

An Introduction to perturbative QCD

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following Jian's Lecture

2022年理论物理前沿讲习班.

暗物质与新物理暑期学校

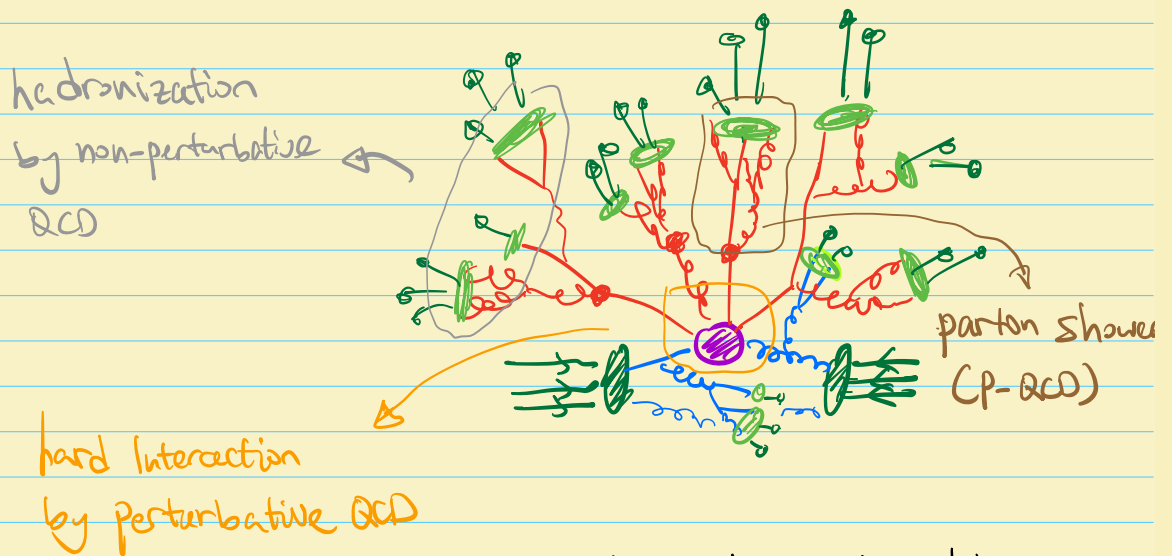
④ 东南师范大学

07/04/2022 - 07/22/2022

Why (perturbative) QCD?

$$\alpha \sim \alpha_s \sim 10^{-2} \ll \alpha_s(M_W) \sim 0.1$$

$\Rightarrow$ more QCD activities at a collider.



a typical event @ LHC

$$\sigma = \hat{\sigma}_{ij}^{e^{+}e^{-}} \cdot F_{i/P} F_{j/P} \cdot D_e(u)$$

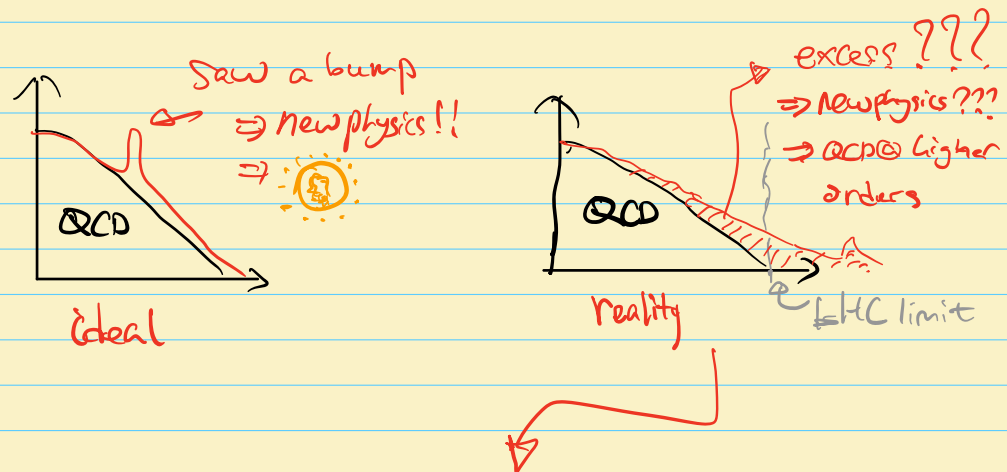
$$\sim \sigma(\mu_0 \sim 100 \text{ GeV}) \hat{\sigma}(u)$$

evolution, resummation, parton shower deal with log of scales

$$\sim \sigma(\mu_{\text{QCD}}) F(u \sim 100) D(u \sim 100)$$

Possible other scales introduced by kinematics or cuts

- dominant background to New physics Searches,



new physics $\sim$ Data - QCD prediction.

- Fundamentals to the Monte-Carlo tools

e.g. Madgraph, pythia, Sherpa, ...

— Outline

• thrust as an example

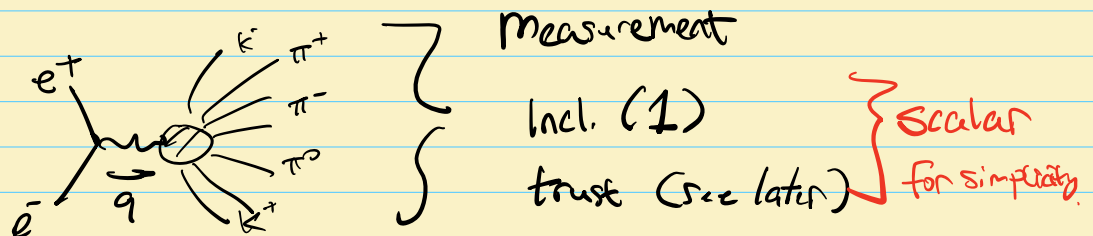
- NLO calculation
- infrared safe
- break-down of fixed order

• Infrared behaviour in QCD/QED

- Coherent branching
- DL Resummation

① Thrust in e^+e^- -annihilation

Consider



$$\sigma = \frac{1}{4} \frac{1}{2S} \int d\Phi_N L_{\mu\nu}(l, \bar{l}) \frac{1}{S^2} H^{\mu\nu}(q, \Phi_N) \Theta(\Phi_N)$$

$\underbrace{\quad}_{\text{Spin ave.}} \quad \underbrace{\quad}_{\text{FLUX}} \quad \underbrace{\quad}_{\text{Phase space of } N \text{ final particles.}}$

scalar $\rightarrow \frac{1}{4} \frac{1}{2S^3} L_{\mu\nu}(l, \bar{l}) \int d\Phi_N \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) H(q, \Phi_N) \Theta(\Phi_N)$

$\nearrow$ gauge

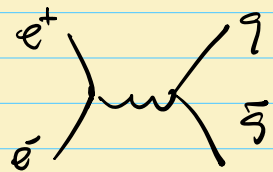
$$= \frac{1}{4} \frac{1}{2S^3} L_{\mu}^{\mu}(l, \bar{l}) \int d\Phi_N H(q, \Phi_N) \Theta(\Phi_N)$$

where

$$H = \frac{1}{d-1} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) H^{\mu\nu}$$

$$L_{\mu}^{\mu} = 4(2l\bar{l} - d l\bar{l}) = -2(d-2)S$$

For large $s = Q^2 \gg \Lambda_{QCD}$, it is well modeled by
perturbative calculation



$$H^{(\gamma)} = 4(\bar{q}\hat{\gamma}^\mu q + q\bar{q}\hat{\gamma}^\mu - g^{\mu\nu}\bar{q}\hat{\gamma}^\nu q)$$

$$L^0, \quad H^{(\gamma)} = -\frac{1}{2-1} 2(d-2)s$$

$$= \frac{2}{f} Q_F^2 N_c e^4 \frac{1}{4} \frac{1}{2s} \frac{[4(1-\epsilon)s]^2}{d-1} \int d\Phi_2 \ominus(\Phi_2) = 1$$

$$= \frac{2}{f} Q_F^2 N_c e^4 \frac{1}{4} \frac{1}{2s} \frac{16(1-\epsilon)^2}{3-2\epsilon} (2\pi) \frac{1}{4} \frac{\Omega_{d-2}}{(2\pi)^{d-1}} s^{-\epsilon} \int_0^1 dx x^{-\epsilon} (1-x)^{-\epsilon}$$

where $x \equiv \frac{-t}{s}$, $\Omega_{d-2} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$ is the solid angle

$$\Rightarrow G_0 = \frac{2}{f} Q_F^2 N_c \frac{4\pi\alpha^2}{s} \frac{1-\epsilon}{3-2\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^\epsilon$$

$$= \frac{2}{f} Q_F^2 N_c \frac{4\pi\alpha^2}{3s} + \mathcal{O}(\epsilon)$$

⊗

$$\int d\Phi_2 = \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \frac{d^d \bar{q}}{(2\pi)^{d-1}} \delta(\bar{q}^2) (2\pi)^d \delta^d(Q - q - \bar{q})$$

$$= (2\pi) \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \delta((Q - q)^2) \stackrel{\sim}{=} \begin{matrix} Q^2 - 2Q \cdot q \\ Q^2 - 2E_Q q - 2\vec{Q} \cdot \vec{q} \\ = s + u + t \end{matrix}$$

$$= (2\pi) \int \frac{d^d q}{(2\pi)^{d-1}} \delta(q^2) \delta(s + u + t)$$

$$\text{let } q^\mu = \frac{2q \cdot \bar{e}}{2\bar{e} \cdot \bar{e}} \bar{e}^\mu + \frac{2q \cdot e}{2e \cdot e} e^\mu + q_\perp^\mu$$

$$= -\frac{u}{s} \bar{e}^\mu + \frac{-t}{s} e^\mu + q_\perp^\mu$$

$$\text{Hence } q^2 = \frac{ut}{s} - \vec{q}_\perp^2$$

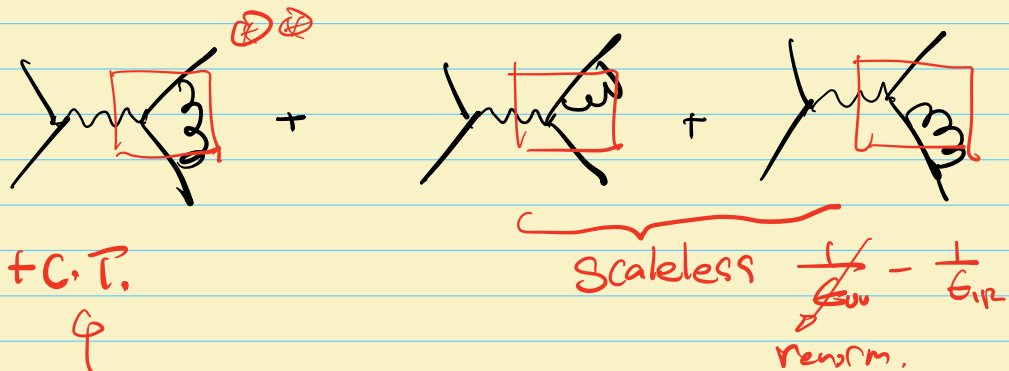
$$\text{and } d^d q = \frac{1}{2} \frac{d(u)}{s} \frac{d(-t)}{s} d^{d-2} q_\perp$$

$$= \frac{1}{4} \frac{d(-u)}{s} \frac{d(-t)}{s} (q_\perp^2)^{\frac{d-2}{2}} dq_\perp d\Omega_{d-2}$$

$$\Rightarrow \int d\Phi_2 = \frac{(2\pi)}{(2\pi)^{d-1}} \frac{1}{4} d\Omega_{d-2} s^{-6} \int_0^1 dx x^6 (1-x)^6$$

for NLO, we have both virtual and real corrections

Virtual :



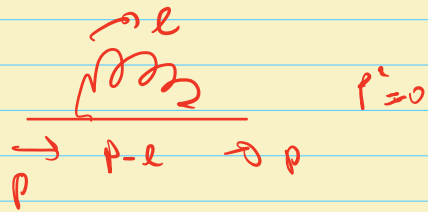
$$\square = M_0^4 \frac{d^3s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-s} \right)^\epsilon \underbrace{\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} - 2 \right)}_{\text{IR Poles}} \frac{T^2(1-\epsilon)T(1+\epsilon)}{T(2-2\epsilon)} e^{\gamma_E \epsilon}$$

$\Rightarrow$ $\mathcal{Q}_{\text{virt}}$

$$= G_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s} \right)^\epsilon e^{\gamma_E \epsilon} \cos \pi \epsilon \left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} - 2 \right) \frac{T^2(1-\epsilon)T(1+\epsilon)}{T(2-2\epsilon)}$$

$$= G_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s} \right)^\epsilon \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2 \right)$$

⊗



$$\propto \int d^d l \frac{1}{e^2 (p-l)^2} \sim [p^\mu]^{\frac{d-4}{2}} = 0 = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

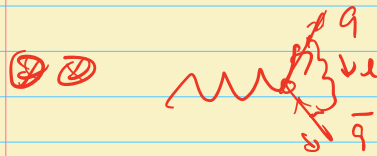
$\int d^d l$ $\xrightarrow{\text{dim. analysis}}$ $\xrightarrow{\text{scaler}}$

$$\int d^d l \frac{1}{l^4} \sim UV \text{ div.}$$

$$\text{Diagram} = -\frac{\alpha_s}{2\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right)$$

$\frac{1}{\epsilon_{UV}}$ $\frac{1}{\epsilon_{IR}}$
 $\uparrow$ $\uparrow$
 UV IR
 ℓ

Cancelled by Counter term


 $[d\ell] \equiv \frac{d^4\ell}{(2\pi)^4}$

$$= \int d\ell \quad \bar{u}_i \quad i g_s \gamma_s^a t_{ik}^a \quad i \frac{\not{x} + \not{\ell}}{(\ell + x)^2} \quad i e \gamma_s^\mu (-i) \frac{\not{\bar{q}} - \not{\ell}}{(\bar{q} - \ell)^2} \quad i g_s t_{kj}^a \gamma_a \psi_j \quad \frac{-i}{\ell^2}$$

$$= +i e g_s^2 (t^a t^a)_{ik} \int [d\ell] \quad \bar{u}_i \gamma^\mu (\not{x} + \not{\ell}) \not{(\bar{q} - \ell)} \gamma_\mu \psi_j \quad \left[\frac{1}{(\ell + x)^2} \frac{1}{(\bar{q} - \ell)^2} \frac{1}{\ell^2} \right]$$

Now using the Feyn. Param. we find

$$\left[\frac{1}{(\ell + x)^2} \frac{1}{(\bar{q} - \ell)^2} \frac{1}{\ell^2} \right] = \Gamma(3) \int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1 + \alpha_2 + \alpha_3 - 1)}{(L^2 - \alpha_1 \alpha_3 S)^3}$$

$$\hookrightarrow \alpha_1 \ell^2 + \alpha_2 \bar{\ell}^2 + 2\alpha_2 \ell \cdot \bar{q} + \alpha_3 \ell^2 - 2\alpha_3 \ell \cdot \bar{q}$$

$$= \ell^2 + 2(\alpha_2 \bar{q} - \alpha_3 \bar{q}) \cdot \ell + (\alpha_2 \bar{q} - \alpha_3 \bar{q})^2 + \alpha_2 \alpha_3 S$$

$$= \underbrace{(\ell + \alpha_2 \bar{q} - \alpha_3 \bar{q})^2}_{\equiv L} + \alpha_2 \alpha_3 S$$

$$\equiv L$$

We simplify the numerator

$$\begin{aligned}
 \square &= \bar{u}_i \gamma^\alpha (1 + \alpha_2 \not{x} + \alpha_3 \not{\bar{x}}) \gamma^\mu (1 - \alpha_2 \not{x} - \alpha_3 \not{\bar{x}}) \gamma_\alpha v_j \\
 &= -2 \bar{u}_i (1 - \alpha_2 \not{x} - \alpha_3 \not{\bar{x}}) \gamma^\mu (1 + \alpha_2 \not{x} + \alpha_3 \not{\bar{x}}) v_j \\
 &\quad + 2\epsilon \bar{u}_i (1 + \alpha_2 \not{x} + \alpha_3 \not{\bar{x}}) \gamma^\mu (1 - \alpha_2 \not{x} - \alpha_3 \not{\bar{x}}) v_j
 \end{aligned}$$

$\propto L^{2\epsilon}$
 odd power
 integrated
 to zero

$$\begin{aligned}
 &\sim 2 \bar{u}_i 1 \gamma^\mu 1 - \alpha_3 \alpha_2 \not{\bar{x}} \gamma^\mu \not{x} v_j \\
 &\quad + 2\epsilon \bar{u}_i 1 \gamma^\mu 1 - \alpha_2 \alpha_3 \not{x} \gamma^\mu \not{\bar{x}} v_j
 \end{aligned}$$

$$\begin{aligned}
 &= -2(1-\epsilon) \bar{u}_i 1 \gamma^\mu 1 v_j \\
 &\quad + (2\alpha_3 \alpha_2 - 2\epsilon \alpha_3 \alpha_2) \bar{u}_i \not{\bar{x}} \gamma^\mu \not{x} v_j
 \end{aligned}$$

$$L^\mu L^\nu \rightarrow \frac{1}{2} L^\mu g^{\mu\nu}$$

$$\begin{aligned}
 &\frac{(d-2)^2}{2} L^2 \bar{u}_i \gamma^\mu v_j \\
 &\quad + 2(\alpha_2 \alpha_3 - \epsilon \alpha_2 \alpha_3) (-S) \bar{u}_i \gamma^\mu v_j
 \end{aligned}$$

$$d_L = 2 - \epsilon$$

Now we use

$$\int [dL] \frac{L^2}{(L^2 + \alpha_2 \alpha_3 S)^3} = \frac{i}{(4\pi)^{d_L/2}} \frac{1}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-\epsilon} (-S)^{-\epsilon} \quad (1)$$

$$\int [dL] \frac{1}{(L^2 + \alpha_2 \alpha_3 S)^3} = \frac{-i}{(4\pi)^{d_L/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-1+\epsilon} (-S)^{-1-\epsilon} \quad (2)$$

$\Rightarrow$

(1) $\Rightarrow$

$$g_s^2 \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (G_S)^{-\epsilon} \frac{(d-2)^2}{d} \int_0^1 dx_2 \int_0^{\bar{x}_2} dx_3 (x_2 x_3)^{-\epsilon} \Gamma(\epsilon)$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s} \right)^{+\epsilon} \left[\frac{1}{2\epsilon} + \frac{1}{2} \right]$$

ϵ UV cancelled by counter term

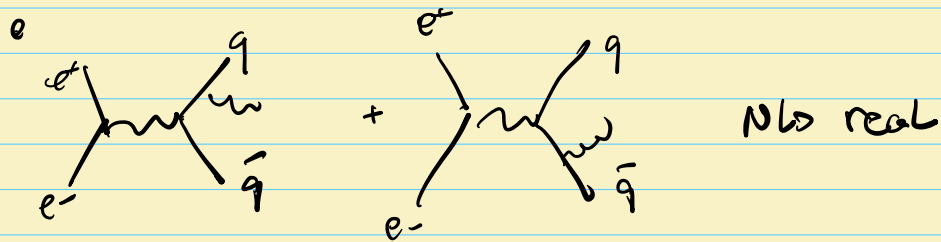
(2) $\Rightarrow$

$$\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s} \right)^{+\epsilon} \left[-\frac{1}{6\epsilon^2} + \frac{2}{6\epsilon} - \frac{9}{2} + \frac{\pi^2}{12} \right]$$

$\hookrightarrow \text{Call } \int_R$

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6}$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s} \right)^{\epsilon} \left[-\frac{1}{6\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{\pi^2}{12} \right]$$



$$H^{(1)} = -C_F g_s^2 \delta(1-\epsilon) \frac{1}{d-1}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2 + y_3 - \epsilon y_3}{y_2} + \frac{-2 + y_2 - \epsilon y_2}{y_3} - 2\epsilon \right\}$$

Where

$$y_1 = \frac{S_{12}}{Q^2}, \quad y_2 = \frac{S_{13}}{Q^2}, \quad y_3 = \frac{S_{23}}{Q^2}$$

$$\text{and } S_{ij} = 2 p_i \cdot p_j$$

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{2^{d+1}} s^{d-3} d\Omega_{d-1} d\Omega_{d-2}$$

$$(y_1 y_2 y_3)^{-\epsilon} \int_0^1 dy_1 dy_2 dy_3 \delta(1-y_1-y_2-y_3)$$

$$\sigma_{\text{real}} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_{\text{E}}}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s} \right)^{-\epsilon}$$

$$\times \int_0^1 dy_1 dy_2 dy_3 (y_1 y_2 y_3)^{-\epsilon} \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$$\times \mathcal{O}(\Phi_3)$$

Inclusive: $\mathcal{O}(\Phi_3) = 1$

$$\text{let } y_1 = (1-z_1), y_2 = z_1 z_2, y_3 = (1-z_2) z_1$$

$$\sigma_{\text{real}} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_{\text{E}}}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s} \right)^{-\epsilon}$$

$$\times \int_0^1 dz_1 dz_2 \bar{z}_1^{-\epsilon} z_1^{-2\epsilon} \bar{z}_2^{-\epsilon} \bar{z}_2^{-\epsilon} z_1 \quad \nearrow \text{Jacobian}$$

$$\times \left\{ \frac{2}{z_1 \bar{z}_1 z_2 \bar{z}_2} + \frac{-2+z_2 \bar{z}_2 - \epsilon z_2 \bar{z}_2}{z_1 z_2} + \frac{-2+z_1 \bar{z}_1 - \epsilon z_1 \bar{z}_1}{z_1 \bar{z}_1} - 2\epsilon \right\}$$

$$\Rightarrow \Delta_{\text{real}} = \Delta_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi M^2}{s} \right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2 \right)$$

$$\Delta_{\text{virt.}} = \Delta_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi M^2}{s} \right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2 \right)$$

$$\Delta = \Delta_0 \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2} \right\}$$

• all IR poles cancelled, KLN Theorem

• Virt & Real almost completely cancelled with each other: $V + R \sim 1 \pm \text{Unity}$

• Origin of the IR poles:

all IR poles from the soft & coll. limit

$$1. z_1 \rightarrow 0. \Rightarrow y_2 \neq y_3 \rightarrow 0$$

$$\Rightarrow p_3 \rightarrow 0 \quad \text{soft}$$

$$\cancel{\phi_{u, \bar{g} \rightarrow 0}}$$

$$2. z_2 \text{ or } \bar{z}_2 \rightarrow 0 \Rightarrow y_2 \text{ or } y_3 \rightarrow 0$$

$$\Rightarrow p_3 \cdot p_1 \rightarrow 0, p_3 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_1$$

$$\text{or } p_3 \cdot p_2 \rightarrow 0, p_2 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_2$$

$$z_1 \rightarrow 0$$

$$\int_0^1 d\alpha d\bar{z}_2 \bar{z}_1^{-2\epsilon} \bar{z}_2^{-\epsilon} \bar{z}_2^{-\epsilon} \times \left\{ \frac{2}{z_1 + \bar{z}_1 \bar{z}_2} \right\} = \frac{2}{6\epsilon} + \dots$$

$\uparrow \quad \uparrow \quad \uparrow$
 soft collinear soft+coll.

$$\bar{z}_2 \rightarrow 0$$

$$\int_0^1 d\alpha d\bar{z}_2 \bar{z}_1^{-\epsilon} \bar{z}_1^{-2\epsilon} \bar{z}_2^{-\epsilon} \times \left\{ \frac{2}{z_1 + \bar{z}_1 \bar{z}_2} + \frac{-2 + \bar{z}_1 - 6\bar{z}_2}{\bar{z}_2} \right\} = \frac{3}{2} \frac{1}{\epsilon}$$

Subtract out to avoid double counting

does reproduce all poles

every order you will have

$$\Delta_{\text{real}}^{(n)} \propto \left(\frac{\alpha_s}{2\pi}\right)^n \left(\underbrace{\frac{\#_{n-1}}{\epsilon^{2n}} + \dots + \frac{\#_1}{\epsilon}}_{n \text{ soft + coll.}} + \text{Finite terms} \right)$$

n-emission

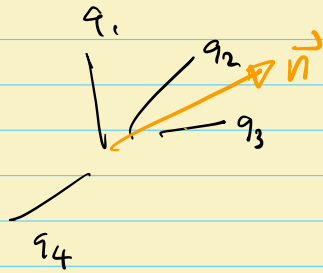
cancel against the IR poles
in the virtual. For inclusive
processes

for exclusive processes,

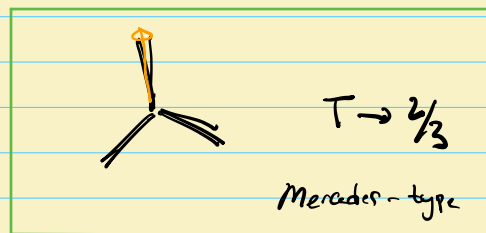
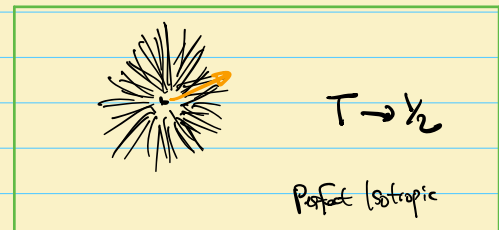
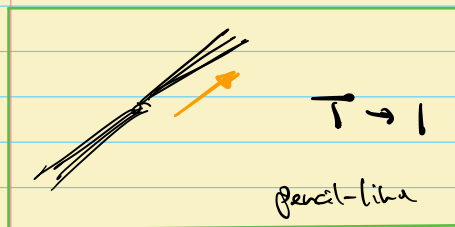
e.g. $pp \rightarrow \dots$
 $e^+e^- \rightarrow \text{hadron} + \dots$

remaining poles to be absorbed
into PDFs or FFs.

thrust: $T = \frac{\sum_i |\vec{q}_i \cdot \vec{n}|}{\sum_i |\vec{q}_i|}$

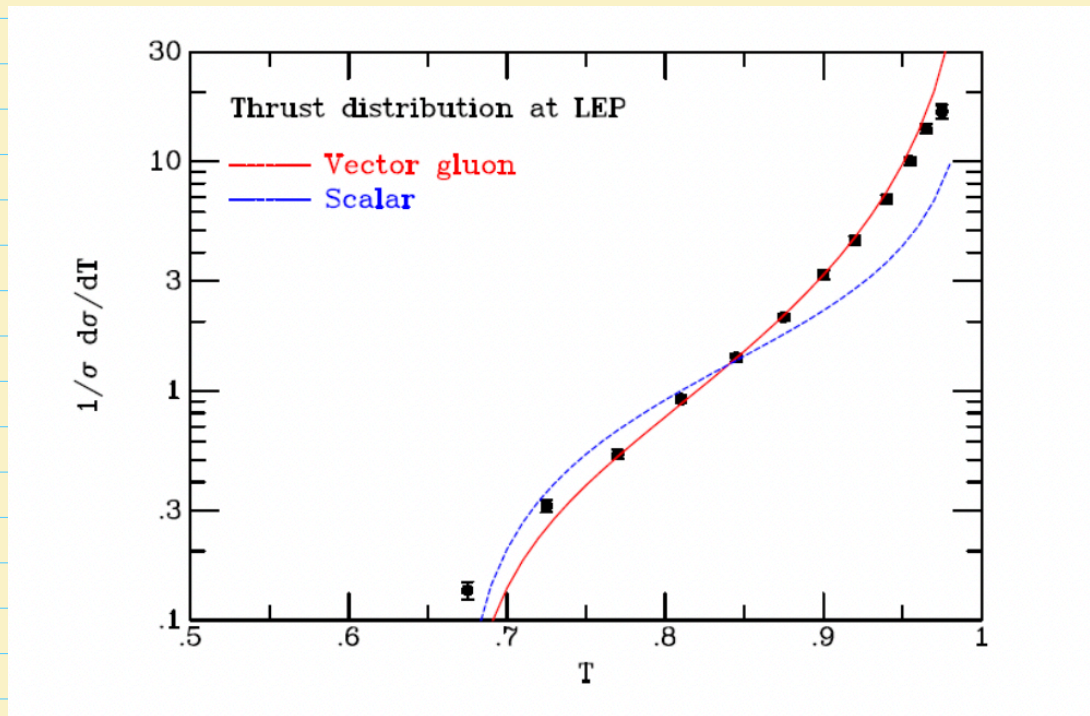


massless case $\sum_i |\vec{q}_i| = Q$



measure the shape of an event

as $T \rightarrow 0$, only soft or coll. radiations
are allowed

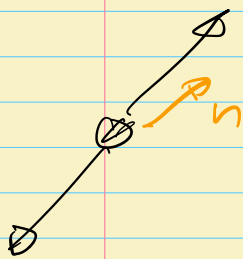


Used for discovery of gluon
+ gluon spin determination
+ precision determination of α_s

restrictions on the phase space

$$\Theta(\mathbb{I}_2) = \delta\left(T - \sum_{\text{max } \vec{n}} \frac{|\vec{q}_1 \cdot \vec{n}| + |\vec{q}_2 \cdot \vec{n}|}{Q}\right)$$

$$= \delta\left(T - \sum_{\text{max } \vec{n}} \frac{|\vec{q}_2 \cdot \vec{n}| + |\vec{q}_1 \cdot \vec{n}|}{Q}\right)$$



$$= \delta\left(T - \sum_{\text{max } \vec{n}} \frac{Q/2 \cos \theta + Q/2 \cos \theta}{Q}\right)$$

$$= \delta\left(T - \sum_{\text{max } \vec{n}} \cos \theta\right)$$

$$= \delta(1-T) \quad \text{always.}$$

no restrictions

does not contribute to $T \neq 1$ case,
which starts from 3-body final
state

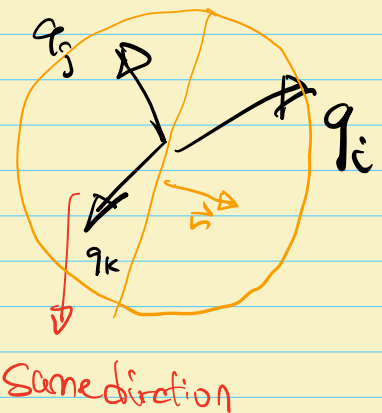
$$\delta^{(6)} + \delta^{(1)}_{\text{int}} \rightarrow (\delta^{(6)} + \delta^{(1)}_{\text{int}}) \delta(1-T)$$

$$\Theta(\bar{\mathcal{A}}_3) = \mathcal{SCT} - \sum_{\max \vec{n}} \frac{|\vec{q}_1 \cdot \vec{n}| + |\vec{q}_2 \cdot \vec{n}| + |\vec{q}_3 \cdot \vec{n}|}{Q}$$

too complicated for analytic calculation

- for an arbitrary given $\vec{n}$, we can separate the phase space to 2 hemispheres, with one covers only one particle q_i

- the max of $\vec{n}$ for such config. is given by



$$\max \left| \frac{\vec{q}_i \cdot \vec{n}}{Q} \right| + \left| \frac{\vec{q}_j \cdot \vec{n}}{Q} \right| + \left| \frac{\vec{q}_k \cdot \vec{n}}{Q} \right|$$

$$= \max \left| \frac{\vec{q}_i \cdot \vec{n}}{Q} \right| + \left| \frac{\vec{q}_j \cdot \vec{n}}{Q} + \frac{\vec{q}_k \cdot \vec{n}}{Q} \right|$$

$$= \max \left| \frac{\vec{q}_i \cdot \vec{n}}{Q} \right| + \left| -\frac{\vec{q}_i \cdot \vec{n}}{Q} \right| = \max \frac{2|\vec{q}_i \cdot \vec{n}|}{Q} = \frac{2E_i}{Q} = x_i$$

$$= \frac{2q_i \cdot Q}{Q^2} = 1 - y_i$$

- We need to maximize all possible configs

$$\Rightarrow T = \max(x_1, x_2, x_3) = \max(1 - y_1, 1 - y_2, 1 - y_3)$$

$$\Rightarrow 1 - T = \min(y_1, y_2, y_3) \equiv \tau$$

Which leads to the phase space restriction

$$\begin{aligned}\Theta(\Phi_3) &= \sum_i \delta(z-y_i) \Theta(y_i < y_j) \Theta(y_i < y_k) \\ &= \sum_i \delta(z-y_i) \Theta(z < y_k < 1-2z) \Theta(z < 1/3)\end{aligned}$$

Plug into the real-emission cross section

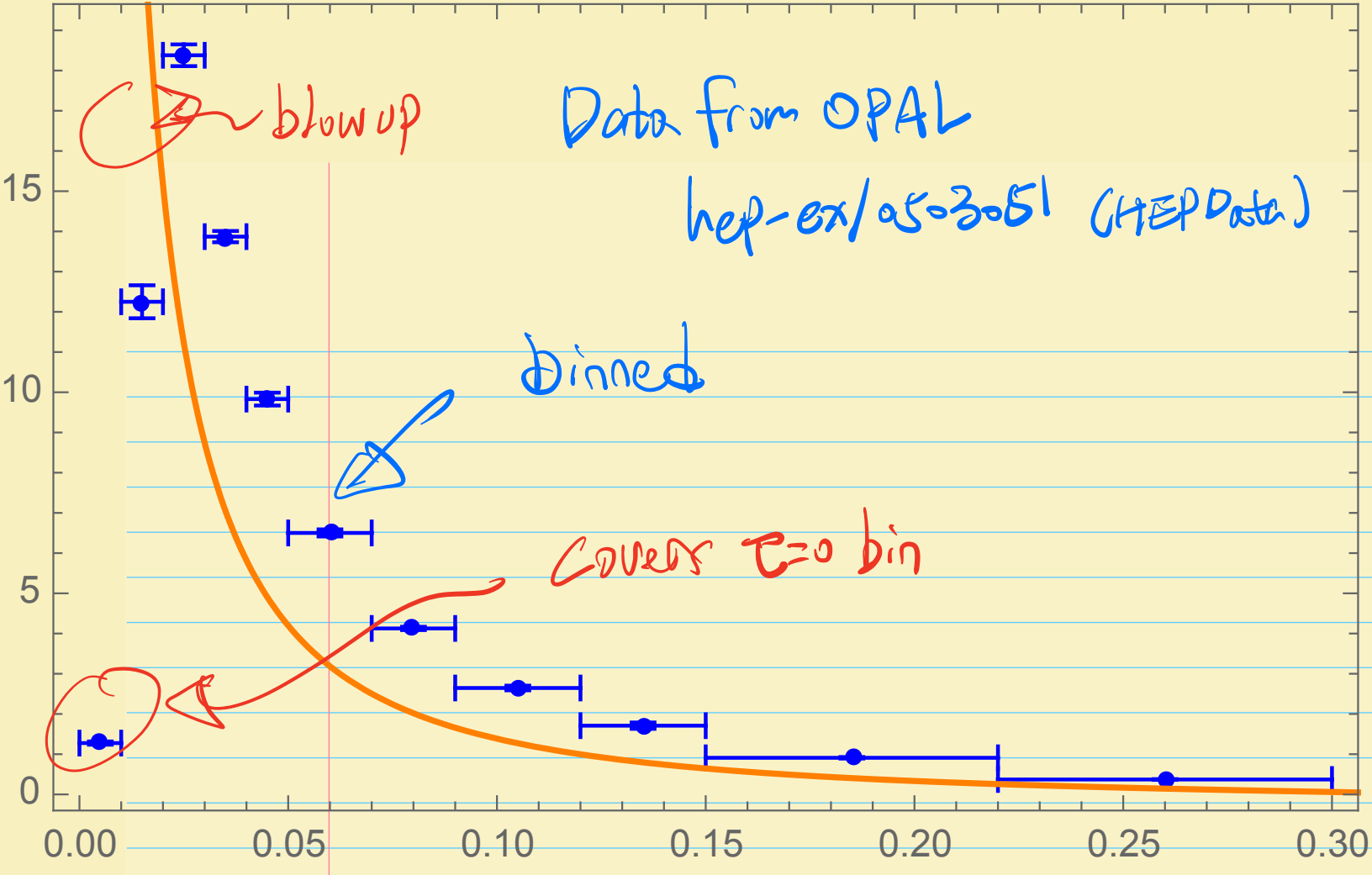
$$\begin{aligned}\sigma_{\text{real}} &= \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_{\text{E}}}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon} \\ &\times \int_0^1 dy_1 dy_2 dy_3 (y_1 y_2 y_3)^{-\epsilon} \delta(1-y_1-y_2-y_3) \\ &\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\} \\ &\Theta(\Phi_3)\end{aligned}$$

You can work out the integration with ϵ kept using Mathematica

For simplicity, we set $\epsilon=0$ to find

$$\frac{d\sigma_{\text{real}}}{d\tau} = \frac{\alpha_s G_F^2}{2\pi} \left\{ \frac{4}{\tau} \log\left(\frac{1-\tau}{\tau}\right) - \frac{3(1+\tau)(1-3\tau)}{\tau} - 6 \log\left(\frac{1-\tau}{\tau}\right) \right\}$$

for $0 < \tau < \frac{1}{3}$, otherwise vanishes,



We can slightly improve our result to cover the " $z=0$ " bin.

Since we know $\sigma_{\text{tot}} = \sigma_0 (1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2})$

hence $\int_0^{1/3} \frac{d\sigma}{dz} = \sigma_{\text{tot}}$

We assume our real calculation is only "valid" for $z > \delta$, with $\delta \rightarrow 0$. Therefore

$$\frac{\sigma_{\text{tot}}}{\sigma_0} = \int_{\delta}^{1/3} \frac{d\sigma_{\text{real}}}{dz} + \int_0^{\delta} \frac{d\sigma_{\text{virt}}}{dz}$$

an effective virtual correction

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{5}{2} - \frac{\pi^2}{3} + 3 \log[\delta] + 2 \log^2[\delta] \right) + \Delta_{\text{virt}}$$

$$= 1 + \frac{3}{2} \frac{\alpha_s}{2\pi} C_F \quad * \quad \Delta_{\text{real}} + \Delta_{\text{virt}} \approx 1$$

this feature is useful for deriving Resummation

$$\Rightarrow \frac{\delta_{\text{virt}}}{\delta_0} = 1 + \frac{\alpha_s}{2\pi} C_F \left(-1 + \frac{\pi^2}{3} - 3 \log[\delta] - 2 \log^2[\delta] \right)$$

Hence the τ -distribution covers $\tau=0$ is given by

$$\frac{1}{\delta_0} \frac{d\delta}{d\tau} = \frac{1}{\delta_0} \left\{ \lim_{\delta \rightarrow 0} \frac{d\delta_{\text{real}}}{d\tau} \theta(\tau-\delta) + \frac{\delta_{\text{virt}}}{\delta} \theta(\delta-\tau) \right\}$$

$$= \lim_{\delta \rightarrow 0} \frac{1}{\delta_0} \frac{d\delta_{\text{real}}}{d\tau} \theta(\tau-\delta) - \frac{1}{\delta} \theta(\delta-\tau) \left(3 \log[\delta] + 2 \log^2[\delta] \right) \frac{\alpha_s}{2\pi} C_F$$

$$\left[1 + \frac{\alpha_s}{2\pi} C_F \left(-1 + \frac{\pi^2}{3} \right) \right] \delta(\tau)$$

now we introduce the "+"-prescription

$$\left(\frac{\log^n z}{z} \right)_+ = \lim_{\delta \rightarrow 0} \theta(\tau-\delta) \frac{\log^n z}{z} + \frac{1}{n+1} \log^n(\delta) \theta(\delta-\tau)$$

You can show that it is equivalent to what you are familiar with

$$\int_0^1 \left(\frac{\log z}{z} \right)_+ f(z) dz = \int_0^1 \frac{\log z}{z} (f(z) - f_0) dz$$

by noting that

$$\delta_{\text{real}} + \delta_{\text{virt}} = \delta_{\text{tot}}$$

$$\Rightarrow \delta_{\text{virt}} = \delta_{\text{tot}} \delta(1-z) - \int_0^1 \frac{d\delta_{\text{real}}}{dz} dz \delta(1-z)$$

$$= \delta_{\text{tot}} \delta(1-z) - \int_0^1 \frac{d\delta_{\text{real}}^{\text{reg.}}}{dz} \delta(1-z) - \int_0^1 \frac{d\delta_{\text{real}}^{\text{sing.}}}{dz} \delta(1-z)$$

$\underbrace{\hspace{10em}}_{\text{Finite } z \rightarrow 0}$
 $\underbrace{\hspace{10em}}_{\text{Div. as } z \rightarrow 0}$

$\underbrace{\hspace{10em}}_{\text{Combine with } \frac{d\delta_{\text{real}}}{dz}}$

$$\frac{d\delta_{\text{real}}}{dz} - \int_0^1 \frac{d\delta_{\text{real}}^{\text{sing.}}}{dz} \delta(1-z) \rightarrow \text{plus prescription}$$

eventually we find

$$\frac{1}{b_0} \frac{db}{dz} = \delta(z) + \frac{\alpha_s C_F}{2\pi} \left\{ -\frac{4}{1-z} \left(\frac{\log z}{z} \right)_+ - 3(1+z)(1-3z) \left(\frac{1}{z} \right)_+ \right. \\ \left. + (-1 + \pi^2/3) \delta(z) \right. \\ \left. + \frac{4}{1-z} \frac{\log(1-z)}{z} - 6 \log\left(\frac{1-z}{z}\right) \right\}$$

The result is equivalent to keep ε in real
do the calc, Laurent expand in ε , + virt,

We can thus have the cumulant cross-section

$$\frac{\Delta(\tau < \delta)}{\Delta_0} = 1 + \frac{\alpha_S}{2\pi} C_F$$

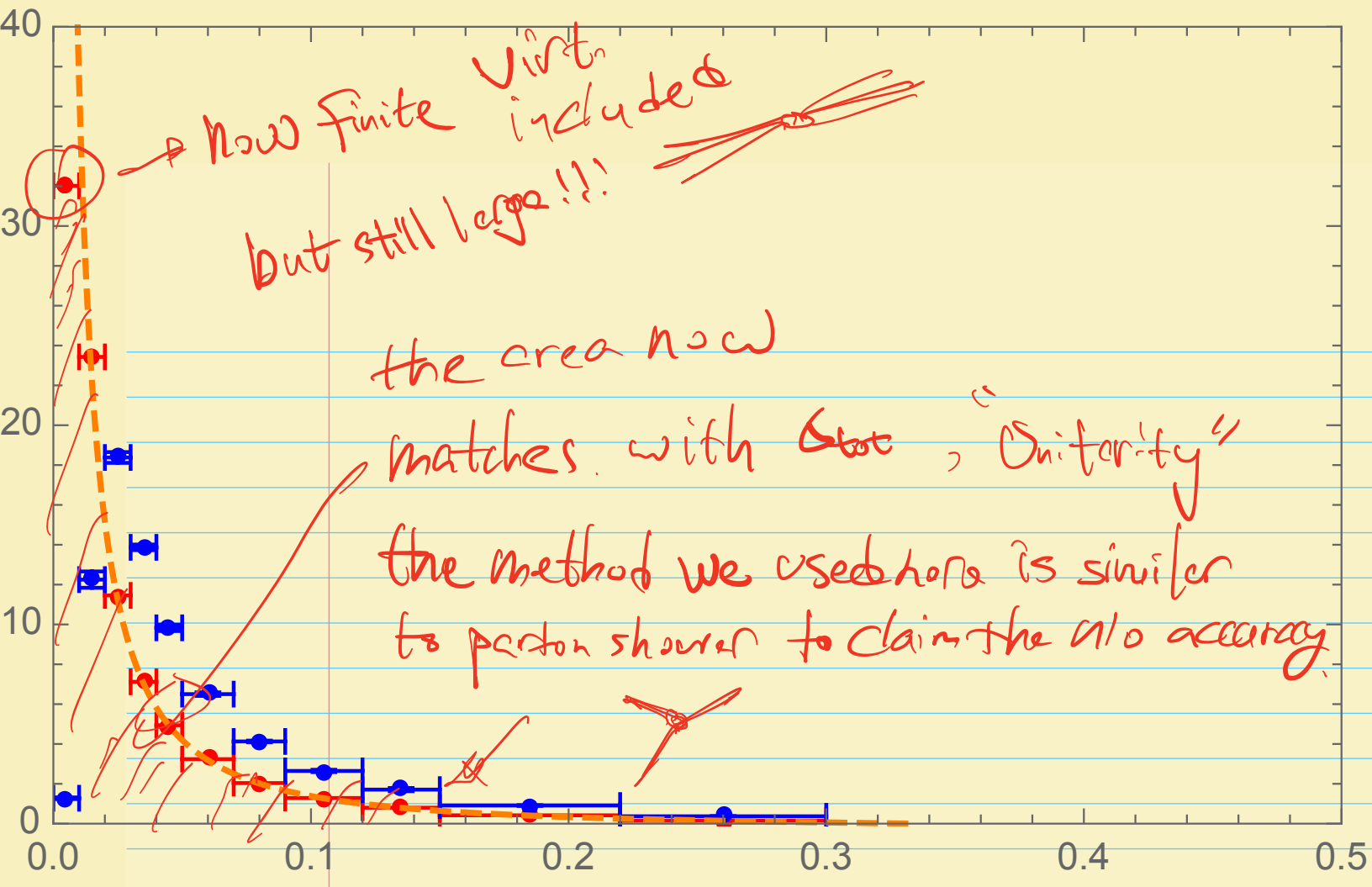
$$\left\{ -2L^2 - 3L - 1 - 4\text{Li}_2(-1+2\delta) \right.$$

$$+ 6\delta \left[\log\left(\frac{\delta}{1-2\delta}\right) + 1 \right] + \frac{9}{2}\delta^2 + 3\log(1-2\delta)$$

$$- 4 \left[\log(1-2\delta) \log(1-\delta) - \log(1-\delta) \log \delta \right.$$

$$\left. - \text{Li}_2(\delta) + \text{Li}_2(2\delta) \right\}$$

$$\text{for } \delta \leq \frac{1}{3}, \quad L \equiv \log \delta$$



• in the " $\tau=0$ "-bin, there is an exact cancellation between real & Virt. poles

$$\frac{1}{\sigma_0} \frac{d\sigma_{\text{virt}}}{d\tau} = \frac{\alpha_S}{2\pi} C_F \left(\frac{4\pi\mu^2}{s} \right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2 \right) \delta(1-\tau)$$

$$\frac{1}{\sigma_0} \frac{d\sigma_{\text{real}}}{d\tau} = \frac{\alpha_S}{2\pi} C_F \left(\frac{4\pi\mu^2}{s} \right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \dots \right) \delta(1-\tau) + \dots$$

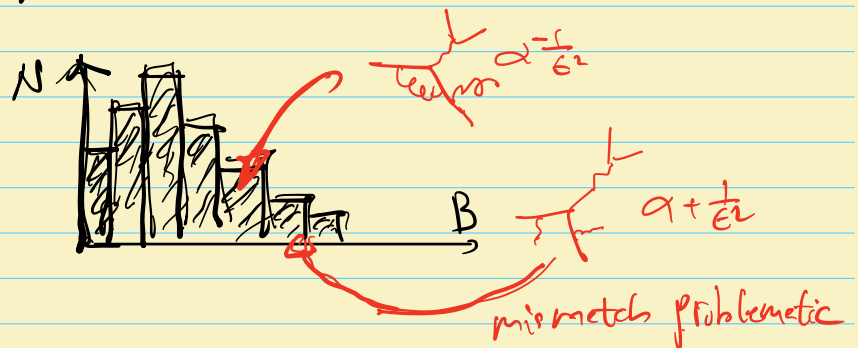
The cancellation ensures the predictive power of a F.O. calculation.

Not all observable has this feature !!)

recall that

- all poles come from the soft & coll. limit of the emitted partons,
- virt correction does not change multiplicities

⇒ to cancel poles, soft & coll. of real should fall into the same bin of virt.



IRC safe observables:

$$\mathcal{O}_N(p_1, p_2, p_3, \dots, p_i, \dots, p_n) \xrightarrow{p_i \rightarrow 0} \mathcal{O}_{N-1}(p_1, p_2, p_3, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$$

$$\mathcal{O}_N(p_1, p_2, p_3, \dots, p_i, \dots, p_j, \dots, p_n) \xrightarrow{p_i \parallel p_j} \mathcal{O}_{N-1}(p_1, p_2, \dots, p_i, p_j, \dots, p_{i+j-1}, p_{i+j+1}, \dots, p_n)$$

You can check that thrust is IR safe.

IRC unsafe:

e.g. particle numbers

$$\mathcal{O}_N(p_1 \dots p_N) = N \xrightarrow{p_i \rightarrow 0} N \neq \mathcal{O}_{N-1}(p_1 \dots p_N) = N-1$$

We need IRC safe observables to make
F.O. predictive.

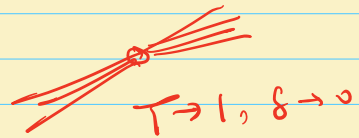
② L indicates the incomplete cancellation between real. & virt.

as $\delta \rightarrow 1/3$, $G^{(1)} \rightarrow \frac{\alpha_s}{2\pi} G_F^{3/2}$, "Complete"

as $\delta \rightarrow 0$, $L \rightarrow \infty$, recovers the divergence

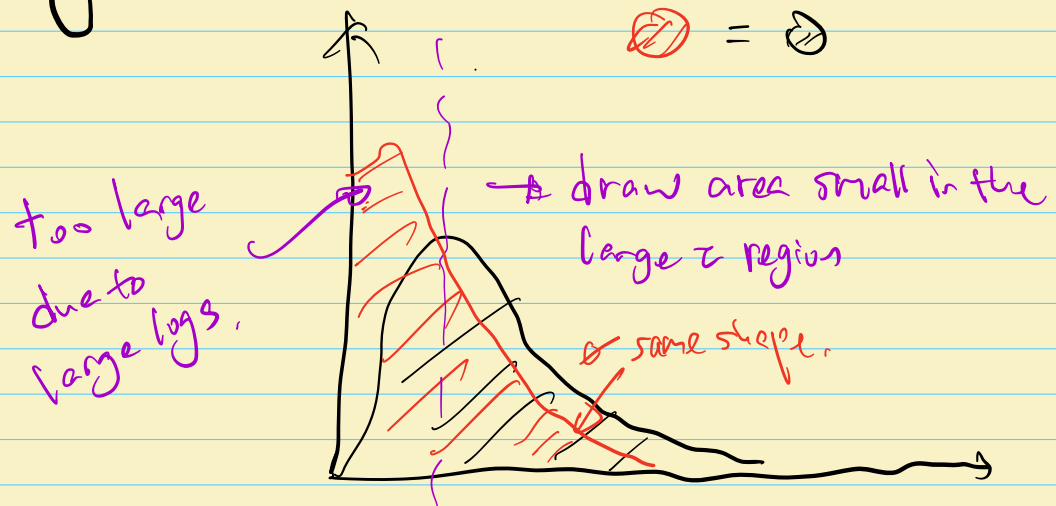


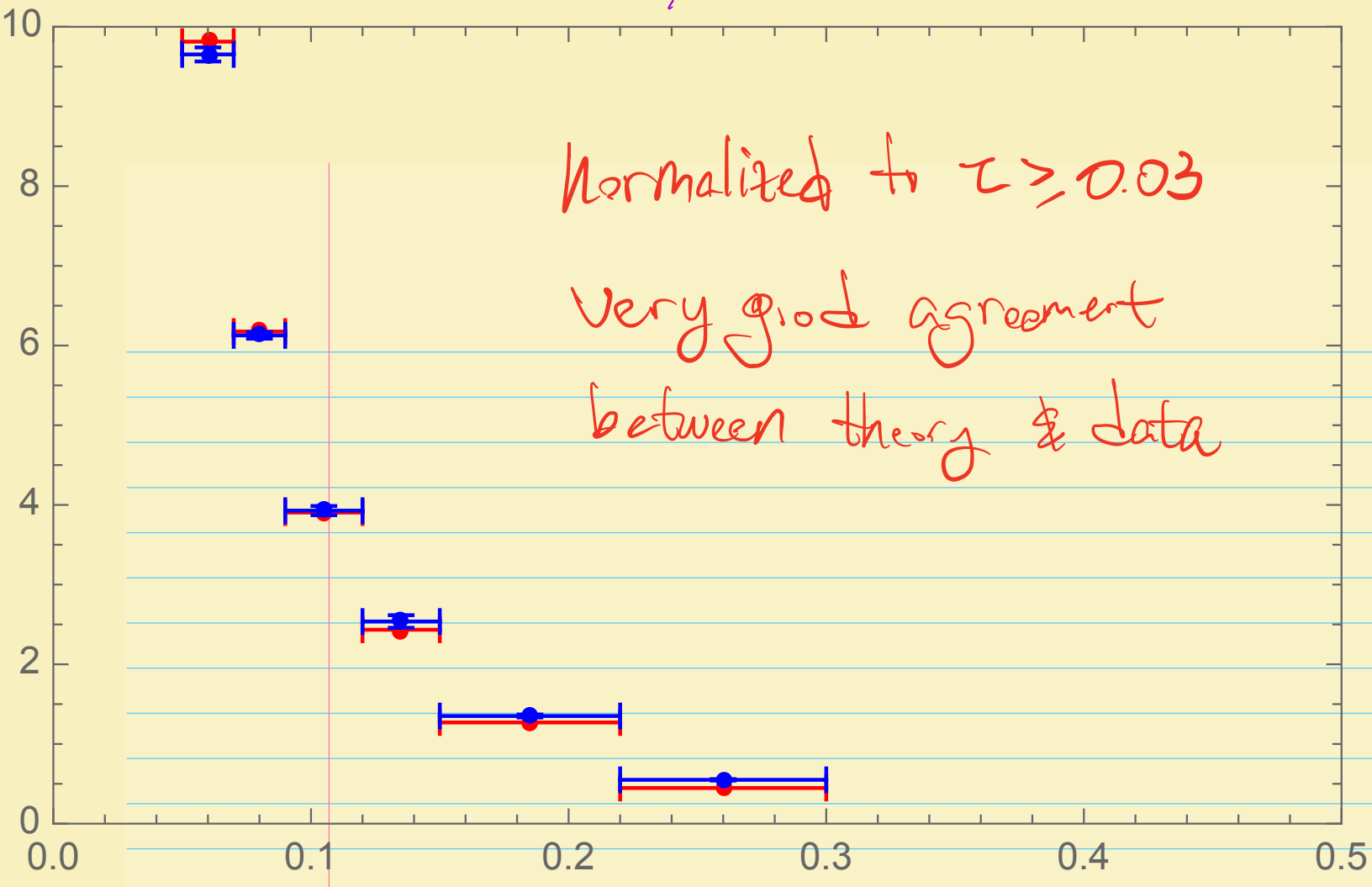
recall only soft and coll. radiations are allowed



Break down of F.O.

The entire $\frac{1}{\epsilon} \frac{d\epsilon}{dz}$ spectrum does not agree with data, all due to large logs in the small z region





Now we focus on the $\delta \ll 1$ region,

When $\delta \ll 1$, only soft & coll. radiations are allowed, and we have

$$\frac{\sigma(\delta)}{\sigma_0} = 1 - \frac{\alpha_S}{2\pi} C_F \left(\frac{4}{2} L^2 + 3 \cdot L + 1 - \frac{\pi^2}{3} \right)$$

recall the poles

$$\frac{\sigma_{\text{rel}}}{\sigma_0} \sim \frac{\alpha_S}{2\pi} C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} \dots \right)$$

soft & collinear nature.

Indeed, recall $\ln = \int \frac{dx}{x}$

Hence, the logs in

$$\sigma_{\text{real}} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_{\text{E}}}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s} \right)^{-\epsilon}$$

$$\times \int_0^1 dy_1 dy_2 dy_3 (y_1 y_2 y_3)^{-\epsilon} \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$\mathcal{O}(\epsilon^3)$ ↗ double log

↖ single log

comes from $y_2 \rightarrow 0, y_3 \rightarrow 0$,

but $y_1 \rightarrow 0$ does not generate any
logs

Pictureially $\Rightarrow$

$$y_2 \rightarrow \text{---} \bullet \text{---} \text{---} + \text{soft } g \Rightarrow \tau = y_2$$

$$\text{real } \tau = \min(y_i)$$

$$y_3 \rightarrow \text{---} \text{---} \bullet \text{---} + \text{soft } g \Rightarrow \tau = y_3$$

$$y_1 \rightarrow \text{---} \text{---} \text{---} \bullet \text{---} + \text{soft quark} \Rightarrow \tau = y_1$$

In this limit $\Rightarrow$

$$\sigma_{\text{real}} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_{\text{E}}}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s} \right)^{-\epsilon}$$

$$\times \int_0^1 dy_1 dy_2 dy_3 (y_1 y_2 y_3)^{-\epsilon} \delta(1 - y_1 - y_2 - y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2 + y_3 - \epsilon y_3}{y_2} + \frac{-2 + y_2 - \epsilon y_2}{y_3} - 2\epsilon \right\}$$

if keep only double log.

$$\Theta(\Phi_3)$$

$$\hookrightarrow [\delta(\tau - y_2) \Theta(\tau < y_3) + y_2 \leftrightarrow y_3 + \log \text{ suppressed}]$$

$$G_{\text{soft}}^{\text{coll}} = -b_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{2 \times 2}{\epsilon} \log \epsilon + \log \text{Suppressed}$$

$$\rightarrow -b_0 \frac{\alpha_s}{2\pi} C_F + \left(\frac{\log \epsilon}{2} \right)_+ \text{ by "Unitarity"}$$

$$\text{cumulative} \rightarrow -b_0 \frac{\alpha_s}{2\pi} C_F \cdot 2 \ln^2 \delta$$

one power from soft
one power from coll.

s-soft-collinear limit

reproduce the leading log structure

To all orders, we have structures

$$\Delta \sim 1$$

$$\text{NLO} \rightarrow \alpha_s L^2 + \alpha_s L + \#$$

$$\text{NNLO} \rightarrow \alpha_s^2 L^4 + \alpha_s^2 L^3 + \alpha_s^2 L^2 + \alpha_s^2 L + \#$$

$$\text{N}^3\text{LO} \rightarrow \alpha_s^3 L^6 + \alpha_s^3 L^5 + \alpha_s^3 L^4 + \alpha_s^3 L^3 + \alpha_s^3 L^2 + \dots$$

$$\vdots$$

When $\alpha_s L^2 \sim 1$, we can not truncate the series at fixed α_s

Since the $\alpha_s^n L^{2n}$ for $n > 1$ are equally important, **We need to resum $\alpha_s L^2$ to all orders.**

$$\Delta \sim 1$$

$$NLO \rightarrow \alpha_s L^2 + \alpha_s L + \#$$

$$NNLO \rightarrow \alpha_s^2 L^4 + \alpha_s^2 L^3 + \alpha_s^2 L^2 + \alpha_s^2 L + \#$$

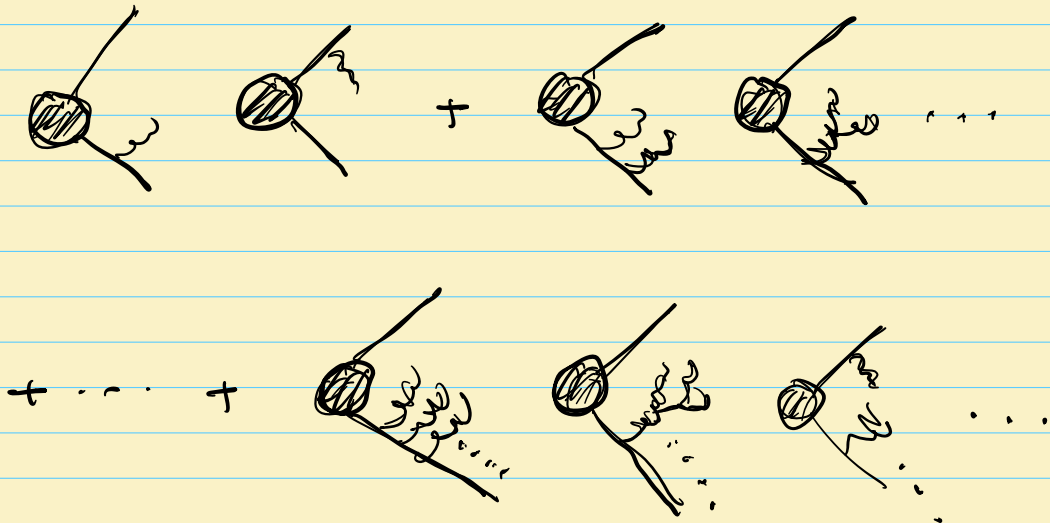
$$N^3LO \rightarrow \alpha_s^3 L^6 + \alpha_s^3 L^5 + \alpha_s^3 L^4 + \alpha_s^3 L^3 + \alpha_s^3 L^2 + \dots$$

⋮
⋮
⋮

**dominant
logs, DL**

Or equivalently.

We sum up the most singular behavior
of the soft & coll. radiations to
all orders



There are different Approaches to Resummation

• CSS Formalism

Collins, Soper, Sterman *Nucl. Phys. B* 250 (1985) 199-224
hep-ph/0409313 (1999)

• Coherent branching algorithm

Catani et al. *nucl. Phys. B* 327 (1989) 323-352
nucl. Phys. B 407 (1993) 3-42

• Soft collinear effective Theory

Bauer et al. hep-ph/0202088, hep-ph/0109045
hep-ph/0107001, hep-ph/0011316
Beneke et al. hep-ph/0206152

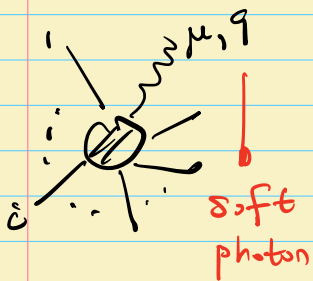
• CAESAR

Banfi et al. hep-ph/0112156

All Approaches Now have the ability to go beyond NLL.

• Infrared behaviour in QCD

→ a lesson from QED $\text{gluon} \rightarrow \text{photon}$



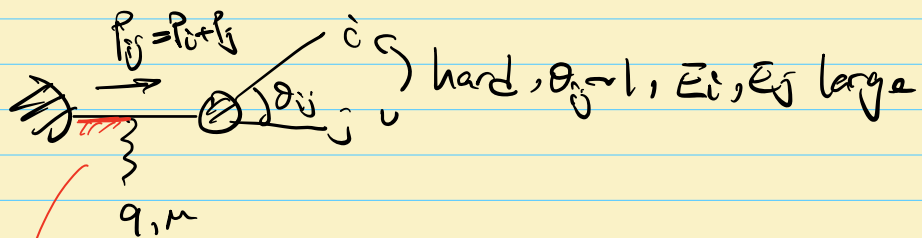
photon phase space:

$$\frac{d^3 q}{2q^0 (2\pi)^3} = \frac{q^2 dq d\Omega}{2q (2\pi)^3} = \boxed{\frac{1}{2(2\pi)^2} q dq d\cos\theta}$$

we are interested in terms $\propto \frac{1}{q^2}$ in the matrix element square.

• possible single emission

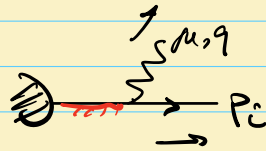
1. photon from internal line



$$\propto \frac{1}{(P_{ij} + q)^2} = \frac{1}{2P_{ij} \cdot q + P_{ij}^2} \xrightarrow{q \rightarrow 0} \frac{1}{P_{ij}^2}$$

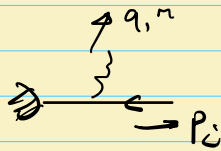
regular as $q \rightarrow 0$

2. photon from external line



$$\begin{aligned}
 & \bar{u}(p_i) (-ie \gamma^\mu) i \frac{\not{p}_i + \not{q}}{2p_i \cdot q} \dots \\
 &= \bar{u}(p_i) e \frac{\gamma^\mu \not{p}_i + \not{p}_i \gamma^\mu}{2p_i \cdot q} \dots \\
 &= \bar{u}(p_i) e \frac{p_i^\mu}{p_i \cdot q} \dots
 \end{aligned}$$

No $\cancel{q}$ enhancement
 $\gamma^\mu \not{p}_i + \not{p}_i \gamma^\mu = 2p_i^\mu$
 will give $\cancel{q}$ enhancement we need



$$\begin{aligned}
 & \bar{u}(p_i) (-ie) \gamma^\mu i \frac{-\not{p}_i}{2p_i \cdot q} \dots \\
 &= \bar{u}(p_i) (-e) \frac{p_i^\mu}{p_i \cdot q} \dots
 \end{aligned}$$

absorb "-" sign into the electric charge.

$$\Rightarrow \text{Diagram with } q^\mu \rightarrow \sum_i e_i \frac{p_i^\mu}{p_i \cdot q} \times \text{Diagram}$$

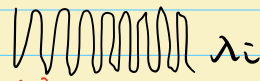
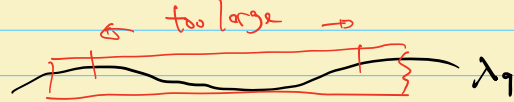
$$- q^\mu \mathcal{M}_\mu = \sum_i e_i \frac{q \cdot p_i}{p_i \cdot q} = \sum_i e_i = 0$$

charge conservation, gauge invariance.

- factorized s-ft current $J_i^\mu = e_i \frac{p_i^\mu}{p_i \cdot q}$
 independent of spin, Low theorem.

- a factorization of the long & short distance physics.

$$\frac{1}{q} \sim \lambda_q \gg \frac{1}{E} \sim \lambda_i$$



Can not resolve the internal structure

Q: How about radiate a soft fermion?

→ Soft leads to logs?

Now we square the Matrix element

$$\left| \text{diagram with a crossed circle} \right|^2 \rightarrow - \sum_{i,j} e_i e_j \frac{p_i \cdot p_j}{p_i \cdot q p_j \cdot q} |\mathcal{M}_0(\{p_i\})|^2$$

for single emission

$$\text{define } d\omega(q) = -d\Phi \sum_{i,j} e_i e_j \frac{p_i \cdot p_j}{p_i \cdot q p_j \cdot q}$$

$$= - \sum_{i,j} \frac{e_i e_j}{8\pi^2} \frac{p_i \cdot p_j}{p_i \cdot q p_j \cdot q} \cdot q dq d\cos\theta$$

$$\frac{1}{2(2\pi)^2} q dq d\cos\theta$$

Single photon emission prob.

Suppose we do a measurement ν on the photon. that imposes the phase space restriction $\Theta_1(\nu)$ the prob. for the photon emission is

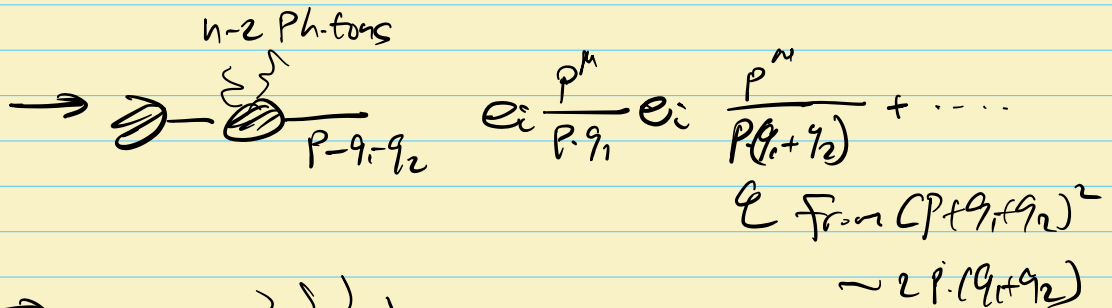
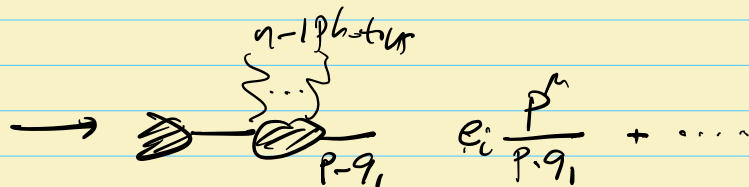
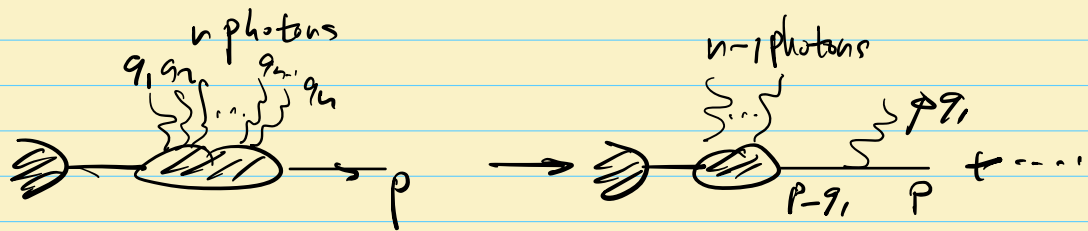
$$W_1(\nu)$$

$$= \int d\nu \left\{ \underbrace{\frac{dW_1(\nu)}{d\nu}}_{\text{real}} \Theta_1(\nu) - \underbrace{\int d\nu \frac{dW_1(\nu)}{d\nu}}_{\text{virt.}} S(\nu) \right\}$$

Since $\int d\nu (\text{virt.} + \text{real}) = 1 + \text{small corrections}$

$$\Rightarrow \boxed{W_1(\nu) = \int d\nu [\Theta_1(\nu) - 1] \frac{dW_1}{d\nu}}$$

Multiple emissions :



$$= \frac{1}{n!} \sum_{\text{All permutations}} e_i \frac{p^\mu}{p \cdot q_1} e_i \frac{p^\mu}{p \cdot (q_1 + q_2)} \dots e_i \frac{p^\mu}{p \cdot (q_1 + \dots + q_n)}$$

$$= \frac{1}{n!} \prod_{j=1}^n e_i \frac{p^\mu}{p \cdot q_j} \rightarrow$$

$$dW_n = \frac{1}{n!} \prod_{j=1}^n dW_j$$

"independent" emission

$$\Rightarrow \sum_n \frac{1}{n!} \int \prod_i d\omega_i |M(\{p_i\})|^2$$


- We note that the emission probability also depends on the phase space restriction which is not necessarily independent

$$\Theta_n(\omega) = \Theta_n(\nu(\{q_i\}))$$

- = if the phase space restriction is also independent then

$$\sigma \rightarrow \sigma_0 \exp[W_1(\nu)]$$

where

w_1 defined before including the "virtual".

$$W_1(\nu) = \int d\nu \left\{ \Theta_1(\nu) - 1 \right\} \frac{1}{1+\nu}$$

you get the resummed form in QED for σ .

$$* \Theta_c(\infty) = 1 \Rightarrow \text{Inclusive} \Rightarrow W_c(0) = 0$$

$$\Rightarrow 0 \rightarrow 0_0, \text{ Unitarity,}$$

$$* \Theta_c(0) = 0 \Rightarrow \text{No emission}$$

$\Rightarrow$

Sudakov Factor

$$\Delta \rightarrow 0_0 \exp \left[- \int dv \frac{dw(q)}{dv} \right] \rightarrow 0$$

$\int_{-\infty}^{\infty} \text{Virtual only}$

$$\Delta(\Theta_{\max}, 0)$$

prob. of no emission below $\Theta_{\max}$

* 2 photons,

$$\frac{P_i^n}{P_i \cdot q_1} \frac{P_i^n}{P_i \cdot (q_1 + q_2)} + \frac{P_i^n}{P_i \cdot q_2} \frac{P_i^n}{P_i \cdot (q_1 + q_2)}$$

$$= \tilde{p}_i^m \tilde{p}_i^r \frac{\tilde{p}_i^r q_2 + \tilde{p}_i^r q_1}{\tilde{p}_i^r q_1 \tilde{p}_i^r q_2 \tilde{p}_i^r (q_1 + q_2)} = \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_1} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_2} \checkmark$$

$$\text{suppose } \sum_{\text{perm.}} \frac{\tilde{p}_i^r}{\tilde{p}_i^r q_1} \frac{\tilde{p}_i^r}{\tilde{p}_i^r (q_1 + q_2)} + \dots \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_{n-1})}$$

$$= \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_1} \dots \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_{n-1}}$$

$$\text{then } \sum_{\text{perm}} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_1} \dots \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + q_2)} \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_n)}$$

$$= \sum_{\text{perm } j \neq n} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j} \dots \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_n)}$$

$$+ \sum_{\text{perm } j \neq 1} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_2} \dots \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + q_n)} \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_n)}$$

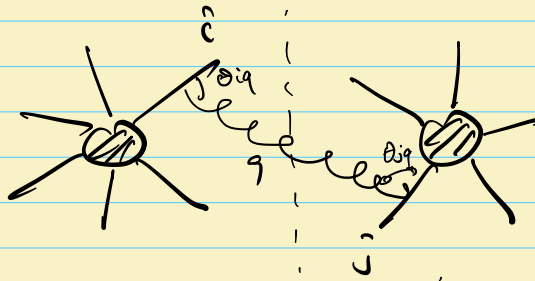
⋮

$$= \left\{ \sum_{j \neq 1} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j} + \sum_{j \neq 2} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j} + \dots + \sum_{j \neq n} \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j} \right\} \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_n)}$$

$$= \sum_j \left(\frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j} (\tilde{p}_i^r q_1 + \tilde{p}_i^r q_2 + \dots + \tilde{p}_i^r q_n) \right) \frac{\tilde{p}_i^m}{\tilde{p}_i^r (q_1 + \dots + q_n)}$$

$$= \sum_j \frac{\tilde{p}_i^m}{\tilde{p}_i^r q_j}$$

QED coherence



$$\propto - \sum_{i \neq j} e_i e_j \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q}$$

suppose θ_{pk} is the smallest among all angles.

$$= - \sum_{i \neq j} e_i e_j \frac{2 p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q}$$

$$= -e_k \sum_{j \neq k} e_j \frac{2 p_k \cdot p_j}{E_k E_j (1 - \cos \theta_{kj})} \frac{1}{E_j E_q (1 - \cos \theta_{jq})} + \text{less singular}$$

$$= -\frac{e_k}{E_q} \frac{2}{1 - \cos \theta_{qk}} \sum_{j \neq k} e_j \frac{p_k \cdot p_j}{E_j E_k (1 - \cos \theta_{kj})} + \text{less singular}$$

$$\sim -\frac{2 e_k}{E_q^2} \frac{1}{1 - \cos \theta_{qk}} \sum_{j \neq k} e_j \frac{E_k E_j (1 - \cos \theta_{kj})}{E_k E_j (1 - \cos \theta_{kj})} + \text{less singular}$$

$-e_k$ Since charge cons.

$$\propto \frac{2e_K^2}{E_q^2} \frac{1}{1 - \cos \theta_{qK}} \quad (*)$$

Here we used the coll. limit to derive the result but it can be derived as long as θ_{qK} is the smallest angle, no need for $\theta_{qK} \rightarrow 0$. See below.

Now we integrate over the photon phase space to find

$$d\omega_i(q) = \frac{\alpha}{\pi} \underbrace{\sum_k e_k^2 \frac{dq}{q}}_{\text{from a single independent emitter}} \underbrace{\frac{d\theta_{qK}^2}{\theta_{qK}^2} \Theta(\theta_{\max} - \theta_{qK})}_{\text{QED coherence due to destructive interference}}$$

from a single independent emitter

QED coherence due to destructive interference

Here $\theta_{\max} = \min \{ \theta_{ij} \}$



$$\begin{array}{c}
 \text{Diagram 1: } \text{photon} \rightarrow e^- \rightarrow e^- \text{ (with } \vec{e} \cdot \hat{j} \text{)} \\
 \text{Diagram 2: } \text{photon} \rightarrow e^+ \rightarrow e^+ \text{ (with } \vec{e} \cdot \hat{j} \text{)}
 \end{array}
 \quad \theta_{ec} \ll \theta$$

$$p_j \parallel p_c$$

$\Rightarrow$  see the overall charge is 0
 does not carry charge $\nearrow$ hence the wide angle emission is suppressed

$$(-e) \frac{p_j^m}{p_j \cdot q} + e \frac{p_c^m}{p_c \cdot q} = (-e + e) \frac{p^m}{p \cdot q} = 0$$

Therefore we find

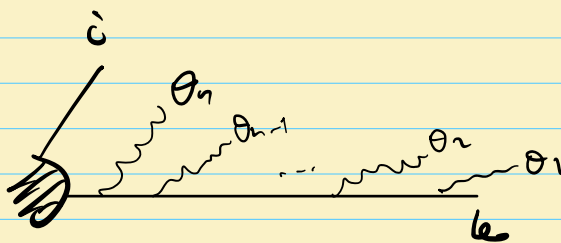
$$\sigma = \sigma_0 \exp \left[\int d\nu [\theta_i(\nu) - 1] \frac{dW_i}{d\nu} \right]$$

$$= \sigma_0 \prod_k \exp \left\{ \frac{\alpha}{\pi} e_k^2 \int d\nu \int \frac{dq}{q} \int \frac{d\theta_{qk}^2}{\theta_{qk}^2} [\theta_i(\nu) - 1] \right\}$$

Shows each leg

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \frac{1}{n!} \prod_n \frac{\alpha^n}{\pi^n} \left(e_k^2 \int \frac{dq}{q} [\theta_i(\nu) - 1] \right)^n \int \frac{d\theta_1^2}{\theta_1^2} \int \frac{d\theta_2^2}{\theta_2^2} \dots \int \frac{d\theta_n^2}{\theta_n^2}$$

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \prod_n \frac{\alpha^n}{\pi^n} \left(e_k^2 \int \frac{dq}{q} [\theta_i(\nu) - 1] \right)^n \underbrace{\int \frac{d\theta_1^2}{\theta_1^2} \int \frac{d\theta_2^2}{\theta_2^2} \dots \int \frac{d\theta_n^2}{\theta_n^2}}_{\text{angular order}}$$



angular ordering to
account for the interference

Very useful
for Monte Carlo
event-generator.

~~*~~

$$\frac{P_i \cdot P_j}{P_i \cdot q \cdot P_j \cdot q} = S_{ij}^{(1)} + S_{ij}^{(2)}$$

$$S_{ij}^{(1)} = \frac{1}{2} \left[\frac{P_i \cdot P_j}{P_i \cdot q \cdot P_j \cdot q} + \frac{1}{E} \frac{1}{1 - \cos \theta_{iq}} - \frac{1}{E^2} \frac{1}{1 - \cos \theta_{jq}} \right]$$

where

$$E_q (1, \sin \theta_{iq} \cos \phi_{iq}, \sin \theta_{iq} \sin \phi_{iq}, \cos \theta_{iq})$$

$$E_j (1, \sin \theta_{ij}, 0, \cos \theta_{ij})$$

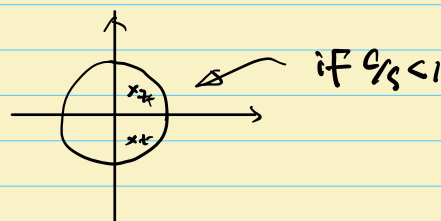
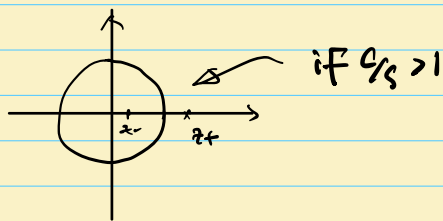
$$1 - \cos \theta_{jq} = \underbrace{1 - \cos \theta_{iq} \cos \theta_{ij}}_C - \underbrace{\sin \theta_{iq} \sin \theta_{ij} \cos \phi_{iq}}_S$$

Therefore

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1 - \cos \theta_{jq}} = \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{C - S \cos \phi_{iq}} \quad \text{let } z = e^{i\phi_{iq}}$$

$$= \frac{-i}{2\pi} \oint_{|z|=1} \frac{dz}{z} \frac{1}{C - S(z + 1/z)} = \frac{i}{2\pi} \oint_{|z|=1} dz \frac{1}{Sz^2 - Cz + S}$$

$$= \frac{i}{2\pi S} \oint_{|z|=1} dz \frac{1}{(z - z_+)(z - z_-)} \quad \text{where } z_{\pm} = \frac{1}{2} \frac{C}{S} \pm \frac{1}{2} \sqrt{\frac{C^2}{S^2} - 1}$$



$$\Rightarrow \text{result} = -\frac{1}{s} \frac{1}{z - z_r}$$

$$= \frac{1}{\sqrt{c^2 - s^2}}$$

$$\Rightarrow \text{result} = 0$$

Therefore

$$\int \frac{d\theta_{ij}}{2\pi} S_{ij}^{(c)} = \frac{1}{2} \frac{1}{c^2} \int \frac{d\theta_{ij}}{2\pi} \left\{ \frac{1 - \cos \theta_{ij}}{(1 - \cos \theta_{ij})(1 - \cos \theta_{iq})} + \frac{1}{1 - \cos \theta_{iq}} - \frac{1}{1 - \cos \theta_{ij}} \right\}$$

$$= \frac{1}{2} \frac{1}{c^2} \int \left\{ \left[\frac{1 - \cos \theta_{ij}}{1 - \cos \theta_{iq}} - 1 \right] \frac{1}{\sqrt{c^2 - s^2}} + \frac{1}{1 - \cos \theta_{iq}} \right\}$$

$$= \frac{1}{2c^2} \left\{ \frac{\cos \theta_{iq} - \cos \theta_{ij}}{1 - \cos \theta_{iq}} \frac{1}{|\cos \theta_{iq} - \cos \theta_{ij}|} + \frac{1}{1 - \cos \theta_{iq}} \right\}$$

$$= \begin{cases} \frac{1}{2c^2} \frac{1}{1 - \cos \theta_{iq}} & \theta_{iq} < \theta_{ij} \\ 0 & \theta_{iq} > \theta_{ij} \end{cases}$$

$z_{p,0}$
 $\frac{z_{p,0}^2}{z_p} \sim z_{p,0}$
 $\frac{1}{z_{p,0}}$
 $\frac{1}{z_p} \sim \frac{1}{z_{p,0}}$

$\tau \theta^{np}$ θ
 $\theta_{ee} > 0$

• summarize for $\Delta E D$

$$W_1 = -4\pi\alpha \sum_{\vec{e}_j} e_j e_j \int \frac{p_i \cdot p_j}{p_i \cdot q p_j \cdot q} (\Theta_i(u) - 1) dv d\vec{e}_q$$

q virtual included

• if phase space factorize, then

$$dG = dG_0 \exp[W_1]$$

• Interference well approximated by time ordering

$$W_1 \rightarrow \frac{\alpha}{\pi} \sum_{\vec{k}} e_k^2 \int \frac{d\vec{q}}{q} \frac{d\omega}{\omega^2} \Theta(\omega_{max} - \omega_{qh}) (\Theta_i(u) - 1) dv$$

and

$$dG = dG_0 \exp[W_1]$$

$$= dG \prod_{k=1}^n \prod_{n=0}^{\infty} \prod_n \left[\frac{\alpha}{\pi} e_k^2 \int \frac{d\vec{q}}{q} (\Theta_i(u) - 1) dv \right]^n \underbrace{\int_{\omega_{min}}^{\omega_{max}} \frac{d\omega_n^2}{\omega_n^2} \int_{\omega_{min}}^{\omega_n} \frac{d\omega_{n-1}^2}{\omega_{n-1}^2} \dots \frac{d\omega_1^2}{\omega_1^2}}_{\text{angular ordering}}$$

→ soft emissions in QCD.

• single emission

Similar to the QED soft photon emission, we have the soft gluon radiated off a quark line.

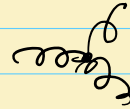
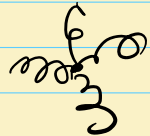
$$\text{Diagram: incoming quark line with a soft gluon emission } \mu, a, q \rightarrow = g_s t_{ij}^a \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j, \dots}$$

↪ with e_i replaced by the color charge, $g_s t^a$, a matrix,

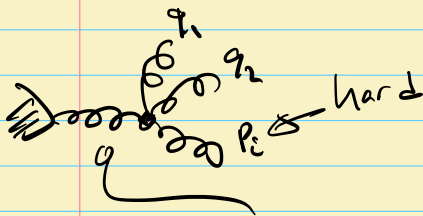
$$\text{Diagram: incoming anti-quark line with a soft gluon emission } \mu, a, q \rightarrow = g_s (-t_{ji}^a) \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j, \dots}$$

↪ From anti-quark

More than that, since now the gluon carries charge



soft gluon can be emitted from hard gluons.



$$\frac{g_s^2}{(p_i + q_1 + q_2)^2} = \frac{g_s^2}{2p_i \cdot (q_1 + q_2) + 2q_1 \cdot q_2} \sim \frac{g_s^2}{2p_i \cdot (q_1 + q_2)}$$

$\begin{matrix} \text{S} & \text{I} & \text{I} \\ \text{E} + \text{E}_2 & \text{E}_2 & \text{E}_2 \end{matrix}$

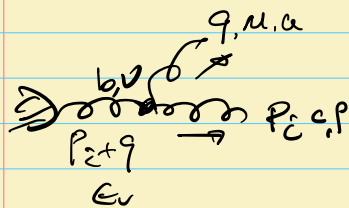
not singular

Since from phase space we have

$q_1 dq_1 q_2 dq_2$, which returns

suppressed results for either

$q_1 \rightarrow 0$ or $q_2 \rightarrow 0$



$$E(p_i + q) \xrightarrow{g_s f_c} E(p_i)$$

$$\begin{aligned} & \propto \frac{1}{2 p_i \cdot q} g_s f^{abc} \left\{ \cancel{(p_i + q)^\mu} g^{\mu\nu} - \cancel{(p_i + q)^\mu} g^{\nu\mu} + \cancel{(p_i - q)^\nu} g^{\mu\nu} \right\} \epsilon_\mu^\nu \mathcal{M}^b \\ & = -\frac{1}{2 p_i \cdot q} g_s f^{abc} 2 p_i^\mu \epsilon_\mu^\nu \mathcal{M}_\nu^{b\dots} \\ & = g_s f^{abc} \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_\mu^{b\dots} \end{aligned}$$

gauge non-singular gauge

C.F. Gluon from quark: $g_s t_{ij}^a \frac{p_i^\mu}{p_i \cdot q}$

- same eikonal factor, spin-independent
- with the color charge $g_s f^{abc} \equiv g_s f_c^a b$

We can Unify the soft approximation for both quark and gluon emission by introducing the color charge $\vec{T}_i^a$ where i is the i -th parton

$$\vec{T}_i^a \rightarrow t^a \text{ if } i \text{ is a quark } (\neq)$$

$$\rightarrow \bar{t}^a = -t^a \text{ if } i \text{ is an anti-quark}$$

$$\rightarrow f_{cb}^a \text{ if } i \text{ is a gluon.}$$

Color algebra: 

We note that $\sum_i \vec{T}_i^a = 0$ color charge conservation

$$\text{cf. } \sum_i e_i = 0 \text{ electric charge conservation}$$

and if we define $\vec{T}_i^2 = \sum_a \vec{T}_i^a \vec{T}_i^a$, then

$$\vec{T}_i^2 = \begin{cases} C_F & \text{For quark \& anti-quark} \\ C_A & \text{For gluon.} \end{cases}$$

(*)

A Feynman diagram showing a quark loop. Two external lines, labeled c_i and c_i' , enter and exit a loop. A gluon line, labeled $t_{c_i' c_i}^a$, is attached to the loop. The diagram is equated to the expression $= g_s t_{c_i' c_i}^a \mathcal{M}^{\dots c_i \dots}$.

$$= g_s t_{c_i' c_i}^a \mathcal{M}^{\dots c_i \dots}$$

Color operator acting on the color space, like a rotation

$$\equiv g_s t_{c_i' c_i}^a |\dots c_i \dots\rangle$$

$$= g_s T_{\bar{c}}^a |\dots c_i \dots\rangle$$

In this way

$$\text{Diagram: } \begin{array}{c} \dots j \\ \nearrow \\ \text{---} i \\ \searrow \\ e_{\text{can Le 9,0rg}} \end{array} \begin{array}{c} n, a, q \\ \text{---} \\ i \end{array} = g_r \sum_i \frac{\overline{T}_i^a}{T_i^a} \frac{p_i^n}{p_i^q} |u(f p_i)\rangle$$

color charge
operator acting
on $|\psi\rangle$

matrix element
becomes a vector in
the color space
of

QED is not truly factorized, there is color correlation remaining, since emitting gluon carries away color charge!!!

c.f. QED

$$\text{Diagram: a horizontal line with a wavy line attached to its left end} = e \tau_i e_i \frac{p_i}{p_i - q} \mu_0(\{p_i\})$$

Now square the matrix element, we find

$$\begin{array}{c} \text{diagram} \end{array} = - \sum_{\vec{c} \neq \vec{d}} g_s^2 \frac{P_{\vec{c} \vec{d}}}{P_{\vec{c}} g_{\vec{d}} g} \langle U_0 | T_{\vec{c}} T_{\vec{d}} | U_0 \rangle$$

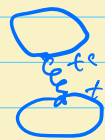
Not entirely Factorized
Color correlation depends
on the color flow of the
hard process:

not proportional to $\langle U_0 | U_0 \rangle = N_c^2$

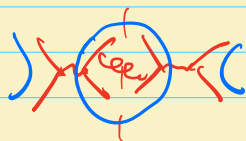
QCD:



Color Flow



$$= \text{Tr}[t^a] \text{Tr}[t^a] = 0$$

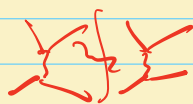


Color Flow

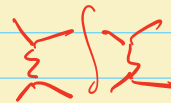


$$= \text{Tr}[t^a t^a] \text{Tr}[1]$$

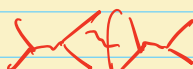
QED:



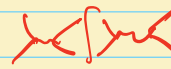
$$\sim e^2 \frac{P_{\vec{c} \vec{d}}}{P_{\vec{c}} g_{\vec{d}} g}$$



exactly same!!



$$\sim e^2 \frac{P_{\vec{c} \vec{d}}}{P_{\vec{c}} g_{\vec{d}} g}$$



multiple emissions

recall QED :

$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(\nu) = -4\pi^2 \sum_{i,j} e_i e_j \int \frac{p_i \cdot k}{p_i \cdot q \cdot q} (Q_1(\omega) - 1) d\nu d\Phi_q$$

direct exponentiation of 1-photo emission. (if the observable ν is factorizable), since in QED,

$$|M|^2 \sim |M_0|^2 e^{2 \frac{p_i \cdot k}{p_i \cdot q \cdot q}} \text{ entirely factorizable,}$$

How about QCD ?

Can we have

$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(\nu) = \int -4\pi^2 \sum_{i,j} \vec{t}_i \cdot \vec{t}_j \int \frac{p_i \cdot k}{p_i \cdot q \cdot q} (Q_1(\omega) - 1) d\nu d\Phi_q \quad ?$$

No. *cannot factorize!!*

- color coherence

similar to QED. the interference can be simplified in the leading IR limit by angular ordering

$$\begin{aligned}
 \text{Diagram: } \text{gluon emission from a quark line} &= -g_s^2 \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} \langle M_0 | \epsilon \cdot T_j | M_0 \rangle \\
 &\approx g_s^2 T_i^2 \frac{2}{E^2} \frac{1}{\theta_{iq}^2} \theta(\theta_{\max} - \theta_{iq}) |M|^2 \\
 &\quad \text{diagonal in the color space, C-A} \\
 &\quad \text{does not "change" color}
 \end{aligned}$$

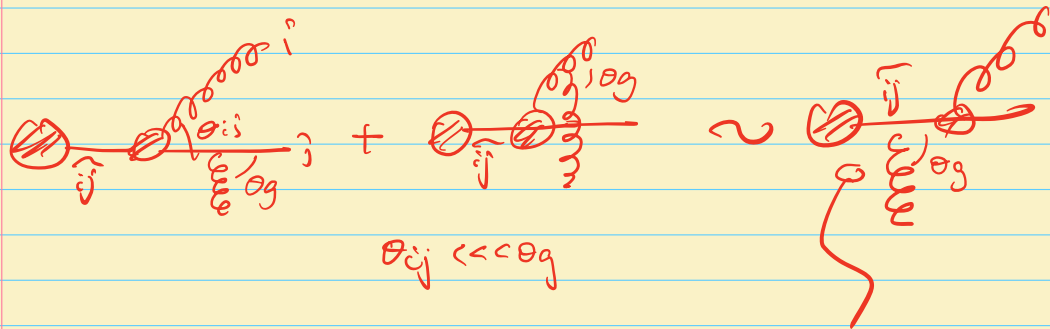
- QED-like Factorization

- angular ordering for interference

$$T_i^2 = \begin{cases} CF & \text{for } q \\ CA & \text{for } g \end{cases}$$

small angle radiation does not "change" the color charge.





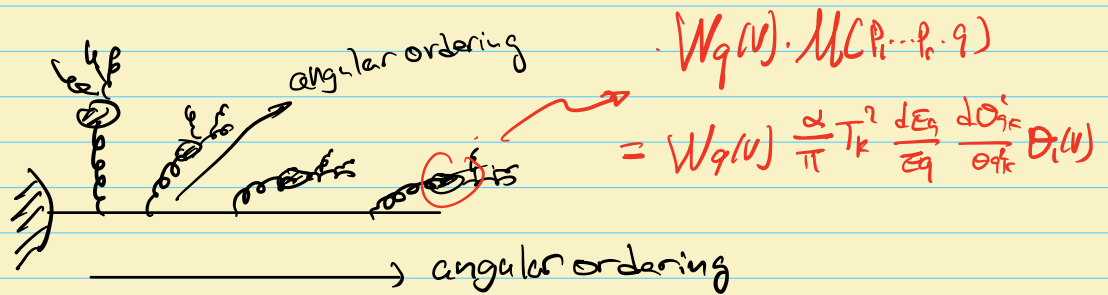
$$T_i \frac{p^u}{p \cdot q} + T_j \frac{p^u}{p \cdot q} = \{ \vec{T}_i + \vec{T}_j \} \frac{p^u}{p \cdot q} = \vec{T}_{ij} \frac{p^u}{p \cdot q}$$

as if radiated from the hard emitter $\tilde{qj}$

wide angle soft radiation only sees
the overall color charge of the partons
at smaller angles



QCD multiple
emission picture $\Rightarrow$
coherent branching



2CD

$$W_K(v) = 1 + \frac{1}{n!} \prod_{i=1}^n \left[\frac{\alpha_s}{\pi} T_K^2 \frac{dE_i}{E_i} \int_{\theta_{qK}^{\min}}^{\theta_{qK}^{\max}} \frac{d\theta_{qK}^2}{\theta_{qK}^2} \left\{ O_1(v) W_{q_i}(v) - 1 \right\} \right]^n$$

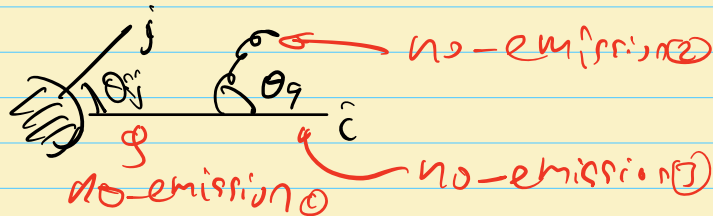
$$= \exp \left[\frac{\alpha_s}{\pi} T_K^2 \int \frac{dE_g}{E_g} \int_{\theta_{qK}^{\min}}^{\theta_{qK}^{\max}} \frac{d\theta_{qK}^2}{\theta_{qK}^2} \left\{ O_1(v) W_g(v) - 1 \right\} \right]$$

- the soft gluon as the new emitter.
- small angle $\theta_{qK} \sim \theta_{gK}$ now

$$W(v) = \prod_K W_K(v), \quad \delta = \delta_0 W(v)$$

- the fundament to parton shower

e.g. Prob. for one-emission



$$W_c(v=0, \theta_j, \theta_2) \underbrace{\frac{\alpha_s}{\pi} \frac{1}{k} \frac{dE}{E} \frac{d\theta_j^2}{d\theta_2^2}}_{\text{1-emission prob.}} W_g(v=0, \theta_1, 0) W_c(v=0, \theta_2, 0)$$

Sudakov ① ② ③

1-emission generated using Markov chain

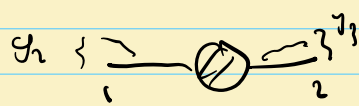
- can be improved to sub-leading logs

$$\frac{\alpha_s}{\pi} \frac{1}{k} \frac{dE}{E} \frac{d\theta^2}{\theta^2} \rightarrow \frac{\alpha_s}{2\pi} \frac{d\theta^2}{\theta^2} P_{\text{ang}}(z) dz$$

(splitting function)

- applied to analytic Resummation

DL resummation for the thrust.



we have two branchings, and we consider $y_2 \rightarrow 0$. $y_3 \rightarrow 0$ can be included in the same way

We first consider the 1-gluon emission

Recall that for the thrust $z \ll 1$

$$\Theta_1(\nu) = \Theta(s-z) \delta(\tau-y_2) \Theta(y_3-y_2) + 2 \leftrightarrow 3$$

+ ~~$1 \leftrightarrow 2$~~ $\rightarrow$ $0 \rightarrow$ does not lead to LL.

$$\approx \Theta\left(s - \underbrace{\frac{\sqrt{E} E_g}{4E^2}}_z \frac{\theta^2}{2}\right) \Theta\left(\frac{4EE_g}{4E^2} - \tau\right)$$

$$= \Theta(s-z) \Theta\left(\frac{E_g}{E} - \tau\right)$$

Therefore we have for one-gluon case:

$$\frac{dE_g}{E_g} \frac{d\theta_g^2}{\theta_g^2} \Theta_1(\nu) = \int_0^1 \frac{d\tau}{\tau} \int_0^1 \frac{dz}{z} \Theta(s-z)$$

For n -emissions we have

$$\Theta_n(V) = \Theta\left(\delta - \sum_i \tau_i\right) \prod_i \delta(\tau_i - y_{\omega}) \Theta(y_{\tau_i} - y_{\bar{\tau}_i})$$

$$= \Theta\left(\delta - \sum_i \tau_i\right) \prod_i (\tau_i - \bar{\tau}_i)$$

Does not factorize! Factorized

Hence we have for n -emission

$$\frac{dE_i}{E_i} \frac{d\theta_i^2}{\theta_i^2} \Theta_n(V)$$

$$= \left\{ \prod_i \int_0^1 \frac{d\tau_i}{\tau_i} \int_{\tau_i}^1 \frac{dz_i}{z_i} \right\} \Theta\left(\delta - \sum_i \tau_i\right)$$

To factorize $\Theta\left(\delta - \sum_i \tau_i\right)$, we note that

$$\Theta\left(\delta - \sum_i \tau_i\right) = \frac{1}{2\pi i} \int \frac{dv}{v} e^{v\delta - \sum_i \tau_i v}$$

to find

$$\frac{d\bar{E}_c}{\bar{E}_c} \frac{d\theta_c^2}{\theta_c^2} \Theta_n(\nu)$$

$$= \frac{1}{2\pi i} \int \frac{dv}{v} e^{vS} \quad \prod_{i=1}^n \int \frac{d\tau_i}{\tau_i} \int_{\tau_i}^1 \frac{dz_i}{z_i} e^{-\tau_i v} \quad \int \Theta_i(v)$$

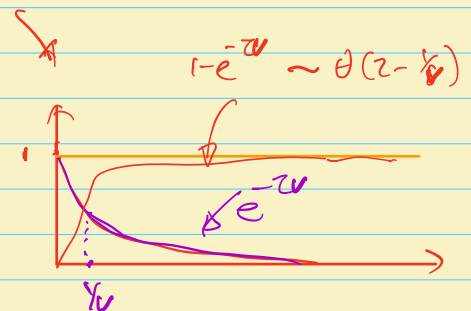
Now we plug it into Wav)

$$W(s) = \frac{1}{2\pi i} \int \frac{dv}{v} e^{vs} \exp \left[2 \int_{\tilde{z}}^z \frac{dz}{z} \frac{\alpha_s}{\pi} G_F \{ e^{-\tilde{v}} - 1 \} \right]$$

We further use the fact that

$$e^{-\tau\delta} - 1 \simeq -\Theta(\tau - e^{\delta_E} \nu^{-1})$$

which holds up to NLL.



If we ignore the τ & z dependence in α_s
which is sub-leading, we can have

$$W(s) = \frac{1}{2\pi i} \int \frac{dv}{v} e^{vs} e^{-\frac{\alpha_s}{\pi} C_F (\chi_E + \log v)^2}$$

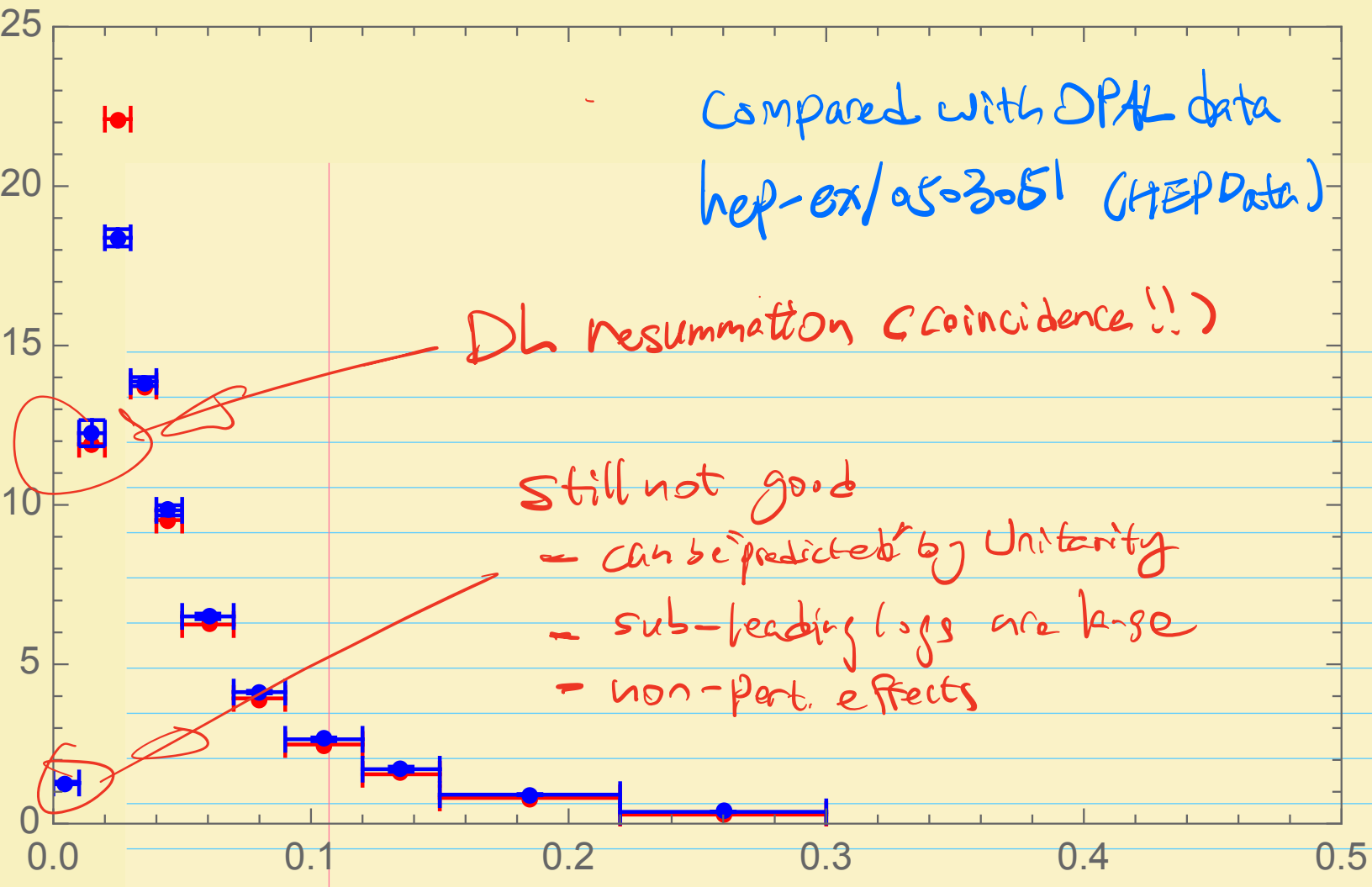
Now we let $u = vs$ to have

$$\begin{aligned} W(s) &= \frac{1}{2\pi i} \int du e^u e^{-\log u} e^{-\frac{\alpha_s}{\pi} C_F (\log u + \log e^{\frac{u}{s}})^2} \\ &= e^{-\frac{\alpha_s}{\pi} C_F \log^2 e^{\frac{1}{s}}} \underbrace{\frac{1}{2\pi i} \int du e^u e^{-\log u} e^{-\frac{\alpha_s}{\pi} C_F (\log^2 u + \dots)}}_{\mathcal{O}(u)} \end{aligned}$$

sub-leading
in logs.

$$\Rightarrow \frac{\delta(s)}{\delta_{\text{tot}}} = e^{-\frac{\alpha_s}{\pi} C_F \log^2 s} + \text{sub-leading terms} \dots$$

$$= 1 - \underbrace{\frac{\alpha_s}{\pi} C_F \log^2 s}_{\substack{\text{reproduce the} \\ \text{full NLO}}} + \underbrace{\left(\frac{\alpha_s}{\pi}\right)^2 C_F^2 \log^4 s + \dots}_{\text{predict higher orders}}$$



Comment on the results:

- Can be improved to NLL by
using the coherent branching to
derive a RGE of the "jet" function

$$\Rightarrow DL = \exp[NLO \text{ double log}]$$

$$\begin{aligned} DL &= \exp \left[\frac{2\alpha_s}{\pi} C_F \int_0^1 \frac{d\theta^2}{\theta^2} \int_{\tau}^1 \frac{dz}{z} \left\{ \Theta(s-z) - 1 \right\} \right] \\ &= 1 + \frac{\alpha_s}{n\pi} \left[\frac{2\alpha_s}{\pi} C_F \int_0^1 \frac{d\theta^2}{\theta^2} \int_{\tau}^1 \frac{dz}{z} \left\{ \Theta(s-z) - 1 \right\} \right]^n \\ &\quad \text{with } \tau = \frac{z\theta^2}{4} \end{aligned}$$

Therefore at DL, we can replace

$$\Theta(s - \sum_{i=1}^n \tau_i) \rightarrow \prod_{i=1}^n \Theta(s - \tau_i)$$

The emissions are all independent!

This holds in general for many observables

