Double-Real-Virtual and Double-Virtual-Real Corrections to the Three-Loop Thrust Soft Function

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Definition

Thrust Brandt et al. (1964), Farhi (1977)

$$T = 1 - \tau = \text{Max}_{\hat{\mathbf{n}}} \frac{\sum_i |\hat{\mathbf{n}} \cdot \vec{k}_i|}{\sum_i |\vec{k}_i|}\,, \qquad (1)$$

where $\vec{k}_i$ are the momenta of the final-state particles or jets, and the maximum is obtained for the Thrust axis $\hat{\mathbf{n}}$.

- One of the fundamental tools to test the theory of strong interactions(such as the asymptotic freedom of QCD)
- Can be measured very accurately
- Infrared safe, theoretically clean, and feasible to high perturbative order computations
- Suitable for the determination of the strong coupling $\alpha_s(M_Z)$ with a high precision

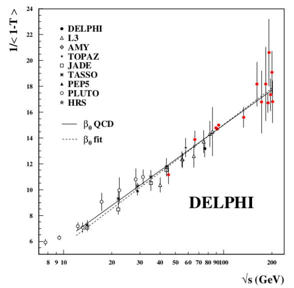

Figure: $\frac{1}{\langle 1-T\rangle}$ as a function of the cms energy. Kluth (2006)

Near the dijet limit ($\tau \to 0$), fixed-order perturbative calculation breaks down due to the presence the large logarithms.

$$\frac{d\sigma}{d\tau} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \alpha_s^n \, C_k^{(n)} \, \frac{1}{\mu} \mathcal{L}_{2n-k} \left(\frac{\tau}{\mu} \right) , \tag{2}$$

where the logarithms

$$\mathcal{L}_{2n-k}(\tau) = \left(\frac{\ln^{2n-k} \tau}{\tau} \right)_+ ,$$
$$\mathcal{L}_{-1}(\tau) = \delta(\tau) \, .$$

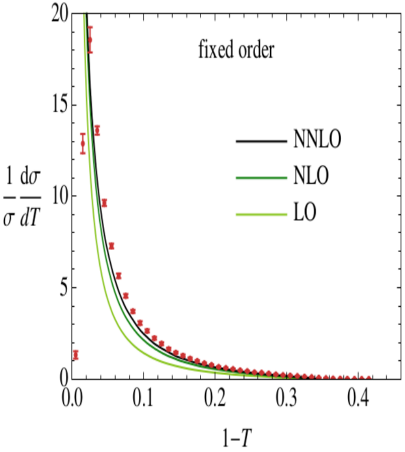

Figure: Fixed-order distribution Becher & Schwartz (2008)

Soft-collinear effective theory (SCET) provides a systematic method to resum large logarithms. Near the dijet region, the thrust distribution is factorized in SCET: Becher & Schwartz (2008)

$$\frac{1}{\sigma_0}\frac{d\sigma_{sing.}}{d\tau} = H(\mu)\int \prod_{i=n,\bar{n},s} d\tau_i \mathcal{J}_n(\tau_n,\mu)\mathcal{J}_{\bar{n}}(\tau_{\bar{n}},\mu)S(\tau_s)\delta(\tau-\tau_n-\tau_{\bar{n}}-\tau_s,\mu)\,,$$

where σ_0 is the leading order cross section for $e^+e^- \to q\bar{q}$. We assume that q is along the light-like direction $n = (1,0,0,1)$ while $\bar{q}$ along $\bar{n} = (1,0,0,-1)$. H is the hard function, $\mathcal{J}$ is the inclusive jet function, and S is the soft function defined as the vacuum expectation of Wilson lines.
The logarithmic terms can be resumed to all orders through the standard resummation techniques.

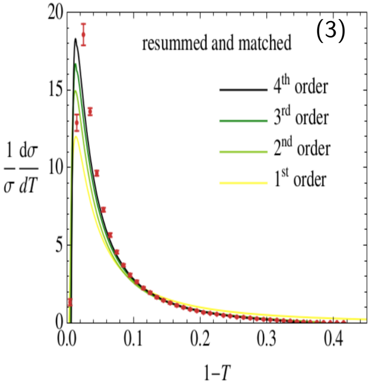

Figure: Resummed distribution Becher & Schwartz (2008)

Soft function

Definition

Soft function

$$S(\tau) = \frac{1}{N_c} \sum_X \mathrm{Tr} \left\{ \langle \Omega | Y_{\bar{n}}^\dagger Y_n | X \rangle \langle X | Y_n^\dagger Y_{\bar{n}} | \Omega \rangle \right\} \delta \left(\tau - \sum_k \min \{ k \cdot n, k \cdot \bar{n} \} \right), \tag{4}$$

where Y_n and $Y_{\bar{n}}$ are the soft Wilson lines, $|\Omega\rangle$ is the vacuum, and the summation is taken over all the final hadronic states of soft scales.

Known results:

- two-loop
  Monni et al. JHEP(2011),
  Kelley et al. PRD(2011),
  Boughezal et al. PRD(2015)
- three-loop
  triple-real:
  Baranowski et al. PRD(2022)
  double-real-virtual & double-virtual-real:
  Chen et al. arXiv:2206.12323

Logarithmic Order	fixed-order matching	Anomalous Dimension γ_i	Γ_{cusp}
LL	1	-	1-loop
NLL	1	1-loop	2-loop
NLL'	α_s	1-loop	2-loop
NNLL	α_s	2-loop	3-loop
NNLL'	α_s^2	2-loop	3-loop
N^{3}LL	α_s^2	3-loop	4-loop
N^{3}LL'	α_s^3	3-loop	4-loop
N^{4}LL	α_s^3	4-loop	5-loop

Table: The definitions for the logarithmic accuracy of the resummation calculation.

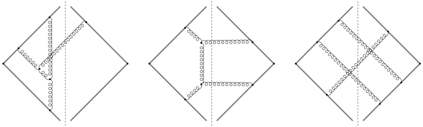

Figure: Representative diagrams for three-loop thrust soft function.

Double-real-virtual (RRV) contribution:

$$S^{(3)}_{RRV}(\tau;\epsilon) = \int \mathrm{d}\Phi_2\, \Theta(\tau; k_1, k_2) \int \frac{d^D l}{(2\pi)^D}\, \omega^{(3)}_{RRV}(l, k_1, k_2; n, \bar{n})\,, \quad (5)$$

where $\mathrm{d}\Phi_n$ denotes the n-body phase-space integration measure, $\omega^{(3)}_{RRV}$ is the squared amplitude, and

$$\Theta(\tau; k_1, k_2) = \int d\tau_1 d\tau_2 \delta(\tau - \tau_1 - \tau_2)\Theta_{\tau_1}(k_1)\Theta_{\tau_2}(k_2),$$

with

$$\Theta_\tau(k) \equiv \theta(k^+ - k^-)\delta(\tau - k^-) + \theta(k^- - k^+)\delta(\tau - k^+)\,.$$

Difficulty: Traditional integration-by-parts (IBP) method and differential-equation (DE) technique break down due to the presence of Heaviside theta functions.

Parametrization

Solution: Integrals with theta functions can be reduced by constructing and solving linear relations in the parametric representation. Chen JHEP(2020), EPJC(2020, 2021)

$$\frac{1}{D_i^{\lambda_i+1}} \quad \Longleftrightarrow \quad \frac{e^{-\frac{\lambda_i+1}{2}i\pi}}{\Gamma(\lambda_i+1)} \int_0^\infty \mathrm{d}x_i \; e^{ix_i D_i} x_i^{\lambda_i}$$

$$\theta(D_i) \quad \Longleftrightarrow \quad -\frac{i}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x_i \frac{e^{ixD_i}}{x_i + i0^+}$$

We generalize theta function and define

$$w_\lambda(u) \equiv e^{-\frac{\lambda+1}{2}i\pi} \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{x^{\lambda+1}} e^{ixu}. \tag{6}$$

Obviously, $w_0(u) = 2\pi\theta(u)$, $w_{-1}(u) = 2\pi\delta(u)$, and $w_{-2}(u) = 2\pi\delta'(u)$.
We consider the parametrization of a general scalar integral:

$$J(\lambda_1, \lambda_2, \ldots, \lambda_n) = \int [\mathrm{d}l_1][\mathrm{d}l_2]\cdots[\mathrm{d}l_L] \frac{w_{\lambda_1}(D_1) w_{\lambda_2}(D_2) \cdots w_{\lambda_m}(D_m)}{D_{m+1}^{\lambda_{m+1}+1} D_{m+2}^{\lambda_{m+2}+1} \cdots D_n^{\lambda_n+1}}, \tag{7}$$

where L is the number of loops, l_i are the loop momenta, D_i are the denominators of propagators, and we denote the integration measure as $[\mathrm{d}l_i] \equiv \frac{1}{\Gamma(1+\epsilon)} \frac{\mathrm{d}^d l_i}{\pi^{d/2}}$ with d the space-time dimension.

Parametrization

$$J = s_g^{-\frac{L}{2}} e^{i\pi\lambda_f}\, I(\lambda_0, \lambda_1, \ldots, \lambda_n) = s_g^{-\frac{L}{2}} e^{i\pi\lambda_f} \int \mathrm{d}\Pi^{(n+1)} \mathcal{I}^{(-n-1)}, \quad \text{with} \tag{8}$$

$$\mathcal{I}^{(-n-1)} = \frac{\Gamma(-\lambda_0)}{\Gamma(1+\epsilon)^L \prod_{i=m+1}^{n+1} \Gamma(\lambda_i + 1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i}\,. \tag{9}$$

Here s_g the determinant of the d-dimensional space-time metric, $\lambda_0 = -d/2$, and $\lambda_f = \frac{1}{2}Ld - \frac{1}{2}m - \sum_{i=m+1}^{n}(\lambda_i + 1)$. The integration measure

$$\mathrm{d}\Pi^{(n+1)} \equiv \prod_{i=1}^{n+1} \mathrm{d}x_i \delta(1 - f(x)), \tag{10}$$

with the positive definite function $f(x)$ satisfying $f(\alpha x) = \alpha f(x)$. The polynomial $\mathcal{F}$ is related to the Symanzik polynomials U and F through

$$\mathcal{F}(x) \equiv F(x) + U(x) x_{n+1}, \quad \text{with} \tag{11}$$

$$U(x) \equiv \det A, \quad \text{and} \quad F(x) \equiv U(x) \left(\sum_{i,j=1}^{L} (A^{-1})_{ij} B_i \cdot B_j - C \right). \tag{12}$$

Here A, B, and C are linear in x and are determined from $\sum_{i=1}^{n} x_i D_i \equiv \sum_{i,j=1}^{L} A_{ij} l_i \cdot l_j + 2\sum_{i=1}^{L} B_i \cdot l_i + C$.

Tensor integrals can be parametrized recursively:

$$
\begin{aligned}
J^{\mu_1\mu_2\cdots\mu_r}_{i_1 i_2\cdots i_r} \equiv & \int [\mathrm{d}l_1][\mathrm{d}l_2]\cdots[\mathrm{d}l_L]\frac{w_{\lambda_1}(D_1)w_{\lambda_2}(D_2)\cdots w_{\lambda_m}(D_m)}{D_{m+1}^{\lambda_{m+1}+1}D_{m+2}^{\lambda_{m+2}+1}\cdots D_n^{\lambda_n+1}} l_{i_1}^{\mu_1} l_{i_2}^{\mu_2}\dots l_{i_r}^{\mu_r} \\
= & s_g^{-L/2}e^{i\pi\lambda_f}\left[P_{i_1}^{\mu_1}P_{i_2}^{\mu_2}\cdots P_{i_r}^{\mu_r} I(\lambda_0,\lambda_1,\lambda_2,\dots,\lambda_n)\right]_{p^\mu=0} ,
\end{aligned} \tag{13}
$$

where

$$
P_i^\mu(p) \equiv -\frac{\partial}{\partial p_{i,\mu}} - \widetilde{B}_i^\mu(\hat{x}) + \frac{1}{2}\sum_{j=1}^{L}\widetilde{A}_{ij}(\hat{x})p_j^\mu, \tag{14}
$$

with $\widetilde{A}_{ij} \equiv \mathcal{D}_0 U(A^{-1})_{ij}$ and $\widetilde{B}_i^\mu \equiv \sum_{j=1}^{L}\widetilde{A}_{ij}B_j^\mu$. p_i^μ are some auxiliary vectors. $\hat{x}_i$ are operators raising or lowering the indices. That is

$$
\begin{aligned}
\hat{x}_i = & \mathcal{D}_i, \quad i = 1,\ 2,\dots,m, \\
\hat{x}_i = & \mathcal{R}_i, \quad i = m+1,\ m+2,\dots,m,
\end{aligned}
$$

with $\mathcal{R}_i$ and $\mathcal{D}_i$ defined by

$$
\begin{aligned}
\hat{\mathcal{R}}_i I(\lambda_0,\dots,\lambda_i,\dots) = & (\lambda_i+1)I(\lambda_0,\dots,\lambda_i+1,\dots), \\
\hat{\mathcal{D}}_i I(\lambda_0,\dots,\lambda_i,\dots) = & I(\lambda_0,\dots,\lambda_i-1,\dots), \\
\hat{\mathcal{A}}_i I(\lambda_0,\dots,\lambda_i,\dots) = & \lambda_i I(\lambda_0,\dots,\lambda_i,\dots).
\end{aligned}
$$

Here $\mathcal{A}_i$ are defined for future convenience.

Linear reduction

By virtue of the homogeneity of the integrands of the parametric integrals, it can be proven that

$$0 = \int \mathrm{d}\Pi^{(n+1)} \frac{\partial}{\partial x_i} \mathcal{I}^{(-n-1)}, \qquad i = 1, 2, \ldots, m, \tag{15a}$$

$$0 = \int \mathrm{d}\Pi^{(n+1)} \frac{\partial}{\partial x_i} \mathcal{I}^{(-n-1)} + \delta_{\lambda_i 0} \int \mathrm{d}\Pi^{(n)} \, \mathcal{I}^{(-n)}\Big|_{x_i=0}, \qquad i = m+1, m+2, \ldots, n+1. \tag{15b}$$

Or equivalently,

$$\left(\mathcal{D}_0 \frac{\partial \mathcal{F}(\hat{x})}{\partial \hat{x}_i} - \hat{z}_i \right) I = 0, \quad i = 1,\ 2, \ldots, n+1. \tag{16}$$

Here $\hat{z}_i$ are defined by

$$\begin{aligned} z_i =& -\mathcal{R}_i, & i =& 1,\ 2, \ldots,\ m, \\ z_i =& \mathcal{D}_i, & i =& m+1,\ m+2, \ldots,\ n, \end{aligned}$$

And we have introduced the operator $\hat{x}_{n+1}$ and $\hat{z}_{n+1}$ formally defined by $\hat{z}_{n+1} I = I$ and $\hat{x}^i_{n+1} I = (\hat{a}_{n+1}+1)(\hat{a}_{n+1}+2)\ldots(\hat{a}_{n+1}+i)I$, with $\hat{a}_{n+1} \equiv -(L+1)\mathcal{A}_0 - \sum_{i=1}^{n} (\hat{a}_i + 1)$, and

$$\begin{aligned} \hat{a}_i =& -\mathcal{A}_i - 1, & i =& 1,\ 2, \ldots,\ m, \\ \hat{a}_i =& \mathcal{A}_i, & i =& m+1,\ m+2, \ldots,\ n. \end{aligned}$$

Linear reduction

Let a_{ij} and b_{ij} be the solutions of

$$\sum_j b_{ij} \frac{\partial A(\hat{x})}{\partial \hat{x}_j} = 0, \qquad \sum_j a_{ij} \frac{\partial B(\hat{x})}{\partial \hat{x}_j} = 0,$$

In general, solutions of these two equations could be linearly dependent. We denote the linearly dependent part of the solutions by c_{ij}. For brevity, we denote

$$\frac{\partial}{\partial a_i} \equiv \sum_j a_{ij} \frac{\partial}{\partial \hat{x}_j}, \qquad \frac{\partial}{\partial b_i} \equiv \sum_j b_{ij} \frac{\partial}{\partial \hat{x}_j}, \qquad \frac{\partial}{\partial c_i} \equiv \sum_j c_{ij} \frac{\partial}{\partial \hat{x}_j},$$

$$\hat{z}_{a_i} \equiv \sum_j a_{ij} \hat{z}_j, \qquad \hat{z}_{b_i} \equiv \sum_j b_{ij} \hat{z}_j, \qquad \hat{z}_{c_i} \equiv \sum_j c_{ij} \hat{z}_j.$$

B_i^μ is of the form $\sum_u B_{iu} Q_u^\mu$, where Q_u^μ are the linearly independent external momenta. We assume that the Gram determinant $Q_u \cdot Q_v$ is invertible. By introducing some auxiliary parameters, we can always make the matrices $\frac{\partial B_j}{\partial b_i}$ and $\frac{\partial B_{ju}}{\partial b_i}$ invertible. That is, there are matrices α and β such that

$$\sum_k \alpha_{ij,k} \frac{\partial A_{mn}}{\partial a_k} = \frac{1}{2} \left(\delta^{im} \delta^{jn} + \delta^{in} \delta^{jm} \right), \qquad \sum_k \beta_{iu,k} \frac{\partial B_{jv}}{\partial b_k} = \delta^{ij} \delta^{uv}.$$

Then we have the following identities

$$\frac{\partial C}{\partial c_i} + \hat{z}_{c_i} = 0, \tag{17a}$$

$$\sum_j \bar{B}_{ju} A_{ij} = B_{iu}, \tag{17b}$$

$$\sum_k \bar{A}_{ik} A_{kj} = \left(A_0 + \frac{E}{2} \right) \delta_{ij}, \tag{17c}$$

where E is the number of external momenta, and

$$\bar{B}_{iu} \equiv \frac{1}{2} \sum_{j,v} g_{uv} \beta_{iv,j} \left(\frac{\partial C}{\partial b_j} + \hat{z}_{b_j} \right), \qquad \bar{A}_{ij} \equiv -\bar{B}_i \cdot \bar{B}_j - \sum_k \alpha_{ij,k} \left(\hat{z}_{a_k} + \frac{\partial C}{\partial a_k} \right),$$

where g_{uv} is the inverse of the Gram matrix $Q_u \cdot Q_v$. In practical calculations, integrals with theta functions are reduced by solving linear relations generated by eqs. (17). By using the operators $\bar{B}_i$, Eq. (14) can be traded by

$$P_i^\mu = -\frac{\partial}{\partial \bar{p}_{i,\mu}} - \bar{B}_i^\mu + \frac{1}{2} \sum_j \widetilde{A}_{ij} \bar{p}_j^\mu, \tag{18}$$

where $\bar{p}_i$ are vectors such that $\bar{p}_i \cdot Q_j = 0$.

By using eq. (18), tensor integrals are parametrized by integrals of the form $f(\widetilde{A})I(-\frac{d}{2},\ldots)$, where $f(\widetilde{A})$ is a sum of chains of $\widetilde{A}_{ij}$. Chains of $\widetilde{A}_{ij}$ can be traded by sums of chains of $\bar{A}_{ij}$ by solving

$$\begin{aligned}\widetilde{A}_{i_2j_2}\widetilde{A}_{i_3j_3}\cdots\widetilde{A}_{i_nj_n}\bar{A}_{i_1j_1} =&\widetilde{A}_{i_1j_1}\widetilde{A}_{i_2j_2}\cdots\widetilde{A}_{i_nj_n}(A_0+\frac{E}{2})\\ &-\frac{1}{2}(\widetilde{A}_{i_1i_2}\widetilde{A}_{j_1j_2}+\widetilde{A}_{i_1j_2}\widetilde{A}_{i_2j_1})\widetilde{A}_{i_3j_3}\cdots\widetilde{A}_{i_nj_n}\\ &-\frac{1}{2}(\widetilde{A}_{i_1i_3}\widetilde{A}_{j_1j_3}+\widetilde{A}_{i_1j_3}\widetilde{A}_{i_3j_1})\widetilde{A}_{i_2j_2}\widetilde{A}_{i_4j_4}\cdots\widetilde{A}_{i_nj_n}\\ &-\cdots\\ &-\frac{1}{2}(\widetilde{A}_{i_1i_n}\widetilde{A}_{j_1j_n}+\widetilde{A}_{i_1j_n}\widetilde{A}_{i_nj_1})\widetilde{A}_{i_2j_2}\widetilde{A}_{i_3j_3}\cdots\widetilde{A}_{i_{n-1}j_{n-1}}.\end{aligned} \quad (19)$$

The right-hand side of this equation is of degree n in $\widetilde{A}_{ij}$, while the left-hand side is of degree $n-1$. Thus, by solving these identities, we can reduce the degrees of $\widetilde{A}_{ij}$ recursively.

Calculation of the amplitude

Steps to calculate the amplitude:

Parametrize tensor integrals by using eqs. (13,18,19).

Reduce the parametric integrals by solving linear relations generated by eqs. (17).

Calculate the master integrals by using the differential-equation method.

Differential equations can be constructed in the parametric representation. We let y be a Lorentz scalar. Then we have

$$\frac{\partial}{\partial y} = -\sum_{i,u,v} \bar{B}_{iu} B_{iv} \frac{\partial Q_u \cdot Q_v}{\partial y} - 2 \sum_{i,u,v} Q_u \cdot Q_v \bar{B}_{iu} \frac{\partial B_{iv}}{\partial y} + \frac{\partial C}{\partial y}. \quad (20)$$

We introduce the dimensionless variables by re-scaling the momenta as $\tau_i \to \hat{\tau}_i \tau$, $k_i \to \hat{k}_i \tau$ and $l \to \hat{l}\tau$, to factorize out the τ dependence from the rest of the integral and find

$$S^{(3)}_{RRV}(\tau;\epsilon) = \tau^{-1-6\epsilon} \int d\hat{\Phi}_2 \, \frac{d^D \hat{l}}{(2\pi)^D} \, \omega^{(3)}_{RRV}(\hat{l}, \hat{k}_1, \hat{k}_2; n, \bar{n}) \, \Theta(1; \hat{k}_1, \hat{k}_2) \,,$$

Thus we see that the τ-dependence of the soft function is trivial. An extra scale can be introduced by inserting into the master integrals a trivial integral of the form $\int \mathrm{d}y \delta(K-y)$, with K a linear combination of Lorentz scalars of the loop momenta. That is

$$J_i \equiv \int [\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}l]\mathcal{J} = \int \mathrm{d}y \int [\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}l]\delta(K-y)\mathcal{J} \equiv \int \frac{\mathrm{d}y}{2\pi} \, J_i'(y),$$

Integrals free of theta functions: $K = k_1 \cdot k_2, \quad y = \frac{1}{2}(1-x)^2,$

Integrals containing theta functions: $K = k_1^{+/-} + k_2^{+/-}, \quad y = \frac{1}{x}.$

Integrals J_i can be obtained from $J_i'(y)$ by integrating out y. For integrals with $K = k_1 \cdot k_2 - y$, the integration is straightforward. For integrals with $K = k_1^{+/-} + k_2^{+/-}$, $J_i(y)$ is singular at $y = \infty$ $(x = 0)$. The integration can be carried out by subtracting the singular part of $J(y)$ (denoted by $\mathrm{Sing}\{J(y)\}$) and integrating the singular part in d dimensions. That is,

Calculation of the amplitude

$$
\begin{aligned}
J_i &= \int_0^\infty \frac{\mathrm{d}y}{2\pi}\ J_i'(y) \\
&= \int_0^\infty \frac{\mathrm{d}y}{2\pi}\ \mathrm{Ser}\{J_i'(y) - \mathrm{Sing}\{J'(y)\}\} + \int_0^\infty \frac{\mathrm{d}y}{2\pi}\ \mathrm{Sing}\{J'(y)\}.
\end{aligned} \tag{21}
$$

Here we use Ser to denote the series expansion of a function with respect to ϵ.

Examples

We consider the following examples:

$$J'_1(y) =(2\pi)^6 \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(k_1^2\right)\delta\left(k_2^2\right)\delta\left(k_1^+ + k_2^+ - 1\right)\delta\left(k_1^- - k_1^+\right)\delta\left(k_2^- - k_2^+\right)\delta(k_1\cdot k_2 - y),$$

$$J'_2(y) =(2\pi)^6 \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(k_1^2\right)\delta\left(k_2^2\right)\delta\left(k_1^+ + k_2^+ - 1\right)\delta\left(k_2^- - k_2^+\right)\theta\left(k_1^- - k_1^+\right)\delta'\left(-y + k_1^- + k_2^-\right),$$

$$J'_3(y) =(2\pi)^6 \int [\mathrm{d}k_1][\mathrm{d}k_2]\delta\left(k_1^2\right)\delta\left(k_2^2\right)\delta\left(k_1^+ + k_2^+ - 1\right)\delta\left(k_2^- - k_2^+\right)\theta\left(k_1^- - k_1^+\right)\delta\left(-y + k_1^- + k_2^-\right),$$

$$J'_4(y) =(2\pi)^6 \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{\delta\left(k_1^2\right)\delta\left(k_2^2\right)\delta\left(k_1^+ + k_2^+ - 1\right)\delta\left(k_2^- - k_2^+\right)\theta\left(k_1^- - k_1^+\right)\delta\left(-y + k_1^- + k_2^-\right)}{k_1^-\left[-(k_1+k_2)^2\right]},$$

$$J'_5(y) =(2\pi)^6 \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{\delta\left(k_1^2\right)\delta\left(k_2^2\right)\delta\left(k_1^+ + k_2^+ - 1\right)\delta\left(k_2^- - k_2^+\right)\theta\left(k_1^- - k_1^+\right)\delta\left(-y + k_1^- + k_2^-\right)}{k_1^+\left(-(k_1+k_2)^2\right)}.$$

The differential-equations system for these integrals are

$$\frac{dJ'_1(x)}{dx} = \left(\frac{1-2\epsilon}{x-1} + \frac{1-\epsilon}{x-2} + \frac{1-\epsilon}{x}\right) J'_1(x),$$

$$\frac{d}{dy}\begin{pmatrix} J'_2 \\ J'_3 \\ J'_4 \\ J'_5 \end{pmatrix} = \begin{pmatrix} -\frac{(5y-3)(\epsilon-1)}{(y-1)y} & \frac{(\epsilon-1)(4\epsilon-5)}{(y-1)y} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{3y^2-2y+3}{2(y-1)^2y^2} & -\frac{(3y-1)(4\epsilon-5)}{2(y-1)^2y^2} & -\frac{(2y-1)(2\epsilon-1)}{(y-1)y} & 0 \\ \frac{3y^2-2y+3}{2(y-1)^2y} & -\frac{(3y-1)(4\epsilon-5)}{2(y-1)^2y} & 0 & -\frac{2\epsilon-1}{y-1} \end{pmatrix} \cdot \begin{pmatrix} J'_2 \\ J'_3 \\ J'_4 \\ J'_5 \end{pmatrix},$$

Examples

The boundary condition for integral J'_1 can easily calculated. For integrals J'_{2-5}, we have

$$J'_i(x) \approx \sum_j \left(C_{1,i,j} x^{\epsilon+j} + C_{2,i,j} x^{2\epsilon+j} + C_{3,i,j} x^{4\epsilon+j} \right) .$$

Among all the $C_{1,i,j}$, only $C_{1,3,-1}$ is independent, which can be determined by calculating the asymptotic expansions of J'_3. And we note that all the $C_{2,i,j}$ and $C_{3,i,j}$ vanish. To see this, we rescale all the loop momenta by a factor of x. For example,

$$J'_3(x) = (2\pi)^6 x^{4\epsilon-5} \int [\mathrm{d}k_1][\mathrm{d}k_2] \delta\left(k_1^2\right) \delta\left(k_2^2\right) \delta\left(k_1^+ + k_2^+ - x\right) \delta\left(k_2^- - k_2^+\right) \\ \times \theta\left(k_1^- - k_1^+\right) \delta\left(-1 + k_1^- + k_2^-\right) .$$

Due to the constraints of the delta functions, it is easy to see that there is only one region, in which $k_1^+ \sim k_2^+ \sim k_2^- \sim x$, and $k_2^- \sim 1$. Thus $J'_3(x) \sim x^{\epsilon-1}$, so only $C_{1,i,j}$ survive. Once the boundary conditions are determined, solving the DES is straightforward.

Analytic results

We decompose the two-gluon-virtual contribution according to the colour structures by

$$\begin{aligned}\hat{S}^{(3)}_{\text{RRV,gg}} =&(4\pi)^6 \cos(\pi\epsilon)\alpha_s^3 C_A C_F \\ &\times\left(N_f \hat{S}^{(3)}_{\text{RRV,gg},a} + C_A \hat{S}^{(3)}_{\text{RRV,gg},b} + C_F \hat{S}^{(3)}_{\text{RRV,gg},c}\right).\end{aligned} \tag{22}$$

Here

$$\begin{aligned}\hat{S}^{(3)}_{\text{RRV,gg},a} =& \frac{1}{54\epsilon^3} + \frac{31}{162\epsilon^2} + \frac{1}{\epsilon}\left(\frac{95}{81} - \frac{1}{9}\zeta_2\right) + \frac{17}{27}\zeta_2 - \frac{41}{27}\zeta_3 + \frac{3778}{729} \\ &+ \epsilon\left[\frac{62}{27}\zeta_2 + \frac{43}{81}\zeta_3 - \frac{149}{18}\zeta_4 + \frac{39568}{2187}\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{S}^{(3)}_{\text{RRV,gg},b} =& -\frac{5}{3\epsilon^5} - \frac{523}{144\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{34}{3}\zeta_2 - \frac{2129}{216}\right) + \frac{1}{\epsilon^2}\left(\frac{409}{24}\zeta_2 + 42\zeta_3 - \frac{76}{3}\right) \\ &+ \frac{1}{\epsilon}\left(-\frac{386}{3}\zeta_2 + \frac{14399}{72}\zeta_3 + \frac{478}{3}\zeta_4 - \frac{101329}{1944}\right) - \frac{596}{3}\zeta_3\zeta_2 \\ &+ \frac{6277}{54}\zeta_2 - \frac{121919}{108}\zeta_3 + \frac{8889}{8}\zeta_4 + \frac{2360}{3}\zeta_5 - 4\log(2)\zeta_2 - \frac{649801}{5832} \\ &+ 1244.294408\epsilon,\end{aligned} \tag{24}$$

and

$$\hat{S}^{(3)}_{\text{RRV,gg},c} = \frac{6}{\epsilon^5} - \frac{60}{\epsilon^3}\zeta_2 - \frac{300}{\epsilon^2}\zeta_3 - \frac{810}{\epsilon}\zeta_4 + 3000\zeta_2\zeta_3 - 8100\zeta_5 + \epsilon\left[7500\zeta_3^2 - 19635\zeta_6\right]. \tag{25}$$

Similarly, for the two-ghost-virtual contribution, we have

$$\hat{S}^{(3)}_{\text{RRV,gh.gh.}} = (4\pi)^6\alpha_s^3 C_A C_F \left(N_f \hat{S}^{(3)}_{\text{RRV,gh.gh.},a} + C_A \hat{S}^{(3)}_{\text{RRV,gh.gh.},b}\right), \tag{26}$$

with

$$\hat{S}^{(3)}_{\text{RRV,gh.gh.},a} = -\frac{1}{108\epsilon^3} - \frac{13}{324\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{18}\zeta_2 - \frac{10}{81}\right) - \frac{17}{54}\zeta_2 + \frac{41}{54}\zeta_3 - \frac{566}{729} + \epsilon\left[\frac{26}{27}\zeta_2 - \frac{457}{162}\zeta_3 + \frac{149}{36}\zeta_4 - \frac{14575}{4374}\right], \tag{27}$$

and

$$\hat{S}^{(3)}_{\text{RRV,gh.gh.},b} = -\frac{5}{288\epsilon^4} - \frac{5}{144\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{5}{144}\zeta_2 - \frac{2}{81}\right) + \frac{1}{\epsilon}\left(-\frac{5}{27}\zeta_2 + \frac{11}{48}\zeta_3 - \frac{167}{3888}\right)$$
$$- \frac{55}{324}\zeta_2 - \frac{289}{216}\zeta_3 - \frac{1}{16}\zeta_4 + 2\log(2)\zeta_2 + \frac{35}{48}$$
$$+ \epsilon\left[\frac{6583\zeta_2}{972} - \frac{655}{144}\zeta_2\zeta_3 - \frac{529}{162}\zeta_3 + \frac{2543}{144}\zeta_4 + \frac{425}{96}\zeta_5 + 4\log^2(2)\zeta_2\right.$$
$$\left.-\frac{59}{6}\log(2)\zeta_2 - 24\text{Li}_4\left(\frac{1}{2}\right) + \frac{20513}{34992} - \log^4(2)\right]. \tag{28}$$

The double-virtual-real contribution reads

$$S^{(3)}_{V^2R}(\tau) = \frac{\alpha^3_{s,\text{r}}}{8\pi^3}\frac{1}{\mu}\left(\frac{\tau}{\mu}\right)^{-1-6\epsilon}\left(C_F C_A N_f S^{(3)}_{V^2R,C_F C_A n_f} + C_F C_A^2 S^{(3)}_{V^2R,C_F C_A^2}\right), \tag{29}$$

with

$$S^{(3)}_{V^2R,C_FC_An_f} = \frac{1}{9\epsilon^4} + \frac{5}{27\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{19}{81} - \frac{23}{18}\zeta_2\right) + \frac{1}{\epsilon}\left(\frac{65}{243} - \frac{115}{54}\zeta_2 - \frac{7}{3}\zeta_3\right) + \frac{211}{729} - \frac{545}{162}\zeta_2 - \frac{35}{9}\zeta_3 - \frac{13}{48}\zeta_4 + \left(\frac{665}{2187} - \frac{2359\zeta_2}{486} - \frac{223\zeta_3}{27} - \frac{65\zeta_4}{144} - \frac{341\zeta_5}{15} + \frac{161\zeta_3\zeta_2}{6}\right)\epsilon\,, \quad (30)$$

and

$$S^{(3)}_{V^2R,C_FC_A^2} = -\frac{1}{3\epsilon^5} - \frac{11}{18\epsilon^4} - \frac{1}{\epsilon^3}\left(\frac{67}{54} + \frac{23}{6}\zeta_2\right) - \frac{1}{\epsilon^2}\left(\frac{193}{81} - \frac{253}{36}\zeta_2 - 2\zeta_3\right) + \frac{1}{\epsilon}\left(-\frac{1142}{243} + \frac{1541}{108}\zeta_2 + \frac{77}{6}\zeta_3 + \frac{409}{48}\zeta_4\right) - \frac{6820}{729} + \frac{4547}{162}\zeta_2 + \frac{469}{18}\zeta_3 + \frac{143\zeta_4}{96} + \frac{52}{3}\zeta_2\,\zeta_3 - \frac{122}{15}\zeta_5\,, + \left(-\frac{40856}{2187} + \frac{13403\zeta_2}{243} + \frac{1441\zeta_3}{27} + \frac{871\zeta_4}{288} + \frac{3751\zeta_5}{30} - \frac{1771\zeta_2\zeta_3}{12} - \frac{637\zeta_6}{1152} - \frac{31}{2}\zeta_3^2\right)\epsilon\,, \quad (31)$$

Cross check

- Some master integrals are numerically checked by using the sector decomposition method.
- The leading pole (ϵ^{-5}) of the $C_F C_A^2$ term is validated by the strongly ordered limit calculation.
- The $C_F C_A^2 \epsilon^{-5}$ term is canceled upon summing the RRR, RRV, and VVR contributions.
- The $C_F^2 C_A$ term agrees with the nonabelian exponentiation theorem.
- Some diagrams (such as the vacuum polarization diagrams) are checked through direct calculations.

Summary

- We analytically calculate the RRV and VVR contribution to the thrust soft function.
- The application of our method to the calculation of the RRR contribution is straightforward. This is the only missing piece of the complete N^{3}LO thrust soft function, which is the key ingredient for the N^{3}LL$'$ and N^{4}LL resummation of the thrust observable in the dijet configuration.
- Our results can be used for the 0-jettiness in Drell-Yan$/ggH$ process as well as the 1-jettiniess observable in DIS through analytic continuation. The results are also indispensable components to fulfill the N^{3}LO calculation using the N-jettiness subtraction scheme.
- The calculation demonstrates the feasibility of our reduction method to future jet and substructure precision calculations.

The End

Thank you !

S. Brandt, C. Peyrou, R. Sosnowski and A. Wroblewski, Phys. Lett. **12**, 57-61 (1964).

E. Farhi, Phys. Rev. Lett. **39**, 1587-1588 (1977).

S. Kluth, Rept. Prog. Phys. **69**, 1771-1846 (2006) [arXiv:hep-ex/0603011 [hep-ex]].

T. Becher and M. D. Schwartz, JHEP **07**, 034 (2008) [arXiv:0803.0342 [hep-ph]].

R. Kelley, M. D. Schwartz, R. M. Schabinger and H. X. Zhu, Phys. Rev. D **84**, 045022 (2011) [arXiv:1105.3676 [hep-ph]].

P. F. Monni, T. Gehrmann and G. Luisoni, JHEP **08**, 010 (2011) [arXiv:1105.4560 [hep-ph]].

R. Boughezal, X. Liu and F. Petriello, Phys. Rev. D **91**, no.9, 094035 (2015) [arXiv:1504.02540 [hep-ph]].

D. Baranowski, M. Delto, K. Melnikov and C. Y. Wang, Phys. Rev. D **106**, no.1, 014004 (2022) [arXiv:2204.09459 [hep-ph]].

W. Chen, F. Feng, Y. Jia and X. Liu, [arXiv:2206.12323 [hep-ph]].

W. Chen, JHEP **02**, 115 (2020) [arXiv:1902.10387 [hep-ph]].

W. Chen, Eur. Phys. J. C **80**, no.12, 1173 (2020) [arXiv:2007.00507 [hep-ph]].

W. Chen, Eur. Phys. J. C **80**, no.12, 1173 (2020) [arXiv:2007.00507 [hep-ph]].