An Introduction to Perturbative QCD

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Following Jian's Lecture

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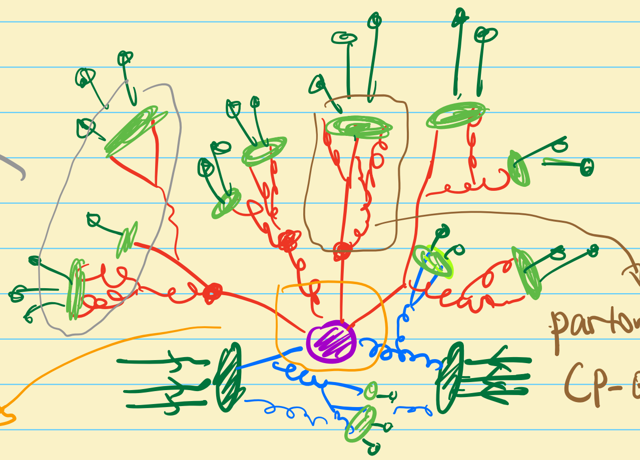
Why (Perturbative) QCD?

- $\alpha \sim \alpha_W \sim 10^{-2} <<< \alpha_s(M_W) \sim 0.1$

$\Rightarrow$ More QCD activities at a collider.

hadronization by non-perturbative QCD



parton shower (P-QCD)

hard Interaction by perturbative QCD

a typical event @ LHC

$$\sigma = \hat{\sigma}_{ij}^{l\ldots}(\mu)\cdot F_{i/p}F_{j/p}(\mu)\cdot D_l(\mu)$$

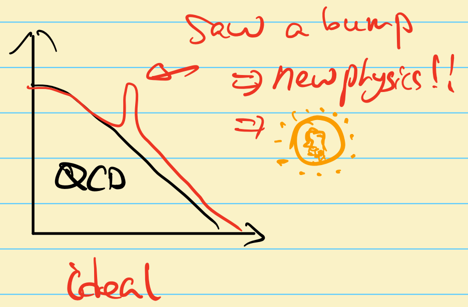
$\gtrsim \mathcal{O}(10\sim100\,\mathrm{GeV})$ $\hat{\sigma}(\mu)$

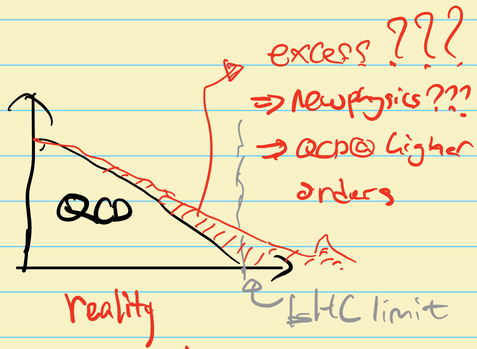
Possible other scales introduced by kinematics & cuts

evolution, resummation, Parton Shower deal with log of scales

$\sim \mathcal{O}(\Lambda_{QCD})$ $F(\mu=\Lambda_{QCD})$, $D(\mu=\Lambda_{QCD})$

- dominant background to new physics searches,





new physics ~ Data − QCD prediction.

- Fundaments to the Monte-Carlo tools

e.g. Madgraph. pythia. Sherpa. ...

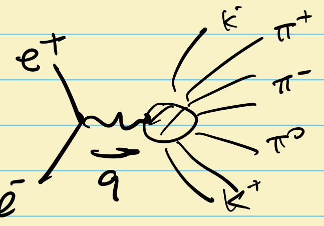
- Outline

- thrust as an example
  - NLO calculation
  - infrared safe
  - break-down of fixed order

- Infrared behaviour in QCD / QED
  - Coherent branching
  - DL Resummation

⊘ Thrust in e^+e^--annihilation

Consider



Measurement
- Incl. (1)
- thrust (see later)

} scalar for simplicity.

$$\sigma = \underbrace{\frac{1}{4}}_{\text{spin ave.}} \underbrace{\frac{1}{2s}}_{\text{Flux}} \int \underbrace{d\Phi_N}_{\text{Phase space of } N \text{ final particles.}} L_{\mu\nu}(l, \bar{l}) \frac{1}{s^2} H^{\mu\nu}(q, \Phi_N)\, \Theta(\Phi_N)$$

scalar

$$\longrightarrow \frac{1}{4}\frac{1}{2s^3} L_{\mu\nu}(l, \bar{l}) \int d\Phi_N \left(g_{\mu\nu} - \cancel{\frac{q^\mu q^\nu}{q^2}}\right) H(q, \Phi_N)\, \Theta(\Phi_N)$$

gauge

$$= \frac{1}{4}\frac{1}{2s^3} L^{\mu}_{\mu}(l\bar{l}) \int d\Phi_N\, H(q, \Phi_N)\, \Theta(\Phi_N)$$

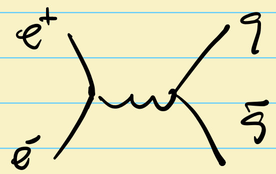
where

$$H = \frac{1}{d-1}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) H^{\mu\nu}$$

$$L^{\mu}_{\mu} = 4(2l\bar{l} - d\, l\bar{l}) = -2(d-2)s$$

For large $s = Q^2 >> \Lambda_{QCD}$, it is well modeled by perturbative calculation

$$H^{\mu\nu}_{(1)} = 4(q^\mu \bar{q}^\nu + q^\nu \bar{q}^\mu - g^{\mu\nu} q \cdot \bar{q})$$



LO, $H^{(0)} = -\frac{1}{d-1} 2(d-2)s$

$$= \sum_f Q_f^2 N_c e^4 \frac{1}{4} \frac{1}{2s^3} \frac{[4(1-\epsilon)s]^2}{d-1} \int d\Phi_2^{\circledast}\, \Theta(\Phi_2) = 1$$

$$= \sum_f Q_f^2 N_c e^4 \frac{1}{4} \frac{1}{2s} \frac{16(1-\epsilon)^2}{3-2\epsilon} (2\pi) \frac{1}{4} \frac{\Omega_{d-2}}{(2\pi)^{d-1}} s^{-\epsilon} \int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon}$$

where $x \equiv \frac{-t}{s}$, $\Omega_{d-2} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$ is the solid angle

$$\Rightarrow \sigma_0 = \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{s} \frac{1-\epsilon}{3-2\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}$$

$$= \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{3s} + O(\epsilon)$$

(*)

$$\int d\Phi_2 = \int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\frac{d^d\bar{q}}{(2\pi)^{d-1}}\,\delta(\bar{q}^2)\,(2\pi)^d\,\delta^{(d)}(Q-q-\bar{q})$$

$$= (2\pi)\int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\delta\left((Q-q)^2\right)$$

$$(Q-q)^2 \rightsquigarrow Q^2 - 2Q\cdot q = Q^2 - 2\ell\cdot q - 2\bar{\ell}\cdot q = s+u+t$$

$$= (2\pi)\int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\delta(s+u+t)$$

let $$q^\mu = \frac{2q\cdot\bar{\ell}}{2\ell\cdot\bar{\ell}}\,\ell^\mu + \frac{2q\cdot\ell}{2\ell\cdot\bar{\ell}}\,\bar{\ell}^\mu + q_\perp^\mu$$

$$= -\frac{u}{s}\,\ell^\mu + \frac{-t}{s}\,\bar{\ell}^\mu + q_\perp^\mu$$

Hence $$q^2 = \frac{ut}{s} - \vec{q}_\perp^{\,2}$$

and $$d^d q = \frac{1}{2}\,\frac{d(-u)}{s}\,\frac{d(-t)}{s}\,d^{d-2}q_\perp$$

$$= \frac{1}{4}\,d\frac{(-u)}{s}\,d\frac{(-t)}{s}\,(q_\perp^2)^{\frac{d-2}{2}}\,dq_\perp\,d\Omega_{d-2}$$

$$\Rightarrow \int d\Phi_2 = \frac{(2\pi)}{(2\pi)^{d-1}}\,\frac{1}{4}\,d\Omega_{d-2}\,s^{-\epsilon}\int_0^1 dx\,x^{-\epsilon}(1-x)^{-\epsilon}$$

For NLO, we have both virtual and real corrections

Virtual:

$+$C.T.

scaleless $\frac{1}{\cancel{\epsilon_{UV}}} - \frac{1}{\epsilon_{IR}}$

renorm.

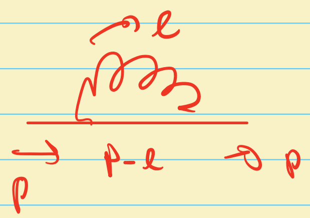
$$\Box = M_0^\mu \frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^\epsilon \left(\underbrace{-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}}_{\text{IR Poles}} -2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} e^{\gamma_E \epsilon}$$

$\Rightarrow \sigma_{virt}$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^\epsilon e^{\gamma_E \epsilon} \cos\pi\epsilon \left(-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}-2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{7}{6}\pi^2\right)$$

⊛



$p^2 = 0$

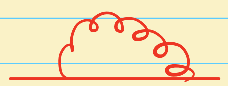
dim. analysis

$$\propto \int d^d\ell \, \frac{1}{\ell^2 (p-\ell)^2} \sim (p^2)^{\frac{d-4}{2}} = 0 = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

scaler

S

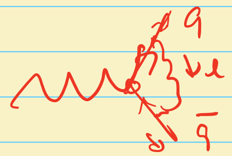
$$\int d^d\ell \, \frac{1}{\ell^4} \sim \text{UV div.}$$



$$= -\frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon} - \frac{1}{\epsilon}\right)$$

UV IR

cancelled by counter term



$$[dl] \equiv \frac{d^d l}{(2\pi)^d}$$

$$= \int dl \; \bar{u}_i \, i g_s \gamma^\alpha t^a_{ik} \; i \frac{\not{q}+\not{l}}{(q+l)^2} \; ie\gamma^\mu \; (-i) \frac{\not{\bar{q}}-\not{l}}{(\bar{q}-l)^2} \; i g_s t^a_{kj} \gamma_\alpha v_j \; \frac{-i}{l^2}$$

$$= +ieg_s^2 (t^a t^a)_{ij} \int [dl] \boxed{\bar{u}_i \gamma^\alpha (\not{l}+\not{q}) \gamma^\mu (\not{l}-\not{\bar{q}}) \gamma_\alpha v_j} \boxed{\frac{1}{(q+l)^2} \frac{1}{(l-\bar{q})^2} \frac{1}{l^2}}$$

Now using the Feyn. Param. we find

$$\boxed{\frac{1}{(l+q)^2} \frac{1}{(l-\bar{q})^2} \frac{1}{l^2}} = \Gamma(3) \int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(L^2 - \alpha_2\alpha_3 s)^3}$$

$$\hookrightarrow \alpha_1 l^2 + \alpha_2 l^2 + 2\alpha_2 l\cdot q + \alpha_3 l^2 - 2\alpha_3 l\cdot\bar{q}$$

$$= l^2 + 2(\alpha_2 q - \alpha_3 \bar{q})\cdot l + (\alpha_2 q - \alpha_3 \bar{q})^2 + \alpha_2\alpha_3 s$$

$$= (\underbrace{l + \alpha_2 q - \alpha_3 \bar{q}}_{\equiv L})^2 + \alpha_2\alpha_3 s$$

We simplify the numerator

$$\boxed{} = \bar{u}_i \gamma^\alpha (\not{t} + \bar{\alpha}_2 \not{q} + \alpha_3 \bar{\not{q}}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \bar{\alpha}_3 \bar{\not{q}}) \gamma_\alpha v_j$$

$$= -2 \bar{u}_i (\not{t} - \cancel{\alpha_2 \not{q}}^{\,0} - \bar{\alpha}_3 \bar{\not{q}}) \gamma^\mu (\not{t} + \bar{\alpha}_2 \not{q} + \cancel{\alpha_3 \bar{\not{q}}}^{\,0}) v_j$$

$$+ 2\epsilon \bar{u}_i (\not{t} + \cancel{\bar{\alpha}_2 \not{q}}^{\,0} + \alpha_3 \bar{\not{q}}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \cancel{\bar{\alpha}_3 \bar{\not{q}}}^{\,0}) v_j$$

$\propto L^{2n}$ odd power integrated to zero

$$\longrightarrow -2 \bar{u}_i \not{t} \gamma^\mu \not{t} - \bar{\alpha}_3 \bar{\alpha}_2 \bar{\not{q}} \gamma^\mu \not{q} v_j$$

$$+ 2\epsilon \bar{u}_i \not{t} \gamma^\mu \not{t} - \alpha_2 \alpha_3 \bar{\not{q}} \gamma^\mu \not{q} v_j$$

$$= -2(1-\epsilon) \bar{u}_i \not{t} \gamma^\mu \not{t} v_j$$

$$+ (2\bar{\alpha}_3 \bar{\alpha}_2 - 2\epsilon \alpha_3 \alpha_2) \bar{u}_i \bar{\not{q}} \gamma^\mu \not{q} v_j$$

$L^\mu L^\nu \to \frac{1}{d} L^2 g^{\mu\nu}$

$$\xrightarrow{\quad} \boxed{\begin{aligned} &\frac{(d-2)^2}{d} L^2 \, \bar{u}_i \gamma^\mu v_j \\ &+ 2(\bar{\alpha}_2 \bar{\alpha}_3 - \epsilon \alpha_2 \alpha_3)(-s) \bar{u}_i \gamma^\mu v_j \end{aligned}}$$

$d/2 = 2 - \epsilon$

Now we use

$\cancel{\delta}$ UV

$$\int [dl] \frac{L^2}{(L^2 + \alpha_2 \alpha_3 s)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-\epsilon} (-s)^{-\epsilon} \quad (1)$$

$$\int [dl] \frac{1}{(l^2 + \alpha_2 \alpha_3 s)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-1-\epsilon} (-s)^{-1-\epsilon} \quad (2)$$

⇒

① ⇒

$$g_s^2 \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\cancel{\Gamma(3)}} (-s)^{-\epsilon} \frac{(d-2)^2}{d} \int_0^1 d\alpha_2 \int_0^{\bar{\alpha}_2} d\alpha_3 \, (\alpha_2 \alpha_3)^{-\epsilon} \, \cancel{\Gamma(3)}$$

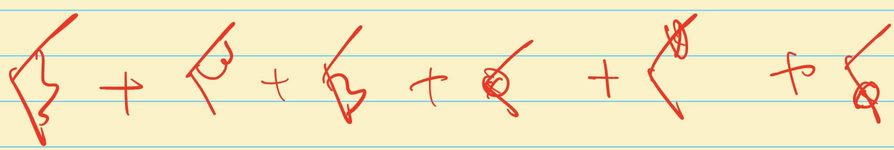
$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[\frac{1}{2\epsilon} + 1/2\right]$$

ϵ_{UV} cancelled by counter term

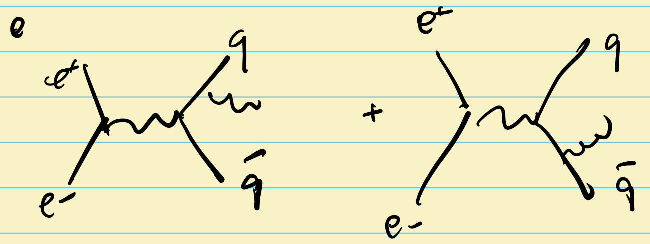
② ⇒

$$\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - 9/2 + \frac{\pi^2}{12}\right]$$

ϵ_{coll} ϵ_{IR}



$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2}\frac{1}{\epsilon} - 4 + \frac{\pi^2}{12}\right]$$



NLO real

$$H^{(1)} = -C_F g_s^2 \delta(1-\epsilon) \frac{1}{d-1}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

Where

$$y_1 = \frac{s_{12}}{Q^2}, \quad y_2 = \frac{s_{13}}{Q^2}, \quad y_3 = \frac{s_{23}}{Q^2}$$

and $s_{ij} = 2 q_i \cdot q_j$

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{2^{d+1}} s^{d-3} \, d\Omega_{D-1} \, d\Omega_{D-2}$$

$$(y_1 y_2 y_3)^{-\epsilon} \int_0^1 dy_1 \, dy_2 \, dy_3 \, \delta(1-y_1-y_2-y_3)$$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$$\times\, \Theta_1(\Phi_3)$$

Inclusive: $\Theta(\Phi_3) = 1$

let $y_1 = (1-z_1)$, $y_2 = z_1 z_2$, $y_3 = (1-z_2) z_1$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dz_1\, dz_2\;\; \bar{z}_1^{-\epsilon}\, z_1^{-2\epsilon}\, z_2^{-\epsilon}\, \bar{z}_2^{-\epsilon}\;\; \overset{\text{Jacobian}}{z_1}$$

$$\times \left\{ \frac{2}{z_1^2 z_2 \bar{z}_2} + \frac{-2+z_1\bar{z}_2-\epsilon z_1\bar{z}_2}{z_1 z_2} + \frac{-2+z_1 z_2-\epsilon z_1 z_2}{z_1 \bar{z}_2} - 2\epsilon \right\}$$

$$\Rightarrow \sigma_{real} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2\right)$$

$$\sigma_{virt.} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2\right)$$

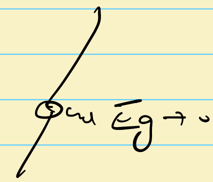
$$\sigma = \sigma_0 \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2} \right\}$$

- all IR poles cancelled, KLN Theorem
- Virt & Real almost completely cancelled with each other : $V + R \sim 1 \leftarrow$ Unity
- Origin of the IR poles :

all IR poles from the soft & coll. limit

1 $z_1 \to 0. \Rightarrow y_2 \,\&\, y_3 \to 0$

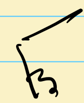
$\Rightarrow p_3 \to 0$ soft



2. z_2 or $\bar{z}_2 \to 0 \Rightarrow y_2$ or $y_3 \to 0$

$\Rightarrow p_3 \cdot p_1 \to 0,\ p_3 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_1$

or $p_3 \cdot p_2 \to 0,\ p_2 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_2$



$z_1 \to 0$

$$\int_0^1 dz_1 dz_2\, z_1^{-2\epsilon}\, z_2^{-\epsilon}\, \bar{z}_2^{-\epsilon} \times \left\{ \frac{2}{\underbrace{z_1}_{\text{soft}}\, \underbrace{z_2 \bar{z}_2}_{\text{collinear}}} \right\} = \underbrace{\frac{2}{\epsilon^2}}_{\text{soft+coll.}} + \cdots$$

$z_2 \to 0$

$$\int_0^1 dz_1 dz_2\, \bar{z}_1^{-\epsilon}\, z_1^{-2\epsilon}\, z_2^{-\epsilon}$$

$$\times \left\{ \cancel{\frac{2}{z_1 \bar{z}_2 z_2}} + \frac{-2 + z_1 - \epsilon z_1}{z_2} \right\} = \frac{3}{2} \frac{1}{\epsilon}$$

subtract out to avoid double counting

does reproduce all poles

every order you will have

$$\Delta^{(n)}_{\text{real}} \propto \left(\frac{\alpha_s}{2\pi}\right)^{\overset{n\text{-emission}}{n}} \left(\underset{n\text{ soft}+\text{coll.}}{\frac{\#_{2n}}{\epsilon^{2n}}} + \cdots + \frac{\#_1}{\epsilon} + \text{Finite terms} \right)$$

cancel against the IR poles in the virtual. For Inclusive processes

for exclusive processes.

e.g. $pp \to \ldots$

$e^+e^- \to$ hadron $+ \ldots$

remaining poles to be absorbed into PDFs or FFs.