 0.0000i$

a_{22}	$\lvert a_{23}\rvert$	Pole position		
		$\text{Re}E < 0$	$0 < \text{Re}E < \Delta$	$\text{Re}E > \Delta$
$a_{22}/\lvert a_{23}\rvert = -0.93, -1, -1.05$	$\lvert a_{23}\rvert = 0.2 \sim 0.8$ fm $\lvert a_{23}\rvert\sqrt{2M_3\Delta} = 0.13 \sim 0.51$	$6^{\text{th}}\text{RS}(-+-)$	$3^{\text{rd}}\text{RS}(--+)$ $4^{\text{th}}\text{RS}(+-+)$	$3^{\text{rd}}\text{RS}(--+)$ $4^{\text{th}}\text{RS}(+-+)$

Threshold Behaviors Case 3: $|a_{23}|\sqrt{2M_3\Delta} \lesssim 1, a_{22} \approx |a_{23}|$

Case 3: Couplings in CH2 & CH3 are weak, and couplings between CH2&3 are strong.

The pole positions for three groups of a_{22}&$|a_{23}|$:

TABLE VIII. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 3. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\lvert a_{23}\rvert$(fm)	Pole position (GeV)		
				4^{th} RS	5^{th} RS	6^{th} RS
0.6	5	0.3	0.3	$-0.0111 \pm 0.0016i$	$0.0130 \pm 0.0147i$	$0.0129 \pm 0.0114i$
0.6	5	0.56	0.6	$-0.0123 \pm 0.0010i$	$0.0043 \pm 0.0082i$	$0.0048 \pm 0.0066i$
0.6	5	0.63	0.6	$-0.0029 \pm 0.0006i$	$0.0135 \pm 0.0084i$	$0.0133 \pm 0.0066i$

a_{22}	$\lvert a_{23}\rvert$	Pole position		
		$\mathrm{Re}E < 0$	$0 < \mathrm{Re}E < \Delta$	$\mathrm{Re}E > \Delta$
$a_{22}/\lvert a_{23}\rvert = 0.93, 1, 1.05$	$\lvert a_{23}\rvert = 0.2 \sim 0.8$ fm $\lvert a_{23}\rvert\sqrt{2M_3\Delta} = 0.13 \sim 0.51$	$4^{\text{th}}\text{RS}(+-+)$	$5^{\text{th}}\text{RS}(++-)$ $6^{\text{th}}\text{RS}(-+-)$	$5^{\text{th}}\text{RS}(++-)$ $6^{\text{th}}\text{RS}(-+-)$

Threshold Behaviors Case 6: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx -|a_{23}|$

Case 6: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole positions for three groups of a_{22}&$|a_{23}|$:

TABLE XII. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 6. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\|a_{23}\|$(fm)	Pole position (GeV)			
				2^{nd} RS	3^{rd} RS	4^{th} RS	6^{th} RS
0.6	5	-3	3	$-0.0063 \pm 0.0005i$	$0.0074 \pm 0.0013i$	$0.0072 \pm 0.0013i$	$-0.0010 \pm 0.0000i$
0.6	5	-3.72	4	$-0.0033 \pm 0.0003i$	$0.0074 \pm 0.0008i$	$0.0072 \pm 0.0008i$	$-0.0008 \pm 0.0000i$
0.6	5	-4.2	4	$-0.0028 \pm 0.0003i$	$0.0076 \pm 0.0007i$	$0.0075 \pm 0.0007i$	$-0.0006 \pm 0.0000i$

a_{22}	$\|a_{23}\|$	Pole position		
		Re$E < 0$	$0 <$Re$E < \Delta$	Re$E > \Delta$
$a_{22}/\|a_{23}\| = -0.93, -1, -1.05$	$\|a_{23}\| = 3 \sim 5.4$ fm	2^{nd}RS$(--+)$	3^{rd}RS$(--+)$	no pole
	$\|a_{23}\|\sqrt{2M_3\Delta} = 1.92 \sim 3.46$	6^{th}RS$(-+-)$	4^{th}RS$(+-+)$	no pole

Threshold Behaviors Case 7: $|a_{23}|\sqrt{2M_3\Delta} \gg 1, a_{22} \approx |a_{23}|$

Case 7: Couplings in CH2 & CH3 are strong, and couplings between CH2&3 are weak.

The pole positions for three groups of a_{22}&$|a_{23}|$:

TABLE XIV. Pole positions for $a_{11} = 0.6$ fm, $|a_{12}| = 5$ fm and three groups of a_{22} and $|a_{23}|$ in Case 7. Here the pole positions are relative to the second threshold, and the third threshold is located at $E = \Delta = 0.00823$ GeV

a_{11}(fm)	a_{12}(fm)	a_{22}(fm)	$\|a_{23}\|$(fm)	Pole position (GeV)			
				4^{th} RS	5^{th} RS	6^{th} RS	8^{th} RS
0.6	5	3	3	$-0.0009 \pm 0.0001i$	$0.0074 \pm 0.0015i$	$0.0074 \pm 0.0012i$	$-0.0061 \pm 0.0000i$
0.6	5	3.72	4	$-0.0008 \pm 0.0001i$	$0.0074 \pm 0.0009i$	$0.0074 \pm 0.0006i$	$-0.0032 \pm 0.0000i$
0.6	5	4.2	4	$-0.0005 \pm 0.0001i$	$0.0076 \pm 0.0008i$	$0.0076 \pm 0.0006i$	$-0.0026 \pm 0.0000i$

a_{22}	$\|a_{23}\|$	Pole position		
		$\text{Re}E < 0$	$0 < \text{Re}E < \Delta$	$\text{Re}E > \Delta$
$a_{22}/\|a_{23}\| = 0.93, 1, 1.05$	$\|a_{23}\| = 3 \sim 5.4$ fm	$4^{\text{th}}\text{RS}(+-+)$	$5^{\text{th}}\text{RS}(++-)$	no pole
	$\|a_{23}\|\sqrt{2M_3\Delta} = 1.92 \sim 3.46$	$8^{\text{th}}\text{RS}(+--)$	$6^{\text{th}}\text{RS}(-+-)$	no pole