NLO Results With Operator Mixing For Fully Heavy Tetraquarks In QCD Sum Rules

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Based on this paper：

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C. Meng(孟策), K.T. Chao(赵光达); arXiv:2201.11714[hep-ph]

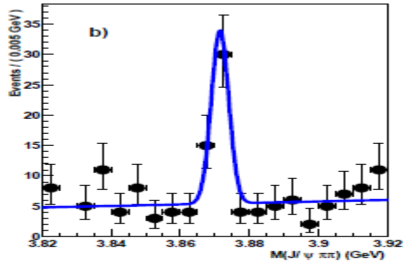
第四届重味物理和量子色动力学研讨会

Outline

- Background
- Fully Heavy Tetraquarks Mass Spectra
  - $\bar{c}c\bar{c}c$ Mass Spectra
  - $\bar{b}b\bar{b}b$ Mass Spectra
- Summary

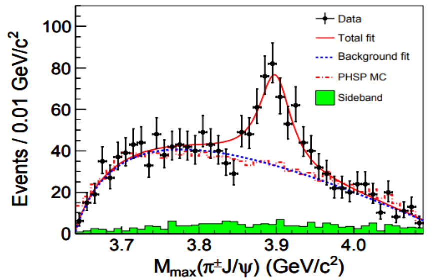
Background

■ New Hadron States



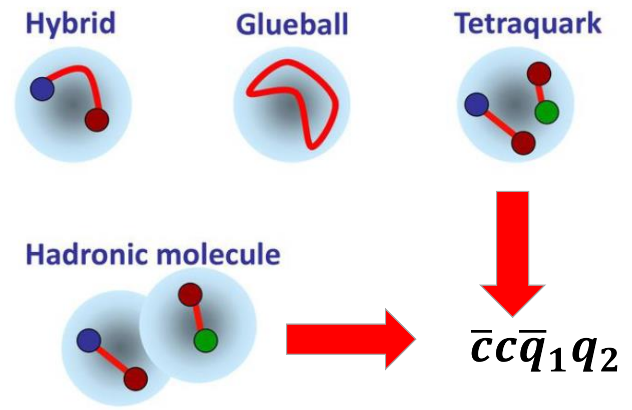
X(3872)

[Belle], PRL(2003).

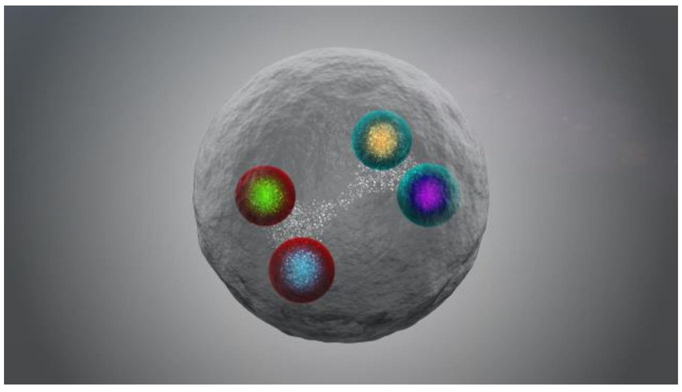


Zc(3900)

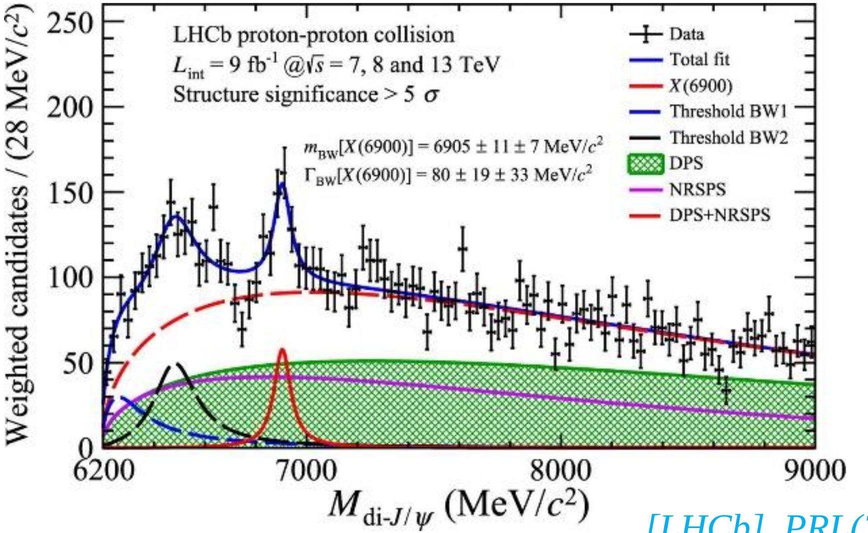
[BES III], PRL (2013)



- **X(6900)**



$\bar{c}c\bar{c}c$



[LHCb], PRL(2020).

■ Theoretical Works for $\bar{Q}Q\bar{Q}Q$ system

- **Models and tools**
  - **QCD sum rules** *W. Chen et. al., (2017); Z.G. Wang (2020); R.M. Albuquerque and S. Narison (2020) ……*
  - **Lattice QCD** *C. Hughes et. al., (2017)*
  - **Potential Models** *Y. Iwasaki (1975); K.T. Chao (1981); Richard J. Lloyd, et. al. (2004); J. Wu, et. al.,(2018); Y. Bai et. al., (2016); M. Karliner, et. al. (2017); V.R. Debastiani (2019); M.S. Liu et. al., (2019) ……*

- **$\bar{c}c\bar{c}c$ System**
  - **Exist bound states below $J/\psi J/\psi$** *L. Heller, et. al., (1985); Z.G. Wang (2020);…*
  - **Do not exist bound states below $J/\psi J/\psi$** *J. Ader, et. al., (1982); W. Chen, et. Al., (2019)…*

- **$\bar{b}b\bar{b}b$ System**
  - **Exist bound states below $\eta_b\eta_b$** *Y. Bai, et. al., (2016); W. Chen, et. al., (2019)…*
  - **Do not exist bound states below $\eta_b\eta_b$** *C. Hughes, et. al., (2017);*

■ $\bar{Q}Q\bar{Q}Q$ System study in QCD sum rules - LO

- **Moment QCD sum rules**

J^{PC}	Currents	m_{X_c}(GeV)	m_{X_b}(GeV)
0^{++}	J_1	6.44 ± 0.15	18.45 ± 0.15
	J_2	6.59 ± 0.17	18.59 ± 0.17
	J_3	6.47 ± 0.16	18.49 ± 0.16
	J_4	6.46 ± 0.16	18.46 ± 0.14
	J_5	6.82 ± 0.18	19.64 ± 0.14
0^{-+}	J_1^+	6.84 ± 0.18	18.77 ± 0.18
	J_2^+	6.85 ± 0.18	18.79 ± 0.18
0^{--}	J_1^-	6.84 ± 0.18	18.77 ± 0.18
1^{++}	$J_{1\mu}^+$	6.40 ± 0.19	18.33 ± 0.17
	$J_{2\mu}^+$	6.34 ± 0.19	18.32 ± 0.18
1^{+-}	$J_{1\mu}^-$	6.37 ± 0.18	18.32 ± 0.17
	$J_{2\mu}^+$	6.51 ± 0.15	18.54 ± 0.15
1^{-+}	$J_{1\mu}^+$	6.84 ± 0.18	18.80 ± 0.18
	$J_{2\mu}^+$	6.88 ± 0.18	18.83 ± 0.18
1^{--}	$J_{1\mu}^-$	6.84 ± 0.18	18.77 ± 0.18
	$J_{2\mu}^-$	6.83 ± 0.18	18.77 ± 0.16
2^{++}	$J_{1\mu\nu}$	6.51 ± 0.15	18.53 ± 0.15
	$J_{2\mu\nu}$	6.37 ± 0.19	18.32 ± 0.17

W. Chen, et. al., 2019

- **Laplace QCD sum rules**

	M_Y(GeV)
$cc\bar{c}\bar{c}(0^{++})$	5.99 ± 0.08
$cc\bar{c}\bar{c}(1^{+-})$	6.05 ± 0.08
$cc\bar{c}\bar{c}(2^{++})$	6.09 ± 0.08
$bb\bar{b}\bar{b}(0^{++})$	18.84 ± 0.09
$bb\bar{b}\bar{b}(1^{+-})$	18.84 ± 0.09
$bb\bar{b}\bar{b}(2^{++})$	18.85 ± 0.09
$cc\bar{c}\bar{c}(1^{--})$	6.11 ± 0.08
$bb\bar{b}\bar{b}(1^{--})$	18.89 ± 0.09

Zhi-Gang Wang, 2018

	M_X (GeV)	M_X (GeV)
0^{++} case A	6.44 ± 0.11	18.38 ± 0.11
0^{++} case B	6.87 ± 0.10	18.50 ± 0.10
0^{++} case C	6.52 ± 0.11	18.44 ± 0.10
0^{++} case D	6.96 ± 0.11	18.59 ± 0.11

Bo-Cheng Yang et. al., 2020

J^{PC}	M_1(GeV)[7]
0^{++}	5.99 ± 0.08
1^{+-}	6.05 ± 0.08
2^{++}	6.09 ± 0.08
1^{--}	6.11 ± 0.08

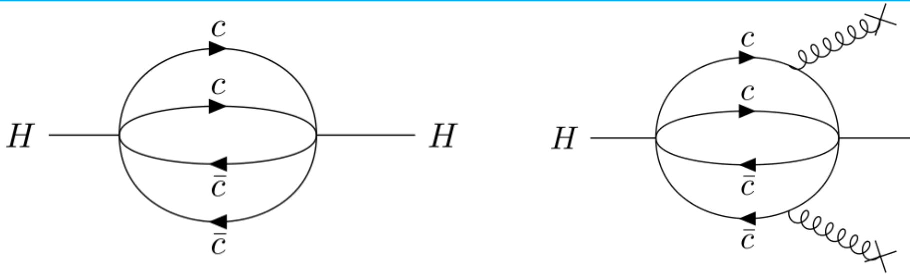
Zhi-Gang Wang, 2020

0^{++} case 1	$6.44^{+0.15}_{-0.16}$
0^{++} case 2	$6.45^{+0.14}_{-0.16}$
0^{++} case 3	$6.46^{+0.13}_{-0.17}$
0^{++} case 4	$6.47^{+0.12}_{-0.18}$

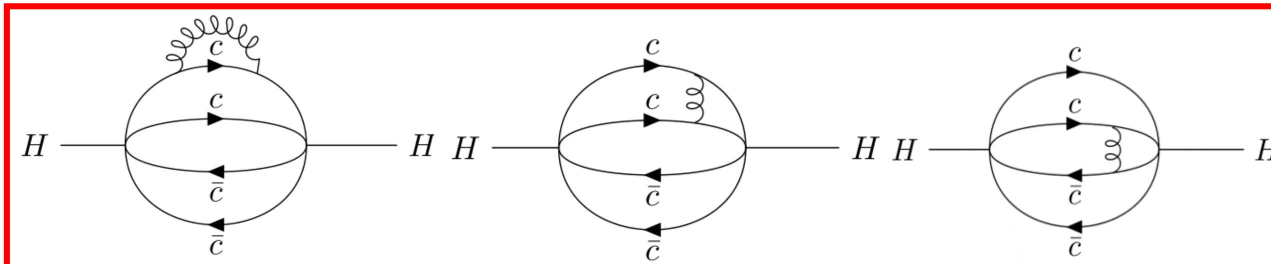
Jian-Rong Zhang, 2020

■ $\bar{Q}Q\bar{Q}Q$ System study in QCD sum rules - NLO

- **NLO corrections are non-negligible** $\left\{ \begin{array}{l} \Xi_{cc}^{++}\text{: C.Y. Wang, et. al., PRD(2019)} \\ \Omega_{QQQ}\text{: R.H. Wu, et. al., CPC(2021)} \end{array} \right.$
- **NLO corrections with $\bar{Q}Q\bar{Q}Q$ System**



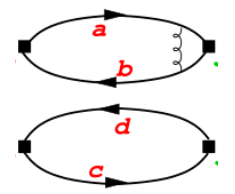
$C_1 - \mathrm{LO}$ $\qquad$ $C_{GG} - \mathrm{LO}$



$C_1 - \mathrm{NLO}$

Lack of complete NLO corrections to C_1 !

Only factorized diagrams!



R.M. Albuquerque and S. Narison, (2020)

M_H [MeV]		
0^{++} Molecule		
$\eta_q\eta_q$	6029 ± 198	19259 ± 88
$J/\psi J/\psi, \Upsilon\Upsilon$	6376 ± 367	19430 ± 145
$\chi_{q1}\chi_{q1}$	6494 ± 66	19770 ± 137
$\chi_{q0}\chi_{q0}$	6675 ± 98	19653 ± 131
0^{++} Tetraquark		
Eq. 25		
S_qS_q	6411 ± 83	19217 ± 120
A_qA_q	6450 ± 75	19872 ± 156
V_qV_q	6462 ± 175	19489 ± 79
P_qP_q	6795 ± 268	19754 ± 79
Eq. 26		
A_qA_q	6471 ± 67	19717 ± 118

QCD sum rules

- Correlation function:

$$\Pi(p^2) = i\int d^4x\, e^{ip\cdot x}\ \langle\Omega|\ \mathcal{T}_1(x)\mathcal{T}_2{}^{+}(0)\,|\Omega\rangle \qquad \mathcal{T} = (\bar{c}_i\Gamma_1 c_i)(\bar{c}_j\Gamma_2 c_j)$$

- Källén – Lehmann representation

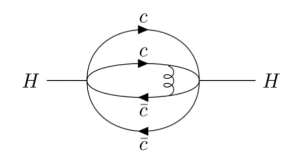
$$\Pi(p^2) = \int_0^\infty ds \frac{\rho(s)}{s - p^2 - i\epsilon}$$

$$\boxed{\begin{aligned}\rho(s) &= \textstyle\sum_i \lambda_i\, \delta(s - M_i{}^2) + \rho_{cont}(s)\ \theta(s - s_h)\\ &\approx \lambda_H\, \delta(s - M_H{}^2) + \tilde{\rho}_{cont}(s)\, \theta(s - s_h)\end{aligned}}$$

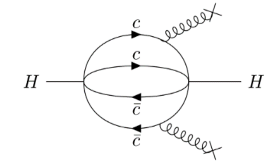
- The mass of ground state

(OPE, Borel transform, Quark-Hadron Duality…)

$$\boxed{M_H^2 = \frac{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\, s\, e^{-s/M_B^2} + \int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\, s\, e^{-s/M_B^2}\langle GG\rangle}{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\ e^{-s/M_B^2} + \int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\ e^{-s/M_B^2}\langle GG\rangle}}$$



C_1 – NLO



C_{GG} – LO

- Borel Platform

The point where the parameter dependence of M_H is weakest within Borel windows

$$\Delta(x, y) = \left(\frac{\partial M_H}{\partial x}\right)^2 + \left(\frac{\partial M_H}{\partial y}\right)^2 \qquad (\,x = s_0\,,\ \ y = M_B^2\,)$$

$\bar{c}c\bar{c}c$ Mass Spectra

- $J^{PC} = 0^{++}$

  - Meson-Meson type Operators $\boxed{\mathcal{J}_{\mathrm{M-M}} = (\bar{Q}_i \Gamma_1 Q_i)(\bar{Q}_j \Gamma_2 Q_j)}$

  - Diquark- Antidiquark type Operator $\boxed{\mathcal{J}_{\mathrm{Di-Di}} = ({Q_i}^T\, \mathcal{C}\, \Gamma_1\, Q_j)(\bar{Q}_i\, \Gamma_2 \mathcal{C}\, {\bar{Q}_j}^T)}$

  - **Diagonalized operator** $\boxed{\boldsymbol{\mathcal{J}_{\mathrm{Dia}} = \mathrm{T}.\mathcal{J}_{\mathrm{M-M}}}}$

$$(\boldsymbol{\Gamma_1}, \boldsymbol{\Gamma_2}) = \begin{pmatrix} (\boldsymbol{\gamma^\mu}, \boldsymbol{\gamma_\mu}) \\ (\boldsymbol{\gamma^\mu\gamma^5}, \boldsymbol{\gamma_\mu\gamma^5}) \\ (1,1) \\ (\boldsymbol{i\gamma^5}, \boldsymbol{i\gamma^5}) \\ (\boldsymbol{\sigma^{\mu\nu}}, \boldsymbol{\sigma_{\mu\nu}}) \end{pmatrix}$$

Reducing renormalization scale μ dependence

$$\delta \begin{pmatrix} -6 & -2 & -12 & -12 & 0 \\ -2 & -6 & 12 & 12 & 0 \\ 0 & 0 & 26 & 6 & \frac{1}{3} \\ 0 & 0 & 6 & 26 & -\frac{1}{3} \\ 0 & 0 & -40 & 40 & -\frac{68}{3} \end{pmatrix}$$

The anomalous dimension matrix of $\mathcal{J}_{M-M}$

Diagonalization

$$\frac{4}{3}\delta \begin{pmatrix} -6 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & -1+\sqrt{241} & 0 \\ 0 & 0 & 0 & 0 & -1-\sqrt{241} \end{pmatrix}$$

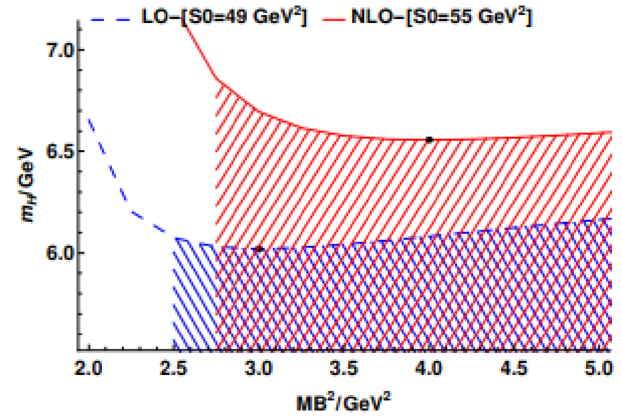
The anomalous dimension matrix of $\mathcal{J}_{Dia}$

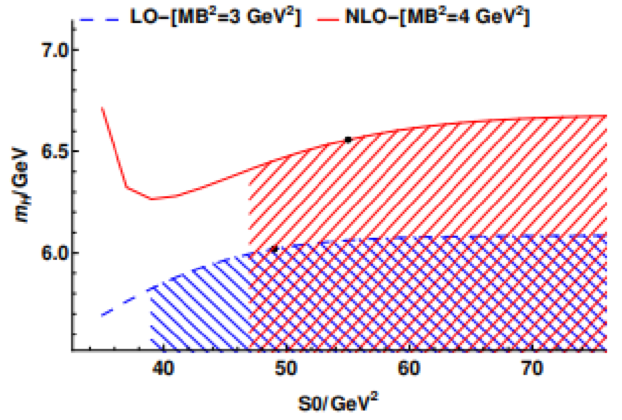
$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{15}{\sqrt{241}} & \frac{15}{\sqrt{241}} & \frac{1}{2} - \frac{8}{\sqrt{241}} \\ 0 & 0 & \frac{15}{\sqrt{241}} & -\frac{15}{\sqrt{241}} & \frac{1}{2} + \frac{8}{\sqrt{241}} \end{pmatrix}$$

■ The Borel Platform Curves - $\bar{c}c\bar{c}c$

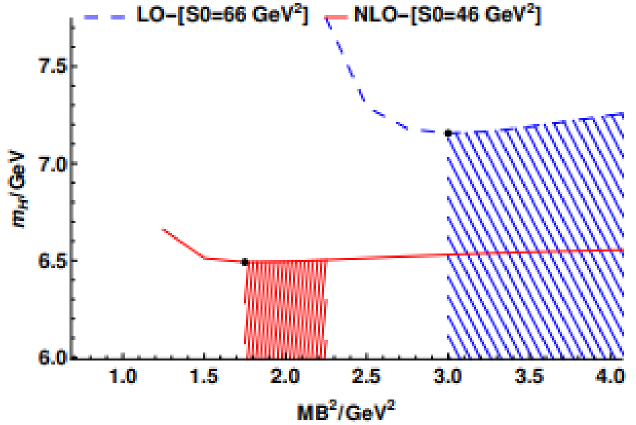
$\boxed{J_{S,4}^{Dia}}$

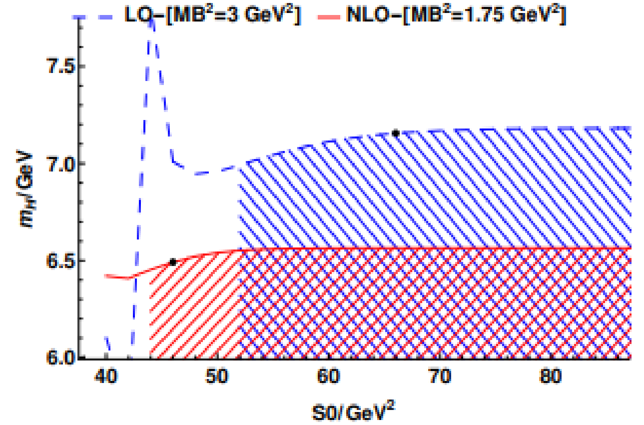
- $\overline{MS}$





- OS





➢ $J^{PC} = 0^{++}$ Diagonalized Operators

- $\overline{MS}$

流算符	LO	NLO($\overline{MS}$)
$J_{S,1}^{Dia}$	$6.18^{+0.08}_{-0.10}$	$7.81^{+0.14}_{-0.16}$
$J_{S,2}^{Dia}$	$6.19^{+0.07}_{-0.12}$	$6.95^{+0.10}_{-0.12}$
$J_{S,3}^{Dia}$	$5.93^{+0.07}_{-0.10}$	$6.35^{+0.06}_{-0.13}$
$J_{S,4}^{Dia}$	$6.02^{+0.05}_{-0.06}$	$6.56^{+0.10}_{-0.12}$
$J_{S,5}^{Dia}$	$6.33^{+0.12}_{-0.14}$	$7.72^{+0.13}_{-0.14}$

- OS

流算符	LO	NLO(OS)
$J_{S,1}^{Dia}$	$7.36^{+0.07}_{-0.10}$	$6.60^{+0.09}_{-0.12}$
$J_{S,2}^{Dia}$	$7.31^{+0.08}_{-0.12}$	$6.58^{+0.08}_{-0.11}$
$J_{S,3}^{Dia}$	$7.06^{+0.07}_{-0.10}$	$6.47^{+0.08}_{-0.10}$
$J_{S,4}^{Dia}$	$7.16^{+0.04}_{-0.05}$	$6.49^{+0.07}_{-0.10}$
$J_{S,5}^{Dia}$	$7.44^{+0.12}_{-0.14}$	$6.62^{+0.09}_{-0.13}$

◆ LO VS NLO

NLO corrections are **significant**.

- $\left|M_H^{NLO} - M_H^{LO}\right| > 0.5$ GeV
- Below or above $\eta_c\eta_c$?

◆ $\overline{MS}$ VS OS

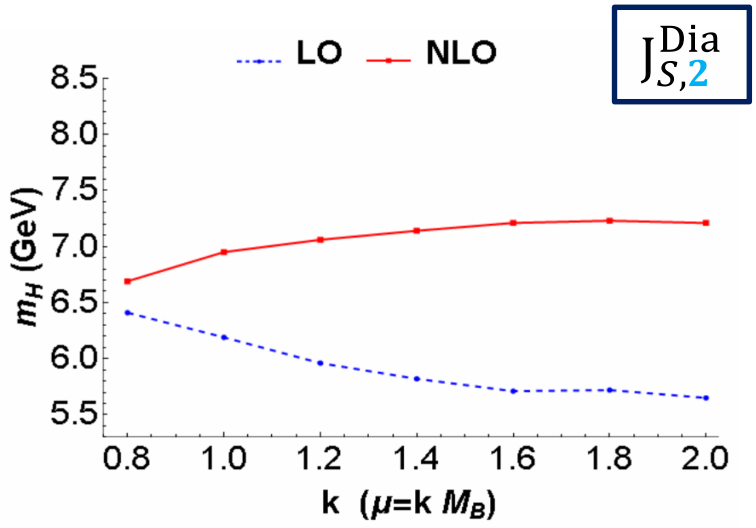
The **scheme dependence** is **reduced** observably.

- $\left|M_H^{\overline{MS},\,LO} - M_H^{OS,\,LO}\right| > 1$ GeV
- $\left|M_H^{\overline{MS},\,NLO} - M_H^{OS,NLO}\right| \sim 0.5$ GeV

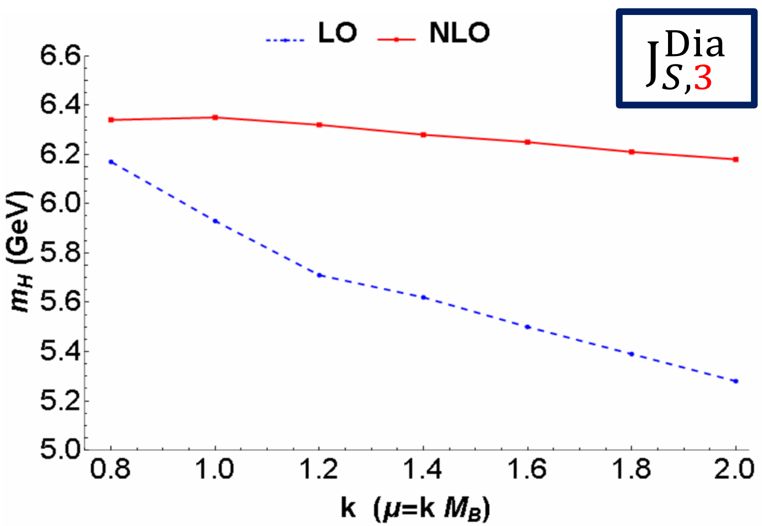
◆ Before VS After (Diagonalization)

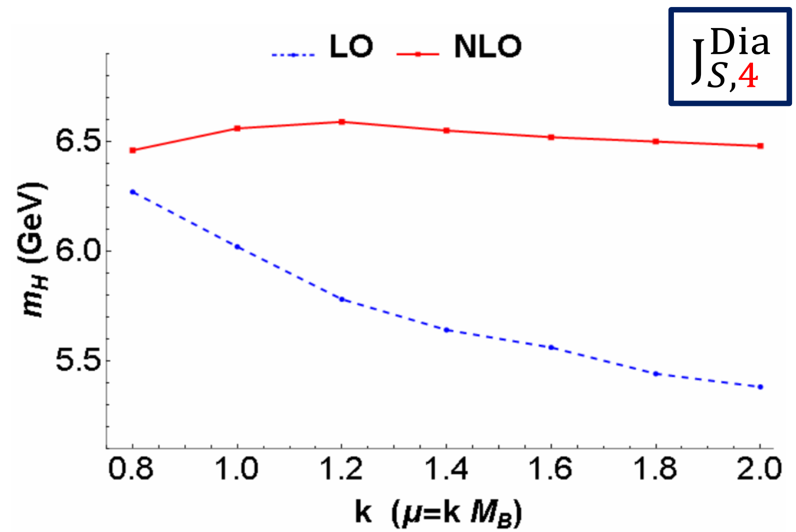
- The problem that Oversize NLO corrections are improved

■ The renormalization scale μ dependence



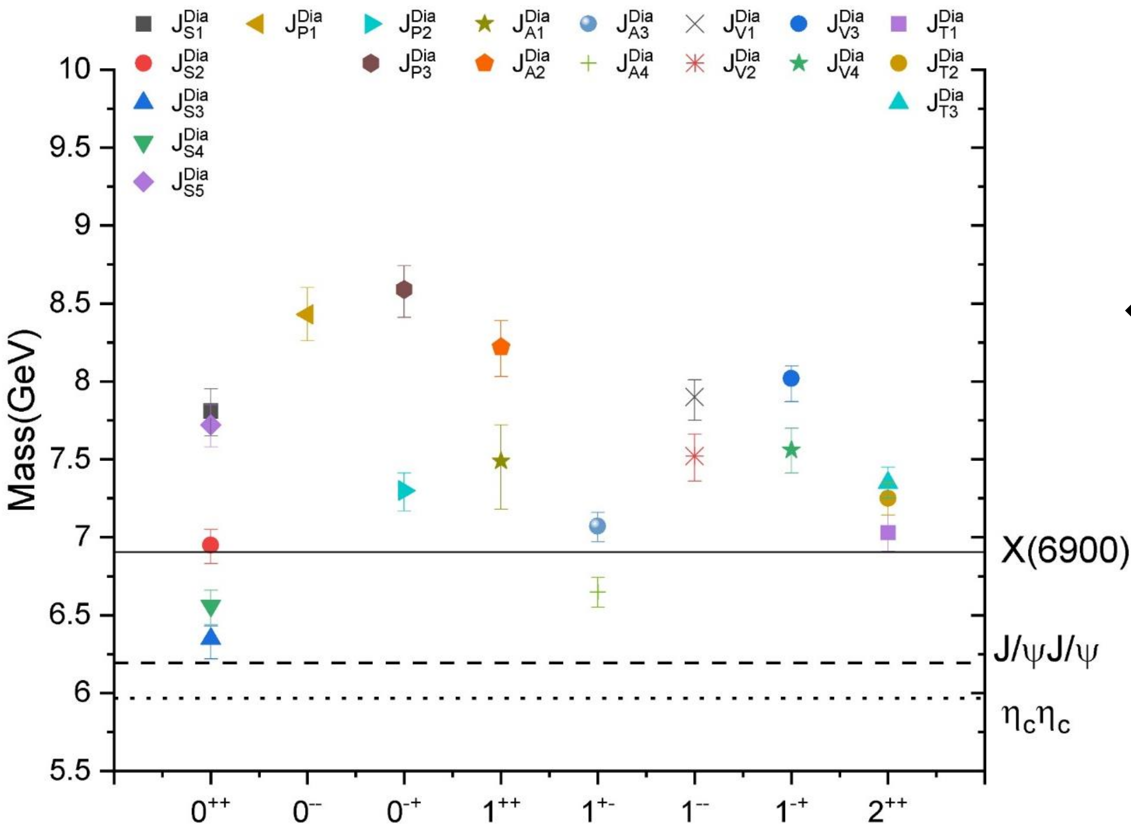
The NLO contributions significantly improve μ dependence of hadron mass m_H





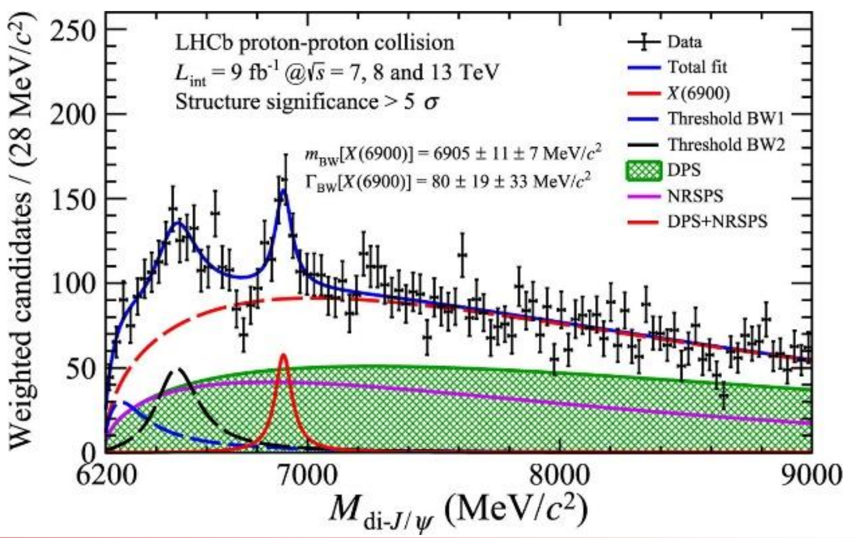
■ $cc\bar{c}\bar{c}$ Mass Spectra

$J_{\text{Dia}}^{\overline{MS},\text{NLO}}$



- Do not exist bound states below $J/\psi J/\psi$.
- $J_{S,3}^{\text{Dia}}$ with $J^{PC} = 0^{++}$ may be assigned to be X(6900) .

  $J_{T,1}^{\text{Dia}}$ with $J^{PC} = 2^{++}$ may also be a candidate for the X(6900) .
- $J_{S,3}^{\text{Dia}}$ and $J_{S,4}^{\text{Dia}}$ may explain the broad structure (6.2~6.8 GeV)



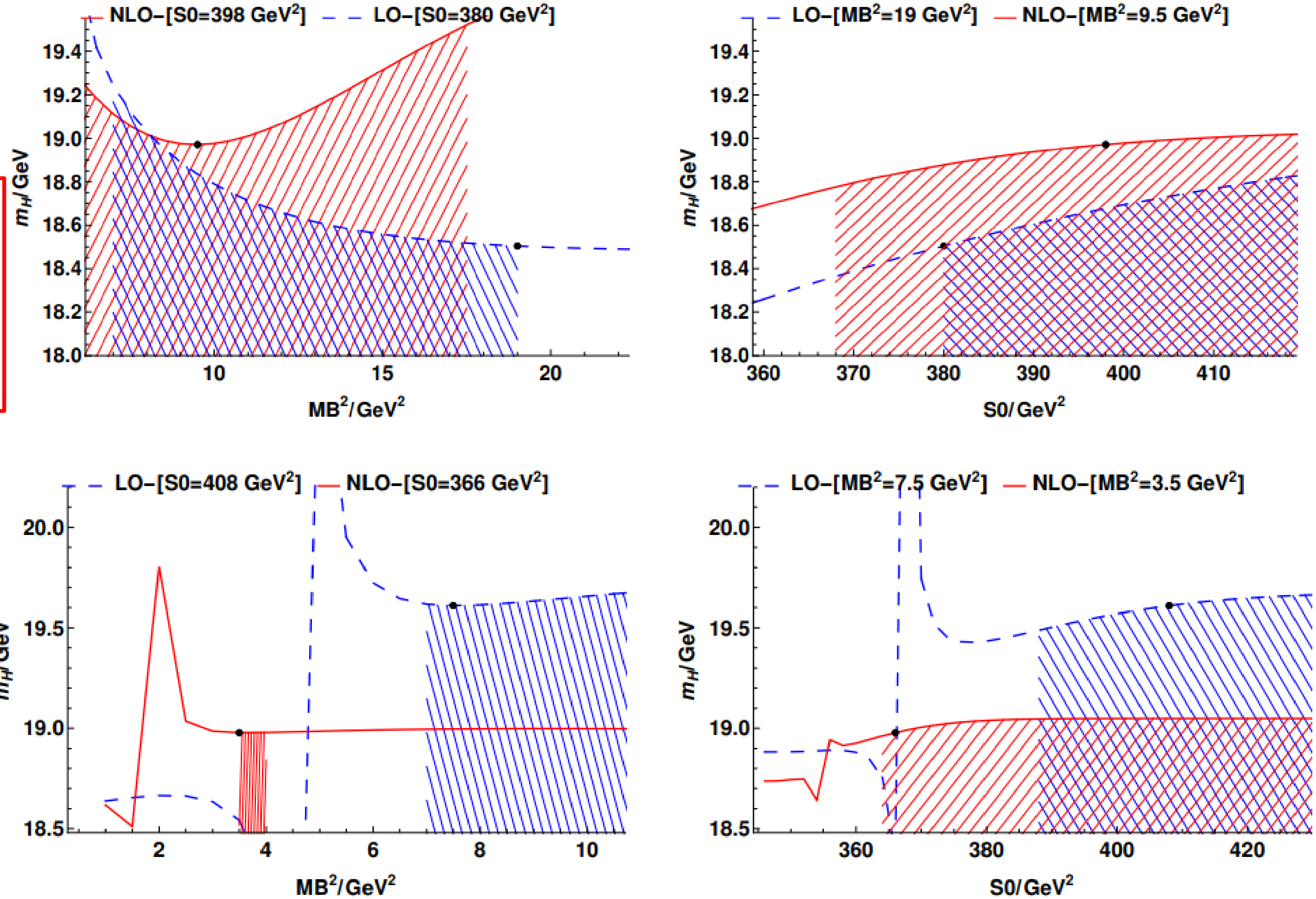
■ The Borel Platform Curves - $\bar{b}b\bar{b}b$

$\boxed{J^{Dia}_{S,4}}$

- $\overline{MS}$

NLO contributions improve the quality of Borel platform evidently

- OS



➢ $J^{PC} = 0^{++}$ Diagonalized Operators

- $\overline{\text{MS}}$

流算符	LO	NLO($\overline{MS}$)
$J_{S,1}^{Dia}$	$18.51^{+0.17}_{-0.26}$	$19.01^{+0.05}_{-0.10}$
$J_{S,2}^{Dia}$	$18.51^{+0.17}_{-0.26}$	$18.97^{+0.06}_{-0.11}$
$J_{S,3}^{Dia}$	$18.50^{+0.18}_{-0.26}$	$18.96^{+0.05}_{-0.11}$
$J_{S,4}^{Dia}$	$18.50^{+0.17}_{-0.26}$	$18.97^{+0.06}_{-0.11}$
$J_{S,5}^{Dia}$	$18.51^{+0.17}_{-0.26}$	$18.95^{+0.08}_{-0.14}$

- OS

流算符	LO	NLO(OS)
$J_{S,1}^{Dia}$	$19.68^{+0.04}_{-0.10}$	$18.98^{+0.07}_{-0.28}$
$J_{S,2}^{Dia}$	$19.67^{+0.04}_{-0.10}$	$18.98^{+0.07}_{-0.28}$
$J_{S,3}^{Dia}$	$19.64^{+0.02}_{-0.06}$	$18.98^{+0.07}_{-0.36}$
$J_{S,4}^{Dia}$	$19.61^{+0.07}_{-0.14}$	$18.98^{+0.07}_{-0.33}$
$J_{S,5}^{Dia}$	$19.66^{+0.08}_{-0.15}$	$18.98^{+0.07}_{-0.26}$

◆ LO VS NLO

NLO corrections are significant.

- $|M_H^{NLO} - M_H^{LO}| \sim 0.5$ GeV
- No oversize NLO corrections
- NLO results are above $\eta_b\eta_b$

◆ $\overline{MS}$ VS OS

The renormalization scheme dependence is reduced.

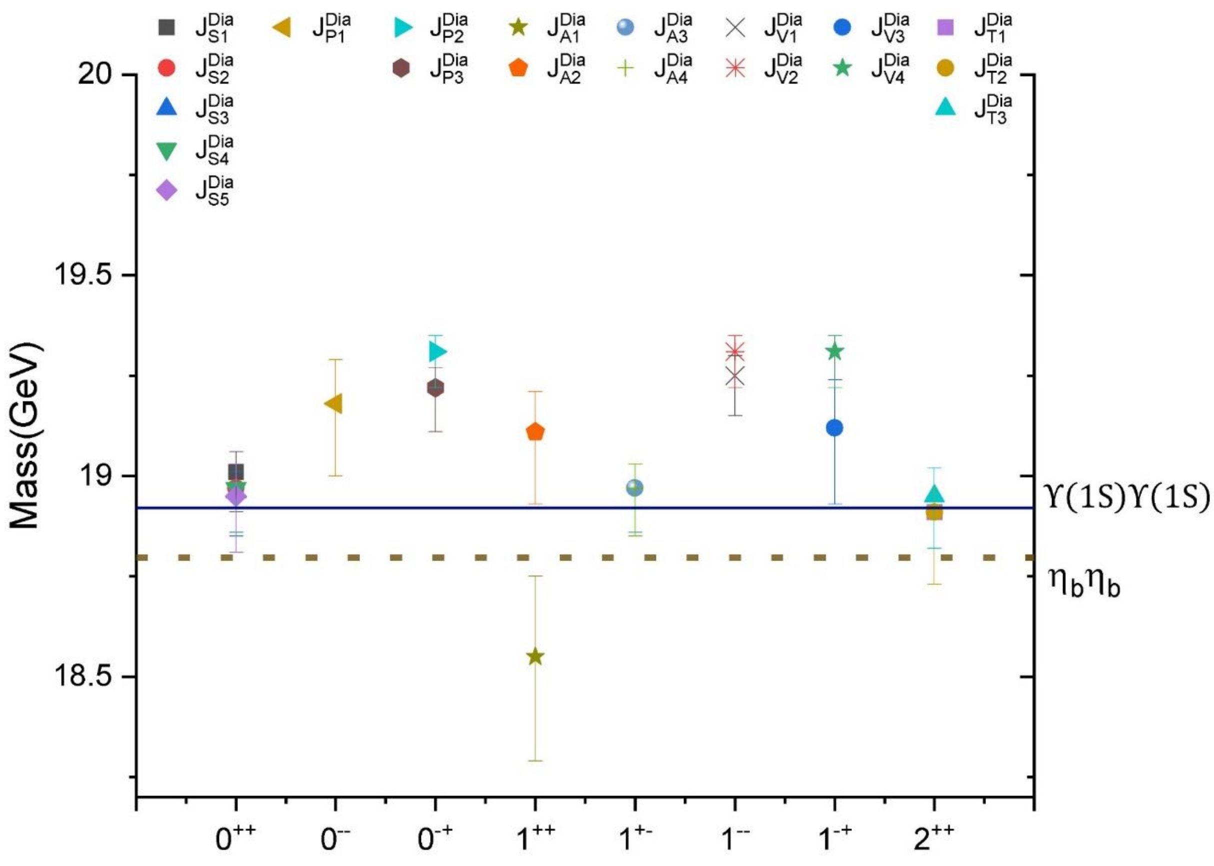
- $|M_H^{\overline{MS},LO} - M_H^{OS,LO}| \sim 1$ GeV
- $|M_H^{\overline{MS},NLO} - M_H^{OS,NLO}| \sim 0.1$GeV

◆ Problem

Perturbative Convergence

■ $\bar{b}b\bar{b}b$ Mass Spectra

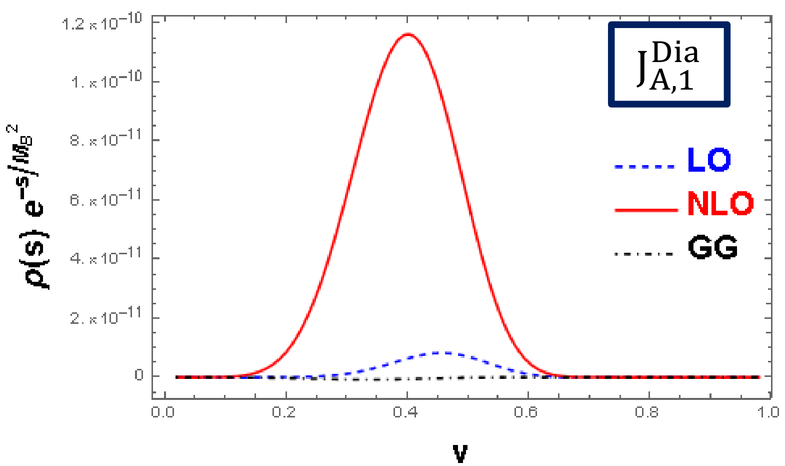
$J_{\text{Dia}}^{\overline{MS},\text{NLO}}$



◆ According to current results, there may not exist bound states below $\eta_b\eta_b$.

◆ The perturbative convergence is bad in $\bar{b}b\bar{b}b$ system.

◆ The errors are large.



Summary

- NLO corrections and operator mixing are important and non-negligibel
  - Large corrections to hadron masses M_H $\left(\left|M_H^{NLO} - M_H^{LO}\right| > 0.5\ GeV\right)$.
  - Improving the quality of the Borel platform evidently. ($\bar{b}b\bar{b}b$ system)
  - Reducing the renormalization scale μ dependence.
  - Reducing the scheme dependence. $\begin{cases} \left|M_H^{\overline{MS},LO} - M_H^{OS,LO}\right| > 1 \text{ GeV} \\ \left|M_H^{\overline{MS},NLO} - M_H^{OS,NLO}\right| \sim 0.5 \text{ GeV} \end{cases}$
- $\bar{c}c\bar{c}c$ Mass Spectra
  - Do not exist bound states below $J/\psi J/\psi$
  - $J_{S,3}^{\mathrm{Dia}}$ and $J_{S,4}^{\mathrm{Dia}}$ with $J^{PC} = 0^{++}$ may explain the broad structure
  - $J_{S,3}^{\mathrm{Dia}}$ with $J^{PC} = 0^{++}$ and $J_{T,1}^{\mathrm{Dia}}$ with $J^{PC} = 2^{++}$ may be candidates of the X(6900) .
- $\bar{b}b\bar{b}b$ Mass Spectra
  - Bad perturbative convergence and large errors . (near-threshold resummation?)
  - There may not exist bound states below $\eta_b\eta_b$ (Based on current results without resummation).

Thanks！

QCD sum rules

Correlation function: $\boldsymbol{\Pi(p^2) = i\int d^4x\, e^{ip\cdot x}\, \langle\Omega|\ \mathcal{T}_1(x)\mathcal{T}_2{}^{+}(0)\ |\Omega\rangle}$

$$\boldsymbol{\mathcal{T} = (\bar{c}_i\Gamma_1 c_i)(\,\bar{c}_j\Gamma_2 c_j)}$$

1. Källén – Lehmann representation

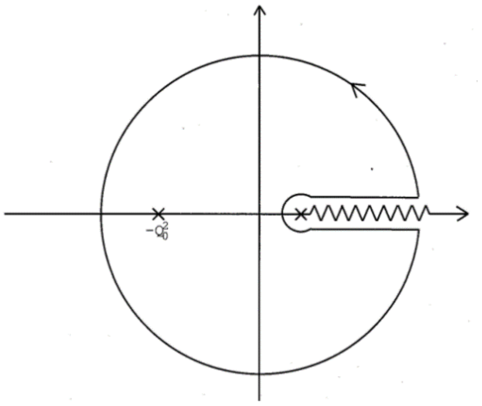
$\rho(s)$: physical spectrum density

$$\Pi(p^2) = \int_0^\infty ds \frac{\rho(s)}{s - p^2 - i\epsilon}$$

$$\boxed{\begin{aligned}\rho(s) &= \textstyle\sum_i \lambda_i\, \delta\big(s - M_i{}^2\big) \ + \rho_{cont}(s)\ \ \theta(s - s_h) \\ &\approx \lambda_H\, \delta\big(s - M_H{}^2\big) + \rho_{cont}(s)\ \theta(s - s_h)\end{aligned}}$$

2. Dispersion relation

$$\Pi(p^2) = \frac{1}{\pi}\int_{s_{th}}^\infty ds \frac{Im\ \Pi(s + i\epsilon)}{s - p^2}$$



$$\boxed{\int_0^\infty ds \frac{\rho(s)}{s - p^2 - i\epsilon} = \frac{1}{\pi}\int_{s_{th}}^\infty ds \frac{Im\ \Pi(s)}{s - p^2 - i\epsilon}}$$

3. Borel Transform

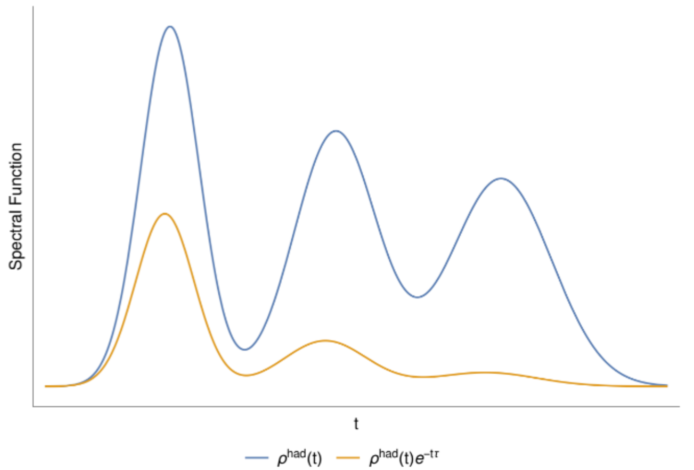
- **Why?**

$$\boxed{\Pi(p^2) = \int_0^\infty ds \frac{\rho(s)}{s - p^2 - i\epsilon} = \frac{1}{\pi}\int_{s_{th}}^\infty ds \frac{\mathrm{Im}\,\Pi(s)}{s - p^2 - i\epsilon}}$$

- ☐ To suppress the contributions from high excited states and continuum
- ☐ To suppress the great circle contribution in contour integral

- **Borel Transform**

$$\hat{\mathcal{B}}[t, M_B^2]\, F(t) = \lim_{\substack{-t,n\to\infty \\ -\frac{t}{n}\to M_B^2}} \frac{(-t)^{n+1}}{n!}\left(\frac{d}{dt}\right)^n F(t)$$



$$\boxed{\hat{\mathcal{B}}[\boldsymbol{p^2}, M_B^2]\, \boldsymbol{\Pi}(\boldsymbol{p^2}) = \int_0^\infty ds\, \boldsymbol{\rho}(\boldsymbol{s})\, \boldsymbol{e^{-s/M_B^2}} = \frac{1}{\pi}\int_{s_{th}}^\infty ds\, \mathrm{Im}[\Pi(s)]\, \boldsymbol{e^{-s/M_B^2}}}$$

4. Operator Product Expand (OPE)

- $\widetilde{\mathcal{T}}(p^2) = i\int d^4x\, e^{ip\cdot x}\, \mathcal{T}_1(x)\mathcal{T}_2{}^{+}(0)$

$$\lim_{p^2\to-\infty} \widetilde{\mathcal{T}}(p^2) = \sum_i C_i(p^2)\, O_i$$

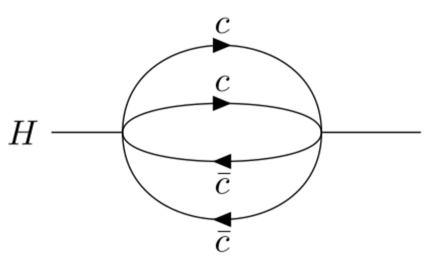
$$= C_1 + C_{qq}\, \bar{q}q + C_{GG}\, GG \cdots$$

- $\Pi(p^2) = \langle\Omega|\widetilde{\mathcal{T}}(p^2)|\Omega\rangle$

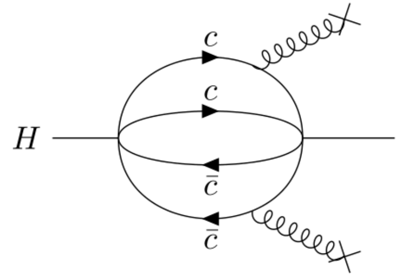
Power suppression: $\dfrac{\Lambda_{QCD}^2}{p^2}$

$$= C_1 + C_{qq}\,\langle\Omega|\bar{q}q|\Omega\rangle + C_{GG}\,\langle\Omega|GG|\Omega\rangle + C_{qqG}\langle\Omega|\bar{q}qG|\Omega\rangle + \cdots$$

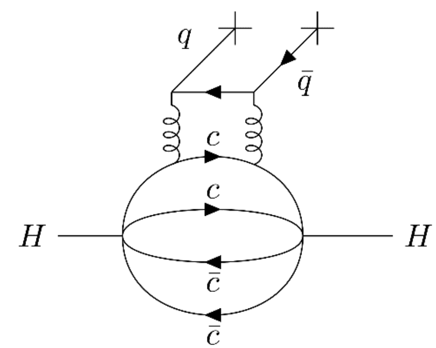
$$\approx C_1 + C_{GG}\,\langle\Omega|GG|\Omega\rangle$$ Up to dimention 4



$C_1 - \mathrm{LO}$



$C_{GG} - \mathrm{LO}$



$C_{qq} - LO$

5. Quark-Hadron Duality (QHD)

QHD :

$$\boxed{\int_{s_h}^{\infty} ds \frac{\rho_{cont}(s)}{s-p^2} = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\boldsymbol{Im[C_1(s)]}}{s-p^2}}$$

$$\boxed{\lambda_H\, \boldsymbol{e}^{-M_H^2/M_B^2} + \int_{s_h}^{\infty} ds\, \rho_{cont}(s) \boldsymbol{e}^{-s/M_B^2} = \frac{1}{\pi}\int_{s_{th}}^{\infty} ds\, (\boldsymbol{Im[C_1(s)]} + \boldsymbol{Im[C_{GG}(s)]}\langle GG\rangle)\, \boldsymbol{e}^{-s/M_B^2}}$$

QHD

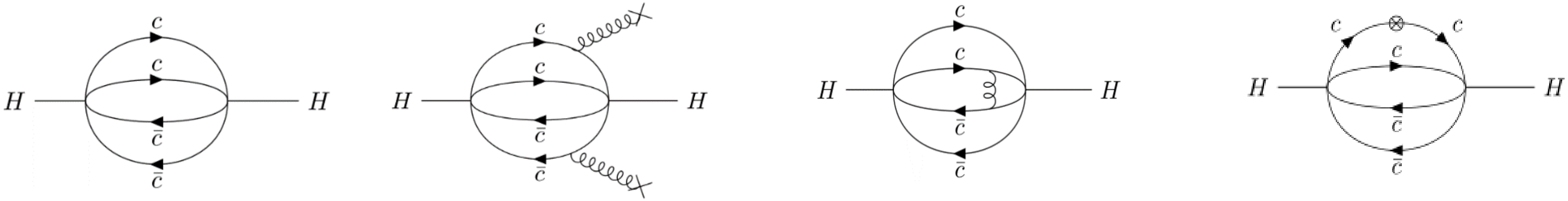
$$\boxed{\lambda_H \boldsymbol{e}^{-M_H^2/M_B^2} = \frac{1}{\pi}\int_{s_{th}}^{s_0} ds\, \boldsymbol{Im[C_1(s)]}\ \boldsymbol{e}^{-s/M_B^2} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds\ \boldsymbol{Im[C_{GG}(s)]}\, \langle GG\rangle\ \boldsymbol{e}^{-s/M_B^2}}$$

6. Hadron Mass

$$\boxed{M_H^2 = \frac{\int_{s_{th}}^{s_0} \boldsymbol{ds\, Im[C_1(s)]\ s\ e^{-\frac{s}{M_B{}^2}}} + \int_{s_{th}}^{\infty} \boldsymbol{ds\, Im[C_{GG}(s)]\ s\ e^{-\frac{s}{M_B{}^2}}\langle GG\rangle}}{\int_{s_{th}}^{s_0} \boldsymbol{ds\, Im[C_1(s)]\ e^{-\frac{s}{M_B{}^2}}} + \int_{s_{th}}^{\infty} \boldsymbol{ds\, Im[C_{GG}(s)]\ e^{-\frac{s}{M_B{}^2}}\langle GG\rangle}}}$$

6. Calculation of $\mathbf{C_i}$

$$M_H^2=\frac{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\ s\ e^{-\frac{s}{M_B{}^2}}+\int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\ s\ e^{-\frac{s}{M_B{}^2}}\langle GG\rangle}{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\ e^{-\frac{s}{M_B{}^2}}+\int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\ e^{-\frac{s}{M_B{}^2}}\langle GG\rangle}$$



$C_1-\mathrm{LO}$ $\quad$ $C_{GG}-\mathrm{LO}$ $\quad$ $C_1-\mathrm{NLO}$ $\quad$ $C_1-\mathrm{NLO}ct$

①. To get amplitudes of $\boldsymbol{C_1}$ and $\boldsymbol{C_{GG}}$ 【FeynArts】 *J. Kublbeck, et al, (1990); T. Hahn (2000)*

②. To simplify spinor structures of $\boldsymbol{C}_{\mathrm{i}}$ 【FeynCalc】 *R. Mertig, et al, (1991); V. Shtabovenko,(2016)*

③. To reduce all loop integrals to a linear combination of master integrals ($\tilde{\mathrm{I}}_i$):

$$Int=\sum_i \mathrm{a}_i\, \tilde{\mathrm{I}}_i$$ 【REDUZE】 *A. von Manteuffel, et al, (2012)*

④. To calculate master integrals: Defferential equation (DE)

$$\tilde{\mathrm{I}}_i(\epsilon,s,r=\frac{m^2}{s})=s^d\sum_{a,b,n} c_{abm}\, r^{a+b\epsilon}\epsilon^n$$

A. V. Kotikov, (1991); Z. Bern, et al, (1993); E. Remiddi, et al, (1997); T. Gehrmann, et al, (2000) X. Liu, et al, (2017)

⑤. Renormalization (for $\boldsymbol{C_1}$ NLO)

7. Borel Windows

$$M_H^2 = \frac{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\; s\; e^{-\frac{s}{{M_B}^2}} + \int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\; s\; e^{-\frac{s}{{M_B}^2}} \langle GG \rangle}{\int_{s_{th}}^{s_0} ds\, Im[C_1(s)]\; e^{-\frac{s}{{M_B}^2}} + \int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\; e^{-\frac{s}{{M_B}^2}} \langle GG \rangle}$$

- the validity of OPE
- the ground-state contribution dominance

➡ To constrain the range of s_0 and M_B^2

(called Borel windows)

$$r_{cont} = \left| \frac{\int_{s_0}^{\infty} ds\, Im[C_1(s)]\; e^{-\frac{s}{{M_B}^2}}}{\int_{s_{th}}^{\infty} ds\, Im[C_1(s)]\; e^{-\frac{s}{{M_B}^2}}} \right| \leq 30\%$$

$$r_{GG} = \left| \frac{\int_{s_{th}}^{\infty} ds\, Im[C_{GG}(s)]\; e^{-\frac{s}{{M_B}^2}} \langle GG \rangle}{\int_{s_{th}}^{\infty} ds\, Im[C_1(s)]\; e^{-\frac{s}{{M_B}^2}}} \right| \leq 30\%$$

8. Borel Platform

The point where the parameter dependence of M_H is weakest within Borel windows

$$\Delta(x, y) = \left(\frac{\partial M_H}{\partial x}\right)^2 + \left(\frac{\partial M_H}{\partial y}\right)^2 \qquad (\, x = s_0\,,\;\; y = M_B^2\,)$$

■ $\bar{c}c\bar{c}c$ Numerical results

➢ $J^{PC} = 0^{++}$ Meson-Meson type operators $(\bar{c}_i\Gamma_1 c_i)(\bar{c}_j\Gamma_2 c_j)$

- $\overline{MS}$

流算符	LO	NLO($\overline{MS}$)
$J_{S,1}^{M-M}$	$6.16^{+0.08}_{-0.10}$	$7.32^{+0.09}_{-0.11}$
$J_{S,2}^{M-M}$	$6.38^{+0.09}_{-0.15}$	$8.33^{+0.13}_{-0.15}$
$J_{S,3}^{M-M}$	$7.11^{+0.13}_{-0.15}$	$7.91^{+0.16}_{-0.19}$
$J_{S,4}^{M-M}$	$5.90^{+0.06}_{-0.08}$	$6.36^{+0.06}_{-0.10}$
$J_{S,5}^{M-M}$	$6.28^{+0.13}_{-0.17}$	$7.78^{+0.13}_{-0.13}$

- OS

流算符	LO	NLO(OS)
$J_{S,1}^{M-M}$	$7.35^{+0.07}_{-0.10}$	$6.60^{+0.09}_{-0.12}$
$J_{S,2}^{M-M}$	$7.44^{+0.12}_{-0.15}$	$6.60^{+0.09}_{-0.15}$
$J_{S,3}^{M-M}$	$8.43^{+0.14}_{-0.18}$	$7.40^{+0.15}_{-0.21}$
$J_{S,4}^{M-M}$	$7.05^{+0.06}_{-0.09}$	$6.44^{+0.08}_{-0.09}$
$J_{S,5}^{M-M}$	$7.45^{+0.10}_{-0.11}$	$6.62^{+0.09}_{-0.13}$

◆ **LO** VS **NLO**

NLO corrections are **significant**.

- $\left|M_H^{NLO} - M_H^{LO}\right| > 0.5$ GeV

  Worst: ~2 GeV

- Below or above $\eta_c\eta_c$?

◆ $\boldsymbol{\overline{MS}}$ VS **OS**

The **scheme dependence** is **reduced** observably.

- $\left|M_H^{\overline{MS},\,LO} - M_H^{OS,\,LO}\right| > 1$ GeV
- $\left|M_H^{\overline{MS},\,NLO} - M_H^{OS,NLO}\right| \sim 0.5$ GeV

- $J^{PC} = 0^{++}$ diquark - antidiquark type operators $(c_i^T \mathcal{C}\, \Gamma_1\, c_j)\ (\bar{c}_i\, \Gamma_2 \mathcal{C}\, \bar{c}_j^{\,T})$

- $\overline{MS}$

流算符	LO	NLO($\overline{MS}$)
$J_{S,1}^{\mathrm{Di-Di}}$	$6.07^{+0.05}_{-0.07}$	$6.60^{+0.09}_{-0.10}$
$J_{S,2}^{\mathrm{Di-Di}}$	$6.19^{+0.07}_{-0.12}$	$6.90^{+0.11}_{-0.12}$
$J_{S,3}^{\mathrm{Di-Di}}$	$6.96^{+0.11}_{-0.14}$	$9.25^{+0.14}_{-0.14}$
$J_{S,4}^{\mathrm{Di-Di}}$	$6.17^{+0.07}_{-0.12}$	$7.36^{+0.10}_{-0.11}$
$J_{S,5}^{\mathrm{Di-Di}}$	$6.07^{+0.08}_{-0.10}$	$6.69^{+0.10}_{-0.12}$

- OS

流算符	LO	NLO(OS)
$J_{S,1}^{\mathrm{Di-Di}}$	$7.23^{+0.04}_{-0.07}$	$6.54^{+0.06}_{-0.08}$
$J_{S,2}^{\mathrm{Di-Di}}$	$7.27^{+0.08}_{-0.11}$	$6.52^{+0.10}_{-0.14}$
$J_{S,3}^{\mathrm{Di-Di}}$	$8.17^{+0.15}_{-0.19}$	$7.19^{+0.16}_{-0.26}$
$J_{S,4}^{\mathrm{Di-Di}}$	$7.31^{+0.08}_{-0.11}$	$6.59^{+0.09}_{-0.12}$
$J_{S,5}^{\mathrm{Di-Di}}$	$7.22^{+0.08}_{-0.12}$	$6.51^{+0.09}_{-0.13}$

◆ LO VS NLO

NLO corrections are **significant**.

- $\left|M_H^{NLO} - M_H^{LO}\right| > 0.5$ GeV

  Worst: ~2 GeV

- Below or above $\eta_c\eta_c$?

◆ $\overline{MS}$ VS OS

The **scheme dependence** is **reduced** observably.

- $\left|M_H^{\overline{MS},\,LO} - M_H^{OS,\,LO}\right| > 1$ GeV
- $\left|M_H^{\overline{MS},\,NLO} - M_H^{OS,NLO}\right| \sim 0.5$ GeV

Error

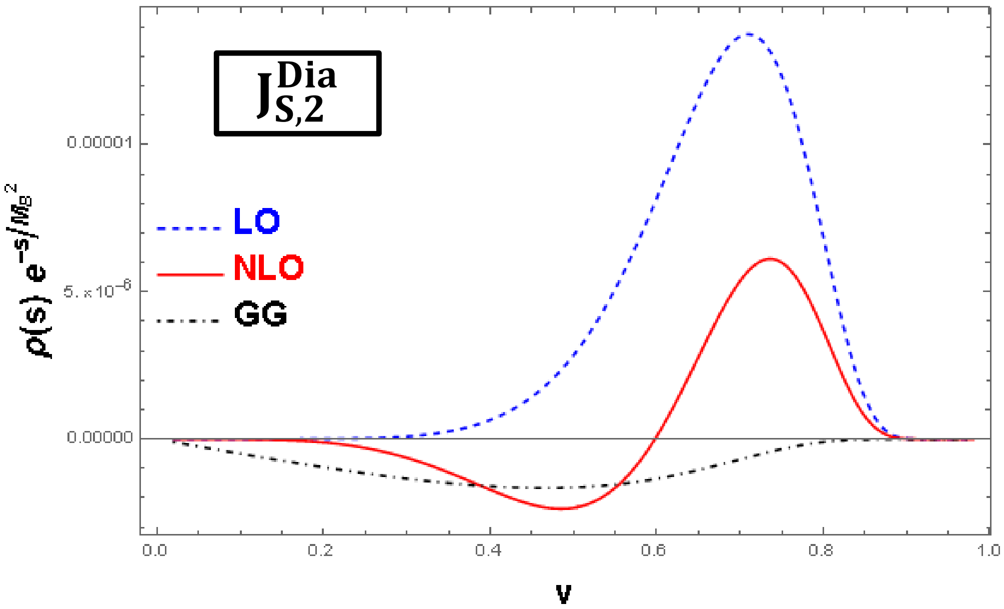
Current	Order	M_H (GeV)	s_0 (GeV^2)	M_B^2 (GeV^2)	Error from s_0 and M_B^2	Error from m_Q	Error from μ
$J_{S,2}^{\rm Dia}$	LO($\overline{\rm MS}$)	$6.19^{+0.26}_{-0.23}$	$51(\pm 10\%)$	$3.50(\pm 10\%)$	$^{+0.07}_{-0.12}$	$^{+0.11}_{-0.14}$	$^{+0.22}_{-0.23}$
	NLO($\overline{\rm MS}$)	$6.95^{+0.21}_{-0.31}$	$61(\pm 10\%)$	$5.00(\pm 10\%)$	$^{+0.10}_{-0.12}$	$^{+0.15}_{-0.13}$	$^{+0.11}_{-0.26}$
	LO(OS)	$7.31^{+0.29}_{-0.24}$	$64(\pm 10\%)$	$3.75(\pm 10\%)$	$^{+0.08}_{-0.12}$	$^{+0.28}_{-0.21}$	
	NLO(OS)	$6.58^{+0.28}_{-0.29}$	$48(\pm 10\%)$	$2.00(\pm 10\%)$	$^{+0.08}_{-0.11}$	$^{+0.27}_{-0.27}$	

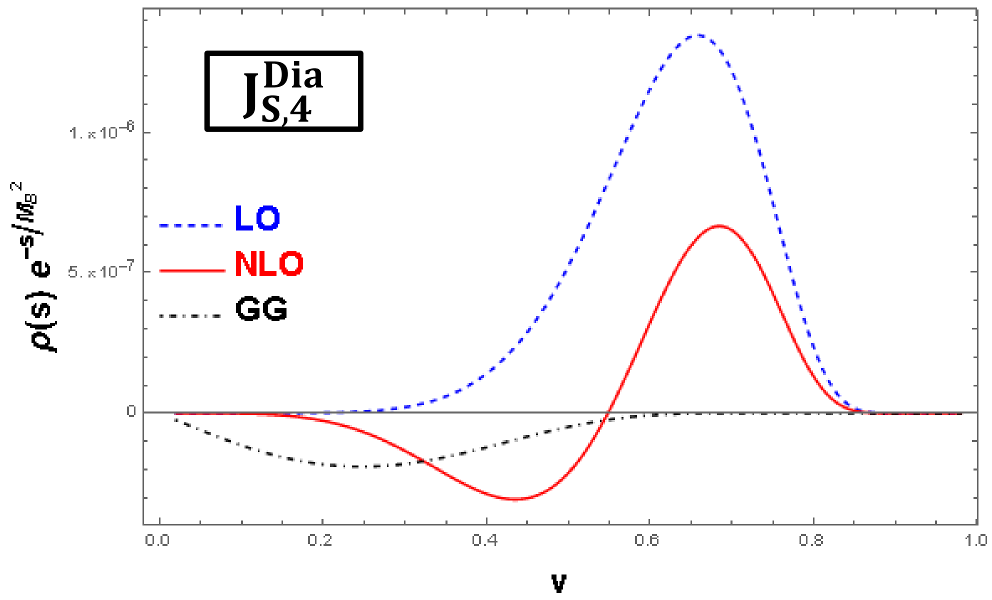
Current	Order	M_H (GeV)	s_0 (GeV^2)	M_B^2 (GeV^2)	Error from s_0 and M_B^2	Error from m_Q	Error from μ
$J_{S,3}^{\rm Dia}$	LO($\overline{\rm MS}$)	$5.93^{+0.31}_{-0.26}$	$45(\pm 10\%)$	$3.00(\pm 10\%)$	$^{+0.07}_{-0.10}$	$^{+0.18}_{-0.09}$	$^{+0.24}_{-0.22}$
	NLO($\overline{\rm MS}$)	$6.35^{+0.20}_{-0.17}$	$51(\pm 10\%)$	$3.50(\pm 10\%)$	$^{+0.08}_{-0.13}$	$^{+0.18}_{-0.11}$	$^{+0.00}_{-0.03}$
	LO(OS)	$7.06^{+0.32}_{-0.26}$	$60(\pm 10\%)$	$3.00(\pm 10\%)$	$^{+0.07}_{-0.10}$	$^{+0.31}_{-0.24}$	
	NLO(OS)	$6.47^{+0.29}_{-0.30}$	$46(\pm 10\%)$	$1.75(\pm 10\%)$	$^{+0.08}_{-0.10}$	$^{+0.28}_{-0.28}$	

Current	Order	M_H (GeV)	s_0 (GeV^2)	M_B^2 (GeV^2)	Error from s_0 and M_B^2	Error from m_Q	Error from μ
$J_{S,4}^{\rm Dia}$	LO($\overline{\rm MS}$)	$6.02^{+0.24}_{-0.28}$	$49(\pm 10\%)$	$3.00(\pm 10\%)$	$^{+0.05}_{-0.06}$	$^{+0.09}_{-0.14}$	$^{+0.22}_{-0.23}$
	NLO($\overline{\rm MS}$)	$6.56^{+0.18}_{-0.20}$	$55(\pm 10\%)$	$4.00(\pm 10\%)$	$^{+0.10}_{-0.12}$	$^{+0.15}_{-0.13}$	$^{+0.03}_{-0.10}$
	LO(OS)	$7.16^{+0.24}_{-0.30}$	$66(\pm 10\%)$	$3.00(\pm 10\%)$	$^{+0.04}_{-0.05}$	$^{+0.24}_{-0.30}$	
	NLO(OS)	$6.49^{+0.29}_{-0.30}$	$46(\pm 10\%)$	$1.75(\pm 10\%)$	$^{+0.07}_{-0.10}$	$^{+0.28}_{-0.28}$	

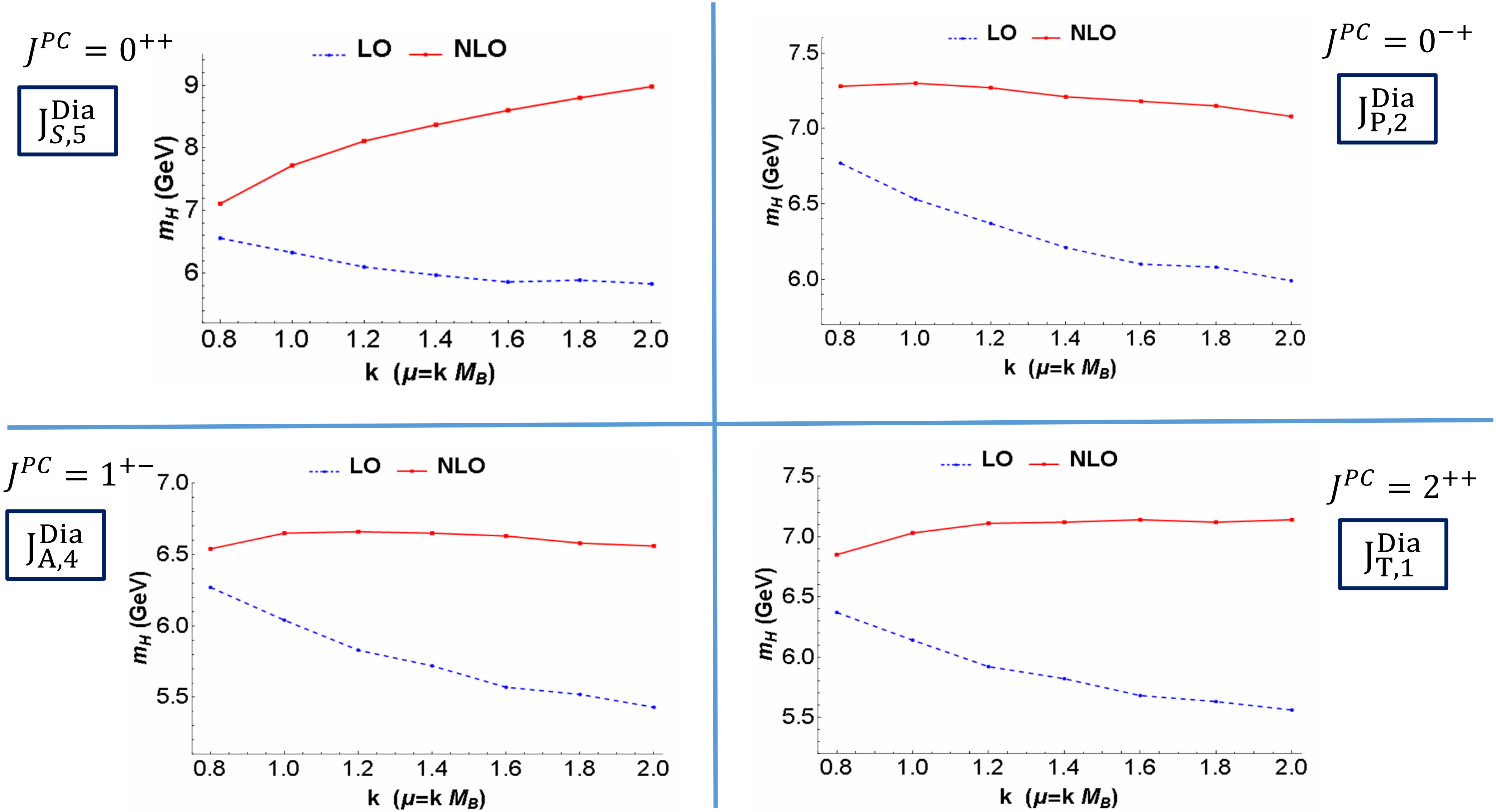
■ Perturbative Convergence—4c

- $\overline{\mathrm{MS}}$





■ The renormalization scale μ dependence



■ $J^{PC} = 0^{++}$ Meson-Meson type operators

• $\overline{MS}$

流算符	LO	NLO($\overline{MS}$)
$J_{S,1}^{M-M}$	$18.51^{+0.17}_{-0.26}$	$19.00^{+0.05}_{-0.10}$
$J_{S,2}^{M-M}$	$18.55^{+0.19}_{-0.26}$	$18.92^{+0.10}_{-0.17}$
$J_{S,3}^{M-M}$	$19.21^{+0.20}_{-0.26}$	$19.66^{+0.05}_{-0.10}$
$J_{S,4}^{M-M}$	$18.50^{+0.17}_{-0.26}$	$18.97^{+0.05}_{-0.11}$
$J_{S,5}^{M-M}$	$18.51^{+0.17}_{-0.26}$	$18.93^{+0.09}_{-0.11}$

• OS

流算符	LO	NLO(OS)
$J_{S,1}^{M-M}$	$19.68^{+0.04}_{-0.10}$	$18.98^{+0.07}_{-0.28}$
$J_{S,2}^{M-M}$	$19.68^{+0.04}_{-0.10}$	$18.98^{+0.07}_{-0.28}$
$J_{S,3}^{M-M}$	$20.51^{+0.05}_{-0.18}$	$19.51^{+0.72}_{-1.55}$
$J_{S,4}^{M-M}$	$19.64^{+0.02}_{-0.06}$	$18.98^{+0.07}_{-0.35}$
$J_{S,5}^{M-M}$	$19.71^{+0.03}_{-0.08}$	$18.98^{+0.07}_{-0.28}$

◆ **LO** VS **NLO**

- 次领头阶修正明显

  $\left|M_H^{NLO} - M_H^{LO}\right| \sim 0.5$ GeV
- 相比于4c体系没有过大修正
- 修正之后，在$\eta_b\eta_b$ 阈值之上

◆ $\boldsymbol{\overline{MS}}$ VS **OS**

结果对重整化方案依赖性显著降低

- $\left|M_H^{\overline{MS},\,LO} - M_H^{OS,\,LO}\right| \sim 1$ GeV
- $\left|M_H^{\overline{MS},\,NLO} - M_H^{OS,NLO}\right| \sim 0.1$GeV

■ The renormalization scale μ dependence

$\boxed{J_{S,4}^{\mathrm{Dia}}}$