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Matching QCD currents/operators onto SCET at 2-loop

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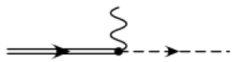
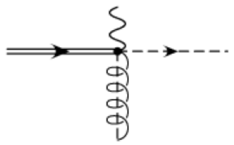
Outline

- ☐ **Matching QCD currents onto SCET at 2-loop**
- ☐ **Matching QCD operators onto SCET at 2-loop**
- ☐ **Phenomenological applications**
- ☐ **Summary**

2-body operators 3-body operators

$$\bar{q}\,\Gamma\, b = \sum_i \int ds\, \tilde{C}_i^{(A)}(s)\, O_i^{(A)}(s) + \sum_i \int ds_1 ds_2\, \tilde{C}_i^{(B)}(s_1, s_2)\, O_i^{(B)}(s_1, s_2) + \dots$$



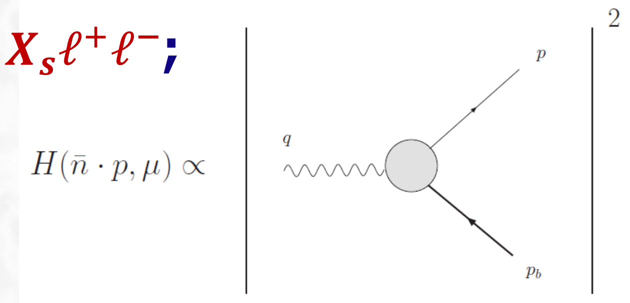


Matching QCD currents onto SCET at 2-loop

Heavy-to-light quark currents

- $\bar{q}\Gamma_i b$ with $\Gamma_i = \{1, \gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, i\sigma^{\mu\nu}\}$ play an important role in B physics:
  - govern **hadronic dynamics** in inclusive $B \to X_s\gamma$, $B \to X_u \ell \nu_\ell$, $B \to X_s \ell^+\ell^-$;

$$d\Gamma \sim H \cdot J \otimes S + \frac{1}{m_b}\sum h \cdot j \otimes S^\wedge + \frac{1}{m_b}\sum h \cdot J^\wedge \otimes S + \ldots$$

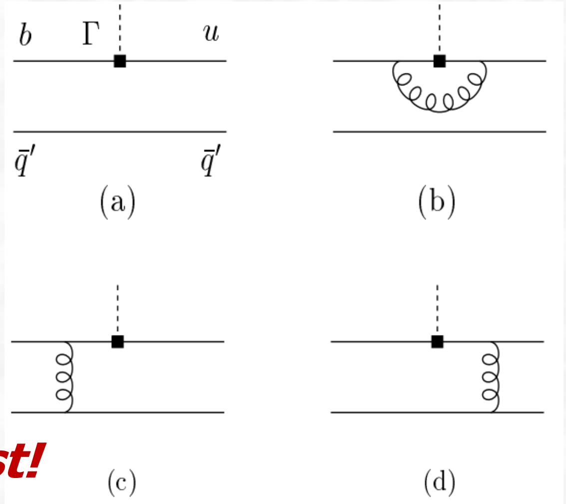


  - in the **shape-function region** (to suppress $B \to X_c \ell \nu_\ell$): hadronic final state has small invariant mass but large energy ⟹ *SCET is the appropriate theoretical framework!*
  - for exclusive B decays, $\langle M|\bar{q}\Gamma_i b|B\rangle$ generally parametrized as $B \to M$ **FFs**, also important for hadronic decays in QCDF;

$$\langle \pi|\bar{u}\gamma^\mu b|B\rangle = \left[(p+p')^\mu - \frac{m_B^2 - m_\pi^2}{q^2} q^\mu\right] f_+(q^2) + \frac{m_B^2 - m_\pi^2}{q^2} q^\mu f_0(q^2)$$

$$f_i(q^2) = C_i\,\xi_P(E) + \Phi_B \otimes T_i \otimes \Phi_P$$

Feynman mechanism dominated or pert. calculatable?



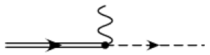
- *How to accurately represent $\bar{q}\Gamma_i b$ in SCET is of particular interest!*

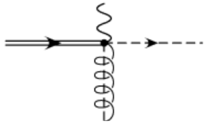
Heavy-to-light currents

□ A generic $\bar{q}\Gamma b$ is represented in SCET as:

2-body operators　　　　　　3-body operators

$$\bar{q}\,\Gamma b \;=\; \sum_i \int ds\; \tilde{C}_i^{(A)}(s)\; O_i^{(A)}(s) \;+\; \sum_i \int ds_1\, ds_2\; \tilde{C}_i^{(B)}(s_1,s_2)\; O_i^{(B)}(s_1,s_2) \;+\; \ldots$$





- $C_i^{(A)}$ and $C_i^{(B)}$: the matching coefficients and perturbatively calculable;
  - 1-loop result known; [*Bauer,Fleming,Pirjol,Stewart'00, Beneke,Kiyo,Yang'04; Becher, Hill'04*]
  - 2-loop result: known only for (V-A) current, [*Bell'08; Bonciani,Ferroglia'08; Asatrian,Greub,Pecjak'08; Beneke,Huber,Li'08*]
  - 2-loop result: now also known for other currents; [*Bell,Beneke,Huber,Li '11*]
- For inclusive processes: $\langle O_i^{(A)}\rangle \to J\otimes S,\qquad \langle O_i^{(B)}\rangle \to \sum_i j_i \otimes s_i;$
- For exclusive processes: $\langle O_i^{(A)}\rangle$ →soft-overlap contribution,
  $\langle O_i^{(B)}\rangle$ →hard spectator-scattering.

□ Some basics of HQET and SCET:

$$\mathcal{L}_{\rm QCD} = \sum_{\psi=u,d,\ldots} \bar{\psi}(i\not{D} - m_\psi)\psi \;-\; \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$Q(x) = e^{-im_b v\cdot x}\left(h_v(x) + H_v(x)\right)$$

with $\quad h_v(x) = e^{im_b v\cdot x}\,\dfrac{1+\not{v}}{2}\,Q(x)$

and $\quad H_v(x) = e^{im_b v\cdot x}\,\dfrac{1-\not{v}}{2}\,Q(x)\,,$

$$\mathcal{L}_{\rm HQET} = \bar{h}_v\, iv\cdot D\, h_v + \frac{1}{2m_b}\,\bar{h}_v\, i\not{\vec{D}}\, i\not{\vec{D}}\, h_v + \cdots$$

b-quark field in HQET

$$\mathcal{L}_{QCD} = \overline{\psi} i\not{D}\psi$$

$$\psi = P_n\psi + P_{\bar{n}}\psi = \hat{\xi}_n + \varphi_{\bar{n}}$$

$$\hat{\xi}_n = P_n\psi = \frac{\not{n}\not{\bar{n}}}{4}\psi$$

$$\varphi_{\bar{n}} = P_{\bar{n}}\psi = \frac{\not{\bar{n}}\not{n}}{4}\psi$$

$$\mathcal{L} = \bar{\hat{\xi}}_n \left(in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D} i\not{D}_\perp \right) \frac{\not{\bar{n}}}{2}\,\hat{\xi}_n$$

collinear field in SCET

General procedure for matching

- **At leading power and for 2-body operator:**

$$[\bar{q}\,\Gamma_i\, b](0) \simeq \sum_j \int ds\; \tilde{C}_i^j(s)\, \left[\bar{\xi} W_{hc}\right](s n_+)\, \Gamma_j'\, h_v(0)$$

gauge-invariant SCET operator

- **Adopt light-cone coordinate:** $\begin{cases} n_+^\mu = (1,0,0,1) \\ n_-^\mu = (1,0,0,-1) \\ v^\mu = (n_+^\mu + n_-^\mu)/2 \end{cases}$

$$p^\mu = \frac{n_+^\mu}{2}(n_- \cdot p) + \frac{n_-^\mu}{2}(n_+ \cdot p) + p_\perp^\mu$$

$$\gamma^\mu = \frac{n_+^\mu}{2}(n_- \cdot \gamma) + \frac{n_-^\mu}{2}(n_+ \cdot \gamma) + \gamma_\perp^\mu$$

- **Matching coefficients C_i^j for various Dirac structures:**

Γ_i	1	γ_5	γ^μ			$\gamma_5\gamma^\mu$			$i\sigma^{\mu\nu}$			
Γ_j'	1	γ_5	γ^μ	v^μ	n_-^μ	$\gamma_5\gamma^\mu$	$v^\mu\gamma_5$	$n_-^\mu\gamma_5$	$\gamma^{[\mu}\gamma^{\nu]}$	$v^{[\mu}\gamma^{\nu]}$	$n_-^{[\mu}\gamma^{\nu]}$	$n_-^{[\mu}v^{\nu]}$
C_i^j	C_S	C_P	C_V^1	C_V^2	C_V^3	C_A^1	C_A^2	C_A^3	C_T^1	C_T^2	C_T^3	C_T^4

- **Example:**

$$J_{\rm QCD} = \bar{u}\gamma^\mu(1-\gamma_5)b \;\Rightarrow\; J_{\rm SCET} = [\bar{\xi} W_{hc}](s n_+)\Gamma_j^\mu h_v,$$

with $\Gamma_1^\mu = \gamma^\mu(1-\gamma_5)$

$\Gamma_2^\mu = v^\mu(1+\gamma_5)$, $\Gamma_1^\mu = n_-^\mu(1+\gamma_5)$

General procedure for matching

- **At leading power and for 2-body operator:**

$$[\bar{q}\,\Gamma_i\, b](0) \simeq \sum_j \int ds\, \tilde{C}_i^j(s)\, [\bar{\xi} W_{hc}]\,(s n_+)\,\Gamma_j'\, h_v(0)$$

- **Perform QCD → SCET matching in momentum space:**

$$C_i^j(n_+ p) = \int ds\; e^{i s n_+ p}\, \tilde{C}_i^j(s)$$

  - **compute on-shell matrix element $\langle q | \cdots | b \rangle$ in both QCD and SCET;**
  - **parameterize QCD result in terms of 12 $F_i^j(q^2)$:**

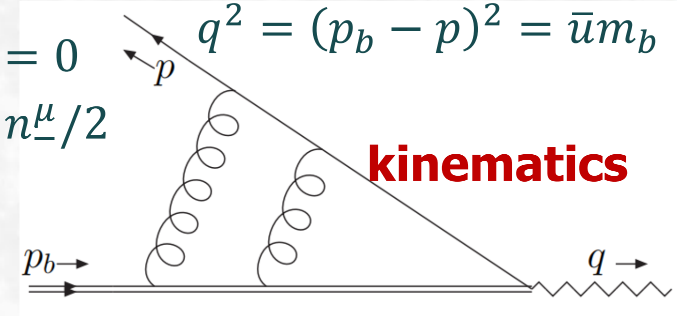
$$\langle q(p)|\bar{q}\Gamma_i b|b(p_b)\rangle = \sum_j F_i^j(q^2)\,\bar{u}(p)\Gamma_j' u(p_b)$$

$q^2 = (p_b - p)^2 = \bar{u} m_b$

$p^2 = 0$

$p^\mu = u m_b n_-^\mu / 2$

$p_b^2 = m_b^2,$



  - **matching quite simplified in dimensional regularization with on-shell quarks, because SCET loop diagrams scaleless and vanish;**

$$\langle q|[\bar{\xi} W_{hc}](s n_+)\Gamma_j' h_v|b\rangle = Z_J\,[\bar{\xi} W_{hc}]\Gamma_j' h_v$$

Z_J: universal renormalization factor of the SCET currents.

  - **final result for matching coefficients:** $C_i^j = Z_J^{-1}\, F_i^j$

General procedure for matching

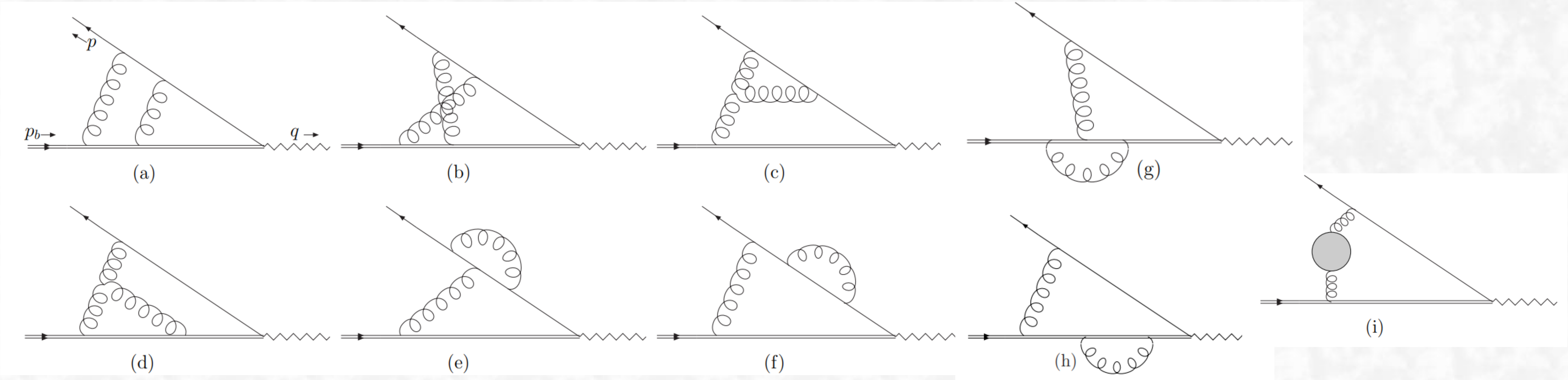
γ^μ			$\gamma_5\gamma^\mu$		
γ^μ	v^μ	n_-^μ	$\gamma_5\gamma^\mu$	$v^\mu\gamma_5$	$n_-^\mu\gamma_5$
C_V^1	C_V^2	C_V^3	C_A^1	C_A^2	C_A^3

- **Note: due to chiral symmetry of QCD, not all C_i^j independent in NDR scheme;** ➡ $C_S = C_P,\ C_A^i = C_V^i$
- **The 12 form factors F_i^j are UV finite after standard renormalization, but IR divergent;**

$$\langle q(p)|\bar{q}\Gamma_i b|b(p_b)\rangle = \sum_j F_i^j(q^2)\,\bar{u}(p)\Gamma_j' u(p_b)$$

canceled exactly by Z_J factor, making sure that C_i^j are finite.

- **On the QCD side, 2-loop Feynman diagrams:**



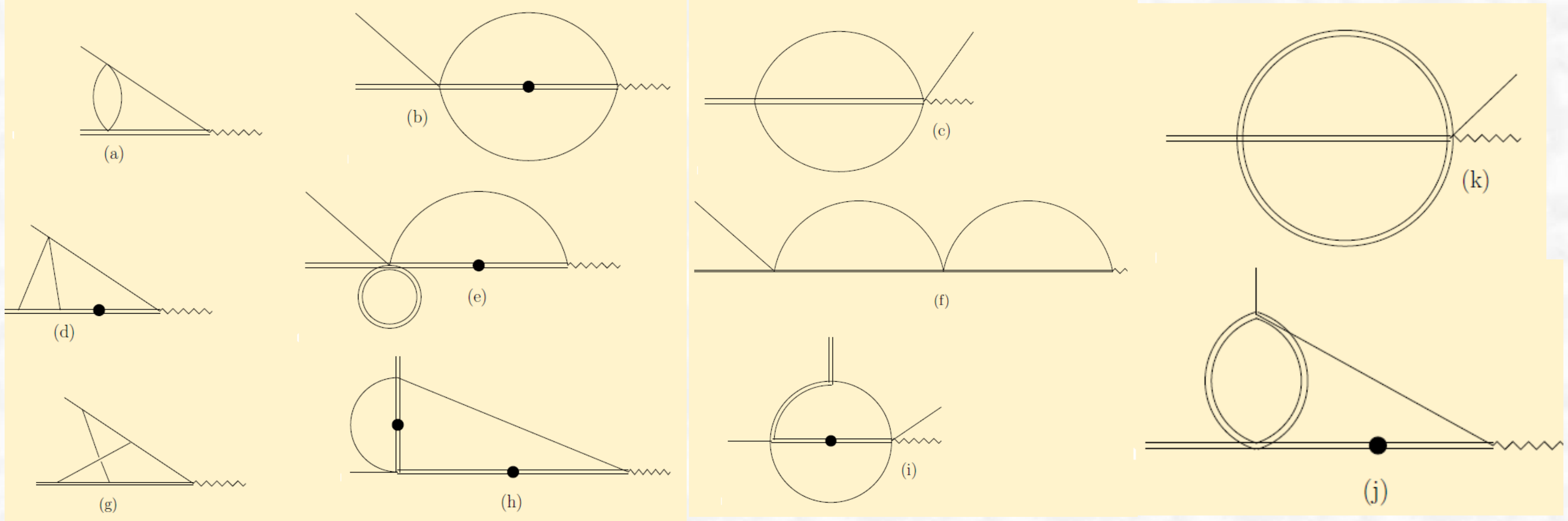
Calculation on QCD side

- Dimensional regularisation with $D = 4 - 2\epsilon$ regulates UV and IR. Poles up to $1/\epsilon^4$.
- Passarino-Veltman reduction of tensor integrals to scalar integrals *[Passarino, Veltman'79]*

thousands of scalar integrals

- Reduction of scalar integrals to a small set of master integrals
  - Integration-by-parts and Lorentz-invariance identities *[Tkachov'81; Chetyrkin, Tkachov'81; Gehrmann, Remiddi'99]*
  - System of equations solved by Laporta algorithm *[Laporta'01; Anastasiou, Lazopoulos'04; Smirnov'08]*

reduce them to totally additional 18 MIs!



double lines: massive;

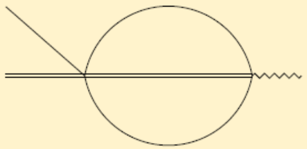
single lines: massless

dots on lines: squared propagators

Calculation on QCD side

- Applied techniques
  - Hypergeometric functions, use HypExp or XSummer for ϵ-expansion

    [Moch,Uwer'05; Maitre,TH'05,'07]



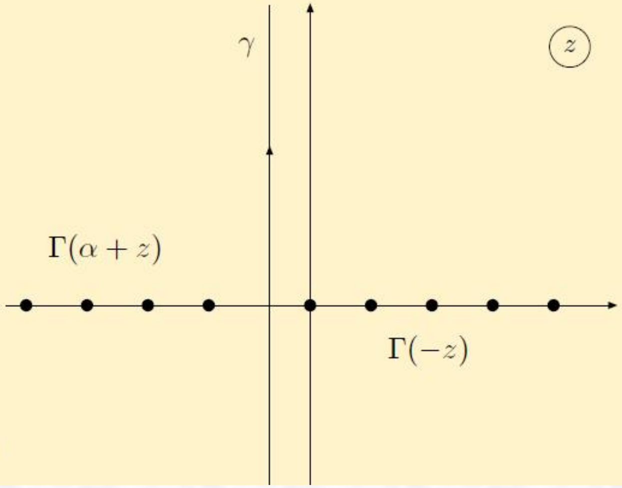
$$= \frac{(m_b^2)^{1-2\epsilon}}{(4\pi)^{4-2\epsilon}} \frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon-1)}{\Gamma(2-\epsilon)} {}_2F_1(\epsilon, 2\epsilon-1; 2-\epsilon; 1-u)$$

  - Differential equations [Kotikov'91; Remiddi'97]

$$\frac{\partial}{\partial u}\mathrm{MI}_i(u) = f(u,\epsilon)\,\mathrm{MI}_i(u) + \sum_{j\neq i} g_j(u,\epsilon)\,\mathrm{MI}_j(u)$$

  - Mellin-Barnes representation [Smirnov'99; Tausk'99]

$$\frac{1}{(A_1+A_2)^\alpha} = \int_\gamma \frac{dz}{2\pi i}\, A_1^z\, A_2^{-\alpha-z}\, \frac{\Gamma(-z)\,\Gamma(\alpha+z)}{\Gamma(\alpha)}$$



    * partially automated
    * Numerical cross checks possible

[Czakon'05; Gluza,Kajda,Riemann'07]

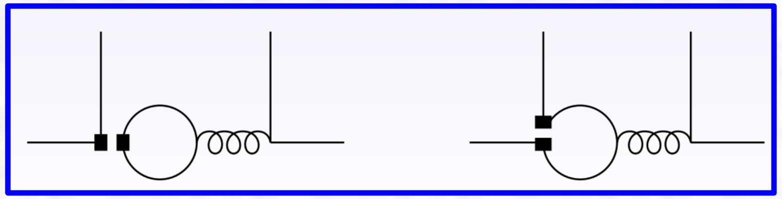
requires Laporta reduction result.

obtain boundary condition in $u = 0$ or $u = 1$ from Mellin-Barnes representation!

AMBRE.m

MB.m

Example of MB representation



- After the loop-momentum integration, we get:

penguin diagrams in QCDF

$$G(s,x) = -4\int_0^1 du\, u(1-u)\ln[s-u(1-u)x]$$

$$G(s-i\epsilon,u) = \frac{2(12s+5u-3u\ln s)}{9u} - \frac{2\xi(2s+u)}{3u}\ln\frac{\xi+1}{\xi-1}$$

with $\xi = \sqrt{1-4s/u}$

(7)

$$\int_0^1 dx \cdot \frac{1}{[s_c - \bar u x \bar x - i\eta]^\epsilon}$$

$$\int_0^1 dx\, x^z \bar x^z = \frac{\Gamma(1+z)^2}{\Gamma(2+2z)}$$

$$= \int_0^1 dx \cdot \frac{1}{\Gamma(\epsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(\epsilon+z)\Gamma(-z) \frac{[-\bar u x \bar x]^z}{[s_c - i\eta]^{\epsilon+z}}$$

$$= \frac{1}{\Gamma(\epsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(\epsilon+z)\Gamma(-z) [-\bar u]^z [s_c - i\eta]^{-\epsilon-z} \times \frac{\Gamma(1+z)^2}{\Gamma(2+2z)}$$

$$\Gamma(2+2z) = \frac{1}{\sqrt{\pi}} 2^{2z+\frac{3}{2}} \Gamma(1+z) \times \Gamma(z+\tfrac{3}{2})$$

$$= \frac{1}{\Gamma(\epsilon)} \frac{1}{2\pi i} \frac{\sqrt{\pi}}{2} \int_{-i\infty}^{+i\infty} dz \cdot [s_c - i\eta]^{-\epsilon} \cdot \times \left[-\frac{\bar u}{4 s_c}\right]^z \frac{\Gamma(\epsilon+z)\Gamma(1+z)\Gamma(-z)}{\Gamma(\frac{1}{2}+z)}$$

$$= \frac{1}{\Gamma(\epsilon)} \frac{\sqrt{\pi}}{2} \cdot [s_c - i\eta]^{-\epsilon} \cdot \frac{\Gamma(\epsilon)\Gamma(1)}{\Gamma(\frac{3}{2})} \times {}_2F_1\left(\epsilon, 1, \tfrac{3}{2}, +\frac{\bar u}{4 s_c}\right)$$

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

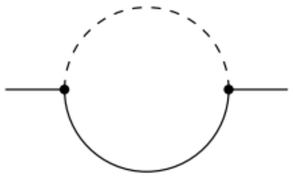
$$= [s_c - i\eta]^{-\epsilon} \cdot {}_2F_1\left(\epsilon, 1, \tfrac{3}{2}, +\frac{\bar u}{4 s_c}\right)$$

Example of MB representation

The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

Example 1



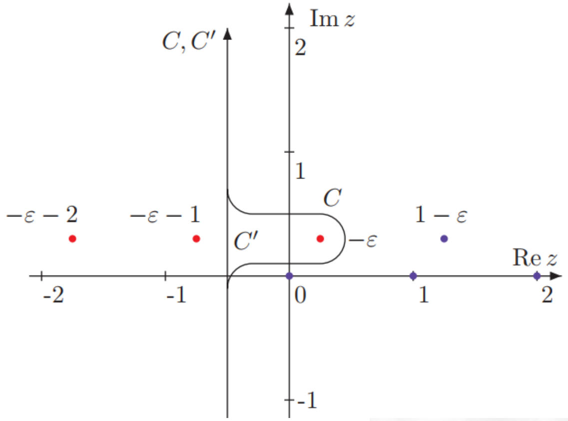
$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2}\frac{G(a_1,a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}},$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \times \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

$\Gamma(\epsilon+z)\Gamma(-z) \to$ a singularity in ϵ



Take a residue at $z=-\epsilon$:

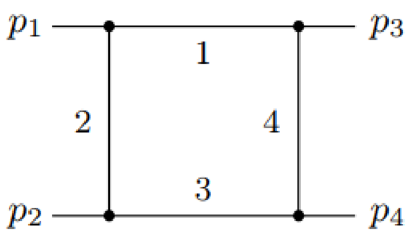
$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon(1-\epsilon)}$$

and shift the contour:

$$\mathrm{i}\pi^2 \frac{1}{2\pi\mathrm{i}}\int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z}$$

Example of MB representation

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0,\ i = 1,2,3,4$



$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_3)^2]^{a_4}}\,,$$

where $s = (p_1+p_2)^2$ and $t = (p_1+p_3)^2$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = (-1)^a \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}$$
$$\times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\epsilon-2}}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\,,$$

where $a = a_1+a_2+a_3+a_4$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

Change the order of integration over z and ξ-parameters, evaluate parametric integrals in terms of gamma functions

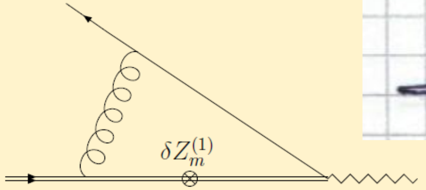
One-dim. MB representation:

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$
$$\times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$
$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

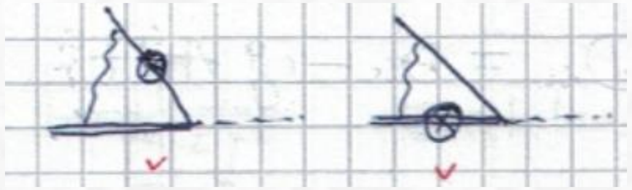
UV renormalization in QCD

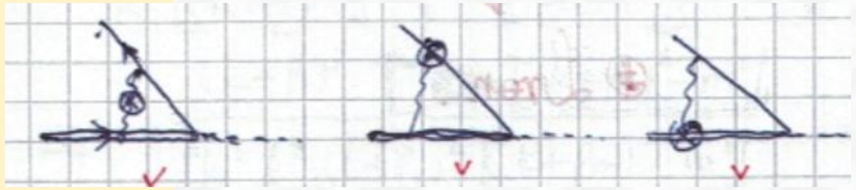
□ F_i^j **are UV & IR divergent, with the UV part renormalized via standard QCD counter-terms;**

- UV conterterms: Apply on-shell scheme for the heavy mass as well as for the heavy (h) and light (l) quark field
  - Z_m and Z_h are known [Broadhurst,Gray,Grafe,Schilcher'90,'91; Melnikov,van Ritbergen'00]
  - Z_l only starts at two loops.
    Derive all-order representation: $Z_l^{\rm os} = 1 + 2C_F t_f \frac{g_0^4 (m^2)^{D-4}}{(4\pi)^D} \frac{(D-1)\Gamma(4-\frac{D}{2})\Gamma(-\frac{D}{2})}{(D-5)(D-7)}$
  - Renormalize α_s in the $\overline{\rm MS}$ scheme
  - Non-vanishing anomalous dimension of scalar and tensor current:
    Additional counterterms Z_S and Z_T [Nanopoulos,Ross'79; Tarrach'81; Broadhurst,Grozin'94]
- All UV renormalizations are simple multiplications except the one-loop mass counterterm



➢ **adopt the standard counter-term method:**

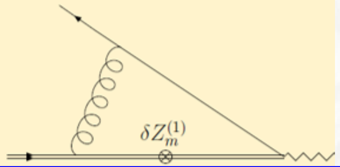




□ **Renormalized on-shell matrix element:**

$$\mathcal{A}^{(1)}_{\rm ren} = \mathcal{A}^{(1)}_{\rm bare} + \frac{1}{2}\,\delta Z_h^{(1)}\,\mathcal{A}^{(0)}\,,$$

$$\mathcal{A}^{(2)}_{\rm ren} = \mathcal{A}^{(2)}_{\rm bare} + \frac{1}{2}\,\delta Z_h^{(1)}\,\mathcal{A}^{(1)}_{\rm bare} + \left[\frac{1}{2}\,\delta Z_h^{(2)} - \frac{1}{8}\,(\delta Z_h^{(1)})^2 + \frac{1}{2}\,\delta Z_l^{(2)}\right]\mathcal{A}^{(0)} \;+$$



IR subtraction

$$F_i^j = \sum_{k=0}^{\infty} \left(\frac{\alpha_s^{(5)}}{4\pi}\right)^k F_i^{j,(k)}, \qquad F_i^{j,(k)} = \sum_l F_{i,l}^{j,(k)} \epsilon^l .$$

- **After UV renormalization, F_i^j still IR divergent:**

$$F_{S,-4}^{(2)}(u) = \frac{1}{2} C_F^2, \qquad \propto 1/\epsilon^4$$

$$F_{S,-3}^{(2)}(u) = C_F^2 \left(L - g_0(u)\right) + \frac{11}{4} C_A C_F - n_l T_F C_F, \qquad \propto 1/\epsilon^3$$

$$Z_J = 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha_s^{(4)}}{4\pi}\right)^k Z_J^{(k)} \qquad C_i^j = \sum_{k=0}^{\infty} \left(\frac{\alpha_s^{(4)}}{4\pi}\right)^k C_i^{j,(k)},$$

- **To obtain finite $C_i^j = Z_J^{-1} F_i^j$ via $\overline{\text{MS}}$ counter-term Z_J of SCET current:**

$$Z_J^{(1)} = C_F \left\{ -\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} \ln \frac{\mu}{n_+ p} - \frac{5}{\varepsilon} \right\}$$

[Bauer, Fleming, Pirjol, Stewart 00]

$Z_J^{(2)}$ from 2-loop calculation of jet and shape function [Becher, Neubert 05,06]

$$C_i^{j,(0)} = F_i^{j,(0)},$$

$$C_i^{j,(1)} = F_i^{j,(1)} - Z_J^{(1)} F_i^{j,(0)},$$

$$C_i^{j,(2)} = F_i^{j,(2)} + \delta\alpha_s^{(1)} F_i^{j,(1)} - Z_J^{(1)} \left(F_i^{j,(1)} - Z_J^{(1)} F_i^{j,(0)}\right) - Z_J^{(2)} F_i^{j,(0)}.$$

indeed IR finite

- **Remember: QCD calculation is in 5-active quark flavors, while the Z_J is given in 4-active flavors, so we need $\alpha_s^{(5)} \to \alpha_s^{(4)}$.**

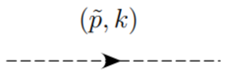
$$\alpha_s^{(5)} = \alpha_s^{(4)} \left[1 + \frac{\alpha_s^{(4)}}{4\pi} \delta\alpha_s^{(1)} + \mathcal{O}(\alpha_s^2)\right]$$

$$\delta\alpha_s^{(1)} = T_F \left[\frac{4}{3} \ln \frac{\mu^2}{m_b^2} + \left(\frac{2}{3} \ln^2 \frac{\mu^2}{m_b^2} + \frac{\pi^2}{9}\right)\epsilon + \left(\frac{2}{9} \ln^3 \frac{\mu^2}{m_b^2} + \frac{\pi^2}{9} \ln \frac{\mu^2}{m_b^2} - \frac{4}{9}\zeta_3\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right]$$

Calculation of Z_J

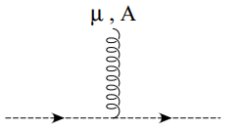
[Bauer, Fleming, Pirjol, Stewart, hep-ph/0011336]

☐ Leading-power Feynman rules in SCET:



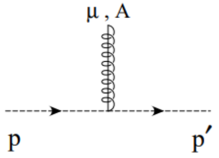
$$= i\frac{\bar{n}\!\!\!/}{2}\frac{\bar n\cdot p}{n\cdot k\,\bar n\cdot p+p_\perp^2+i\epsilon}$$

collinear-quark propagator



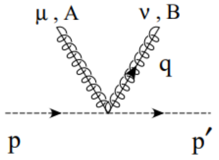
$$= igT^A\, n_\mu \frac{\bar{n}\!\!\!/}{2}$$

collinear-quark interaction with one soft gluon



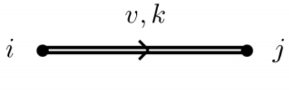
$$= igT^A\left[n_\mu+\frac{\gamma_\mu^\perp p\!\!\!/_\perp}{\bar n\cdot p}+\frac{p\!\!\!/'_\perp\gamma_\mu^\perp}{\bar n\cdot p'}-\frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar n\cdot p\,\bar n\cdot p'}\bar n_\mu\right]\frac{\bar{n}\!\!\!/}{2}$$

collinear-quark interaction with one collinear gluon



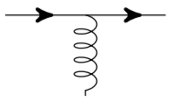
$$= \frac{ig^2T^AT^B}{\bar n\cdot(p-q)}\left[\gamma_\mu^\perp\gamma_\nu^\perp-\frac{\gamma_\mu^\perp p\!\!\!/_\perp}{\bar n\cdot p}\bar n_\nu-\frac{p\!\!\!/'_\perp\gamma_\nu^\perp}{\bar n\cdot p'}\bar n_\mu+\frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar n\cdot p\,\bar n\cdot p'}\bar n_\mu\bar n_\nu\right]\frac{\bar{n}\!\!\!/}{2}$$
$$+\frac{ig^2T^BT^A}{\bar n\cdot(q+p')}\left[\gamma_\nu^\perp\gamma_\mu^\perp-\frac{\gamma_\nu^\perp p\!\!\!/_\perp}{\bar n\cdot p}\bar n_\mu-\frac{p\!\!\!/'_\perp\gamma_\mu^\perp}{\bar n\cdot p'}\bar n_\nu+\frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar n\cdot p\,\bar n\cdot p'}\bar n_\mu\bar n_\nu\right]\frac{\bar{n}\!\!\!/}{2}$$

collinear-quark interaction with two collinear gluons



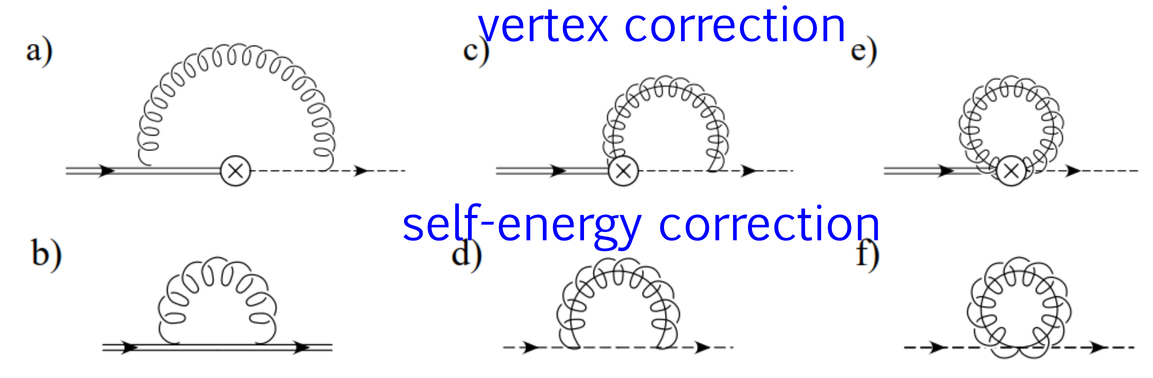
$$= \frac{i}{v\cdot k+i\eta}\frac{1+v\!\!\!/}{2}\delta_{ji}$$

Feynman rules in HQET



$$= -igv^\mu T^A$$

☐ One-loop Feynman diagrams for $\bar\xi_{n,p}\Gamma h_v$:



☐ Use off-shell-ness $p^2 = p_\perp^2 \neq 0$ of collinear quark to regulate IR divergences;

☐ Explicit results for each diagram:

$$\text{Fig. 6a} = i\bar\xi_{n,p}\Gamma h_v\frac{C_F\alpha_s(\mu)C(\mu)}{4\pi}\left[-\frac{1}{\epsilon^2}-\frac{2}{\epsilon}\ln\left(\frac{\mu\,\bar n\cdot p}{-p_\perp^2-i\epsilon}\right)-2\ln^2\left(\frac{\mu\,\bar n\cdot p}{-p_\perp^2-i\epsilon}\right)-\frac{3\pi^2}{4}\right]$$

$$\text{Fig. 6b} = iv\cdot k\frac{\alpha_s(\mu)C_F}{4\pi}\left[-\frac{2}{\epsilon}-4-4\ln\left(\frac{\mu}{-2v\cdot k-i\epsilon}\right)\right],$$

$$\text{Fig. 6c} = i\bar\xi_{n,p}\Gamma h_v\frac{C_F\alpha_s(\mu)C(\mu)}{4\pi}\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}+\frac{2}{\epsilon}\ln\left(\frac{\mu^2}{-p_\perp^2-i\epsilon}\right)+\ln^2\left(\frac{\mu^2}{-p_\perp^2-i\epsilon}\right)\right.$$
$$\left.+2\ln\left(\frac{\mu^2}{-p_\perp^2-i\epsilon}\right)+4-\frac{\pi^2}{6}\right],$$

$$\text{Fig. 6d} = \frac{i\bar{n}\!\!\!/}{2}\frac{p_\perp^2}{\bar n\cdot p}\frac{\alpha_s(\mu)C_F}{4\pi}\left[\frac{1}{\epsilon}+1+\ln\left(\frac{\mu^2}{-p_\perp^2-i\epsilon}\right)\right],$$

$$\text{Fig. 6e,f} = 0\,.$$

Calculation of Z_J

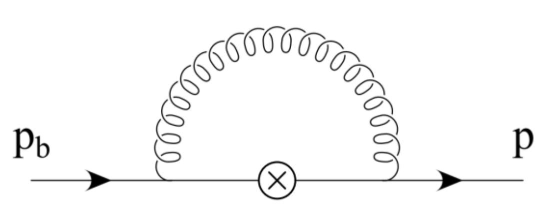
[Bauer, Fleming, Pirjol, Stewart, hep-ph/0011336]

- **Renormalization factor Z_J for SCET $\bar{\xi}_{n,p}\Gamma h_v$:**

$$Z_J = 1 - \frac{\alpha_s^{(4)}(\mu) C_F}{4\pi}\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(\frac{\mu}{u m_b}\right) + \frac{5}{2\epsilon}\right]$$

> ➢ **Z_J is independent of the spin structure of SCET currents, and hence universal!**

- **One-loop Feynman diagram of the full QCD heavy-to-light current $\bar{q}\Gamma b$:**



+ the wavefunction renormalization factors

+ the current renormalization factors

- **Matching coefficients at one-loop:**

$$L \equiv \ln\left(\frac{\mu^2}{m_b^2}\right)$$

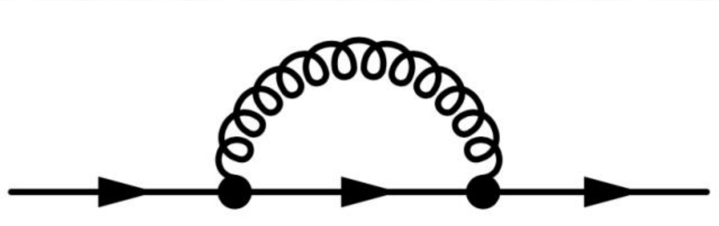
$$f_{-1}(u) = 2\ln(u) - \frac{5}{2}$$

$$F_1^{(1)} = C_F\left\{-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left[f_{-1}(u) - L\right] - \frac{L^2}{2} + L\,f_{-1}(u) + f_0(u) + \epsilon\left[-\frac{L^3}{6} + \frac{1}{2}L^2 f_{-1}(u) + L\,f_0(u) + f_1(u)\right] + \epsilon^2\left[-\frac{L^4}{24} + \frac{1}{6}L^3 f_{-1}(u) + \frac{1}{2}L^2 f_0(u) + L\,f_1(u) + f_2(u)\right]\right\}$$

➡ **$C_i^j = Z_J^{-1} F_i^j$: finite**

Self-energy diagram in QCD and EFT

□ **One-loop diagram in QCD:**

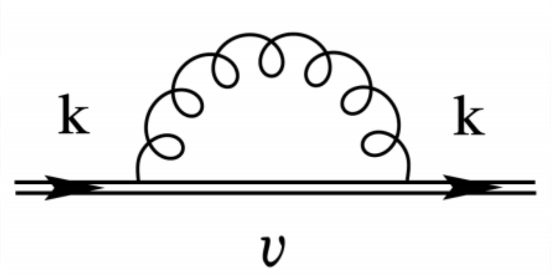


$$\Sigma_{\rm QCD}(p) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+i\epsilon)} \frac{\gamma_\mu(\not{p}+\not{l}+m_Q)\gamma^\mu}{(p+l)^2 - m_Q^2 + i\epsilon}$$

wave function renormalization of quark field in full QCD in $\overline{\rm MS}$ scheme:

$$Z_Q = 1 - C_F \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}$$

□ **One-loop diagram in HQET:**



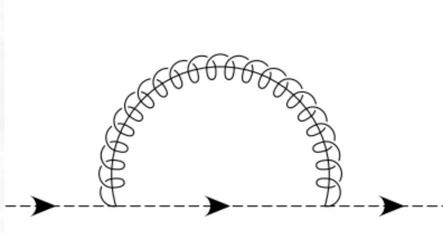
$$\Sigma(v\cdot k) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+i\epsilon)(v\cdot k + v\cdot l + i\epsilon)} P_+$$

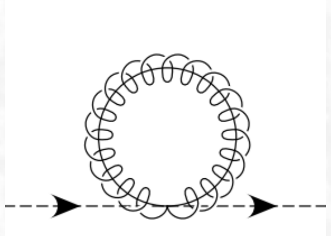
wave function renormalization of quark field in HQET in $\overline{\rm MS}$ scheme:

$$\frac{1}{A^n B^m} = 2^m \frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} \int_0^\infty d\lambda \frac{\lambda^{m-1}}{(A+2\lambda B)^{m+n}}$$

$$Z_{\rm HQET} = 1 + C_F \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}$$

□ **One-loop diagram in SCET:**





$$i\Sigma_c(p) = g^2 C_F \frac{\not{\bar{n}}}{2} \int \frac{d^d l}{(2\pi)^d} \Big\{ (n\cdot\bar{n}) \frac{p_\perp^2 + \not{p}_\perp \not{l}_\perp}{\bar{n}\cdot p (p+l)^2 l^2} + (n\cdot\bar{n}) \frac{p_\perp^2 + \not{l}_\perp \not{p}_\perp}{\bar{n}\cdot p (p+l)^2 l^2}$$
$$+ 2(d-4)\frac{p_\perp^2 + l_\perp\cdot p_\perp}{\bar{n}\cdot p(p+l)^2 l^2} - (d-2)\left(\frac{(p_\perp + l_\perp)^2}{[\bar{n}\cdot(p+l)]^2} + \frac{p_\perp^2}{[\bar{n}\cdot p]^2}\right)\frac{\bar{n}\cdot(p+l)}{(p+l)^2 l^2}\Big\}$$

Pheno. applications

□ Inclusive $B \to X_u \ell \nu$ and $|V_{ub}|$ extraction:

- experiments impose cuts to suppress background from $B \to X_c \ell \nu$
- measurements restricted to shape-function region: $E_X \sim m_b,\ m_X^2 \sim m_b \Lambda_{QCD}$
- multi-scale OPE in terms of non-local light-cone operators [Korchemsky, Sterman 94]

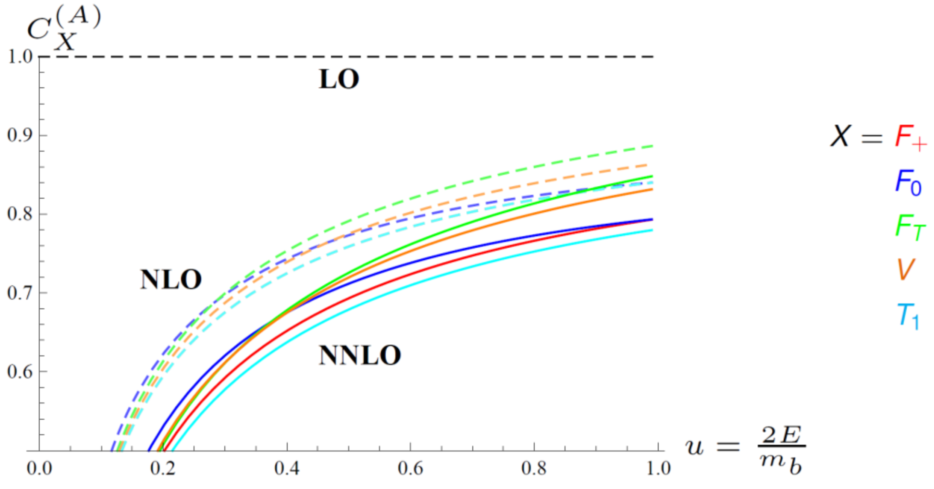
$$\Gamma_u \simeq \sum_{i,j} H_{ij}(n_+ p) \int d\omega\, J(p_\omega^2)\, S(\omega)$$

Exp.	Method	$\Delta\mathcal{B}^{\rm exp}$ $[10^{-4}]$	$\lvert V_{ub}\rvert$ $[10^{-3}]$ NLO	$\lvert V_{ub}\rvert$ $[10^{-3}]$ NNLO
CLEO	$E_l > 2.1$ GeV	$3.3 \pm 0.2 \pm 0.7$	$3.56 \pm 0.40^{+0.48\,+0.31}_{-0.27\,-0.26}$	$3.81 \pm 0.43^{+0.33\,+0.31}_{-0.21\,-0.26}$
BABAR	$E_l > 2.0$ GeV	$5.7 \pm 0.4 \pm 0.5$	$3.97 \pm 0.22^{+0.37\,+0.26}_{-0.23\,-0.25}$	$4.30 \pm 0.24^{+0.26\,+0.28}_{-0.20\,-0.27}$
BELLE	$E_l > 1.9$ GeV	$8.5 \pm 0.4 \pm 1.5$	$4.27 \pm 0.39^{+0.32\,+0.25}_{-0.19\,-0.22}$	$4.65 \pm 0.43^{+0.27\,+0.27}_{-0.18\,-0.24}$
BELLE	$M_X < 1.7$ GeV	$12.3 \pm 1.1 \pm 1.2$	$3.55 \pm 0.24^{+0.22\,+0.21}_{-0.13\,-0.19}$	$3.87 \pm 0.26^{+0.21\,+0.21}_{-0.13\,-0.19}$
BABAR	$M_X < 1.55$ GeV	$11.7 \pm 0.9 \pm 0.7$	$3.67 \pm 0.18^{+0.29\,+0.26}_{-0.17\,-0.24}$	$3.96 \pm 0.19^{+0.20\,+0.26}_{-0.13\,-0.24}$

NNLO corrections shift $|V_{ub}|$ upwards by 10% compared to NLO. [Greub, Neubert, Pecjak 09]

□ Heavy-to-light FFs in large-recoil regime:

$$F_X^{B\to M}(E) \simeq C_X^{(A)}(E)\, \xi_M(E) + \int d\omega \int du\, \phi_B(\omega)\, T_X(E,\omega,u)\, \phi_M(u)$$



→ "universal" corrections to heavy-to-light form factors

□ Semi-inclusive $B \to X_s \ell^+ \ell^-$:

$$m_X \le m_X^{\rm cut} = 1.8 \ldots 2.0\,\text{GeV} \text{ and } 1\,\text{GeV}^2 \le q^2 \le 6\,\text{GeV}^2$$

⇒ "shape function region"

$$d\Gamma^{[0]} = h^{[0]} \times J \otimes S$$

Zero point of $A_{FB}(q^2)$:

$$\frac{q_0^2}{2m_b(m_B - \langle p_X^+ \rangle)} = -\frac{\text{Re}\,[C_7^{\rm incl}(q_0^2)]}{\text{Re}\,[C_9^{\rm incl}(q_0^2)]}\, \underbrace{\frac{c_1^7(u_0)}{c_1^9(u_0)}}_{=R_\perp}$$

$$q_0^2 = [(3.34 \ldots 3.40)^{+0.22}_{-0.25}]\ \text{GeV}^2 \text{ for } m_X^{\rm cut} = (2.0 \ldots 1.8)\ \text{GeV}$$

$$Q = \int d\hat{t}\, \tilde{T}^{\mathrm{I}}(\hat{t}) O^{\mathrm{I}}(t) + \int d\hat{t} d\hat{s}\, \tilde{H}^{\mathrm{II}}(\hat{t}, \hat{s}) O^{\mathrm{II}}(t, s)$$

Matching QCD operators onto SCET at 2-loop

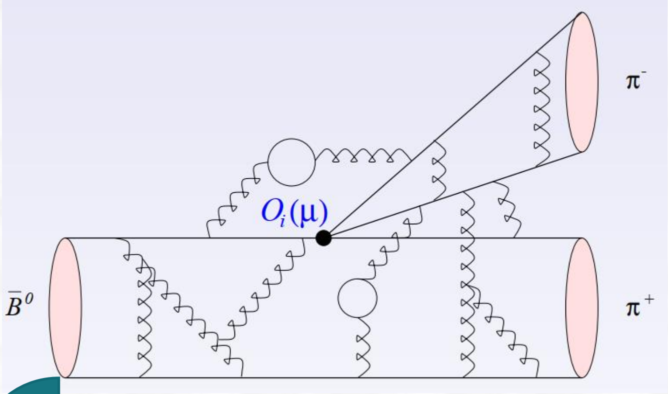
QCD factorization

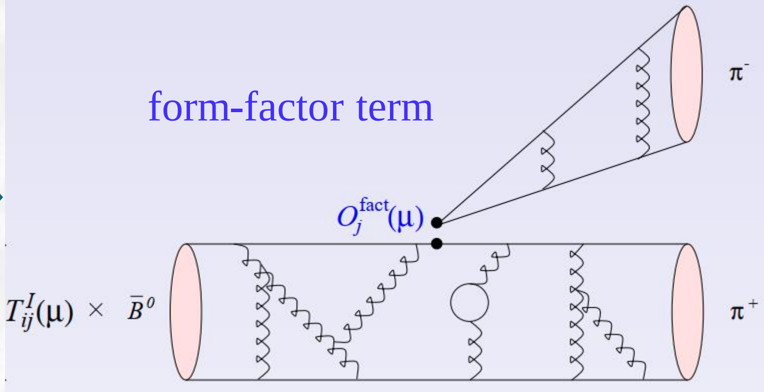
[BBNS, '99-'03]

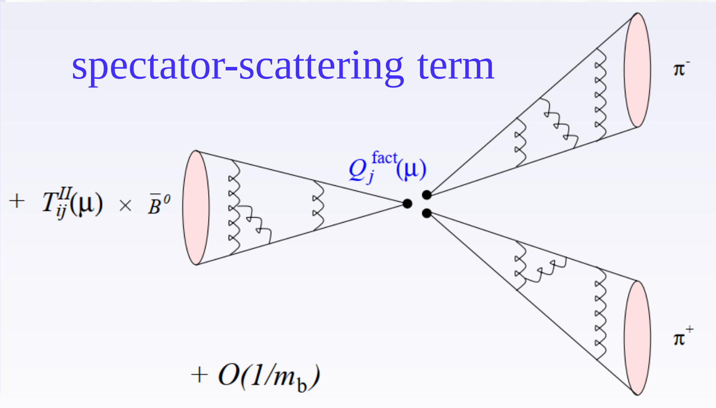
- **QCDF for $\langle M_1 M_2|\mathcal{O}_i|\bar{B}\rangle$**: systematic framework to all orders in α_s, but limited by $1/m_b$ corrections.

$$\langle M_1M_2|Q_i|\bar{B}\rangle = F^{BM_1}(0)\int_0^1 du\, T_i^I(u)\Phi_{M_2}(u) + \int_0^\infty d\omega \int_0^1 dudv\, T_i^{II}(\omega,u,v)\,\Phi_B(\omega)\Phi_{M_1}(v)\Phi_{M_2}(u)$$

- At LO in α_s and $1/m_b$, reproduces the naïve factorization result.
- Higher-order pert. corrections in α_s could be calculated systematically.
- Factorization generally broken at higher-order power in $1/m_b$.







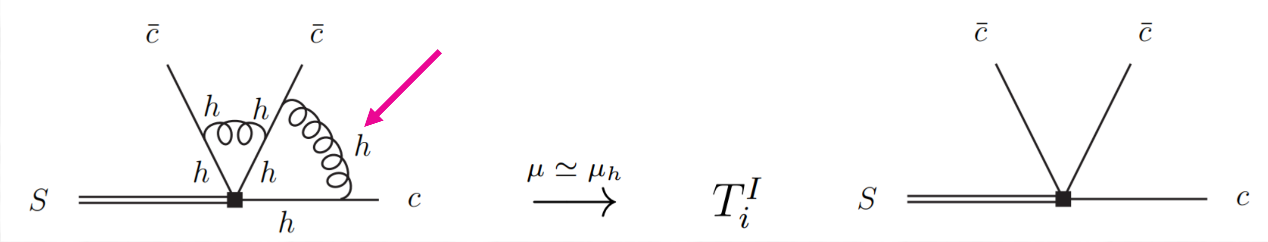
reduces $\langle M_1 M_2|\mathcal{O}_i|\bar{B}\rangle$ to simpler $\langle M|j_\mu|\bar{B}\rangle$ (form factors), $\langle 0|j_\mu|\bar{B}\rangle$, $\langle M|j_\mu|0\rangle$ (decay constants & light-cone distribution amplitudes), all can be obtained from exp. data, lattice-QCD, or LCSR.

Soft-collinear factorization from SCET

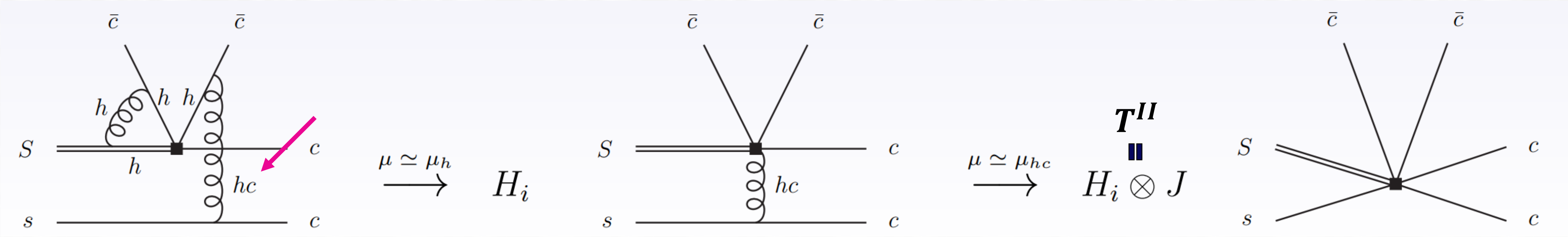
- **SCET diagrams reproduce precisely QCD diagrams in collinear and soft momentum region**

→ **QCD - SCET = short-distance coefficients**

- **For hard kernel T^I:** one-step matching, QCD $\rightarrow \mathrm{SCET_I(hc, c, s)}$!



- **For hard kernel T^{II}:** two-step matching, QCD $\rightarrow \mathrm{SCET_I(hc, c, s)} \rightarrow \mathrm{SCET_{II}(c, s)}$!



- **SCET result exactly the same as QCDF, but more apparent & efficient;** [Beneke,1501.07374]

General procedure

$\lambda_u = V_{ub}V_{ud}^* \sim \mathcal{O}(\lambda^3)$

$\lambda_c = V_{cb}V_{cd}^* \sim \mathcal{O}(\lambda^3)$

$B \to \pi\pi$: tree-dominated!

α_4 is loop suppressed vs $\alpha_{1,2}$!

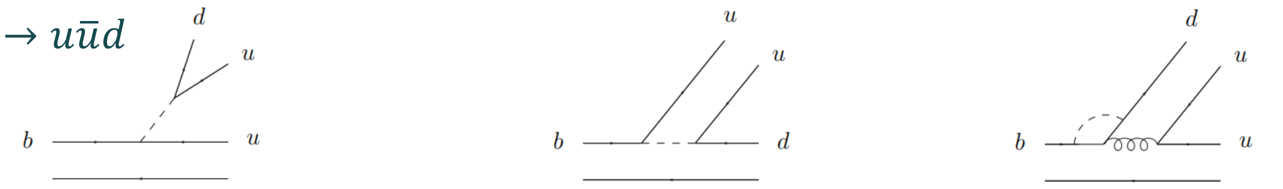
$\lambda_u = V_{ub}V_{us}^* \sim \mathcal{O}(\lambda^4)$

$\lambda_c = V_{cb}V_{cs}^* \sim \mathcal{O}(\lambda^2)$

$B \to K\pi$: penguin-dominated!

□ QCDF/SCET for $B \to M_1 M_2$ decay amplitudes:

$b \to u\bar{u}d$



colour-allowed tree α_1 colour-suppressed tree α_2 QCD penguins α_4

□ Start with

$$\mathcal{H}_{\rm eff} = -\frac{4G_F}{\sqrt{2}} \sum_{p=u,c} V_{pD}^* V_{pb} \left(C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{10} C_i Q_i + C_{7\gamma} Q_{7\gamma} + C_{8g} Q_{8g} \right)$$

□ Hard kernels $T^{I,II}$ by QCD → SCET matching;

$$Q = \int d\hat{t}\, \tilde{T}^{\rm I}(\hat{t}) O^{\rm I}(t) + \int d\hat{t} d\hat{s}\, \tilde{H}^{\rm II}(\hat{t},\hat{s}) O^{\rm II}(t,s)$$

$$T^{\rm I}(u) = \int d\hat{t}\, e^{iu\hat{t}}\, \tilde{T}^{\rm I}(\hat{t})$$

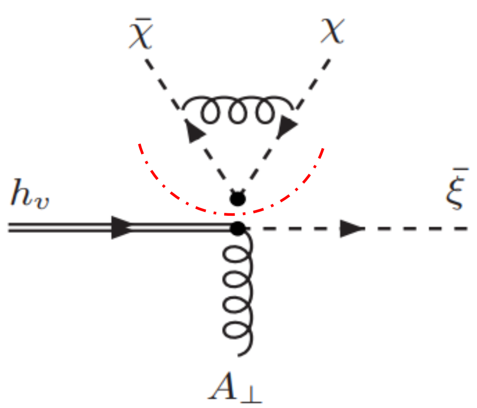
$$H^{\rm II}(u,v) = \int d\hat{t} d\hat{s}\, e^{i(u\hat{t}+(1-v)\hat{s})}\, \tilde{H}^{\rm II}(\hat{t},\hat{s})$$

$$O^{\rm I}(t) = (\bar{\chi} W_{c2})(tn_-) \frac{\not{n}_-}{2} (1-\gamma_5)(W_{c2}^\dagger \chi) \left[\bar{q} \not{n}_+ (1-\gamma_5) b \right]$$

$$O^{\rm II}(t,s) = \frac{1}{m_b} \left[(\bar{\chi} W_{c2})(tn_-) \frac{\not{n}_-}{2} (1-\gamma_5)(W_{c2}^\dagger \chi) \right] \left[(\bar{\xi} W_{c1}) \frac{\not{n}_+}{2} [W_{c1}^\dagger i \not{D}_{\perp c1} W_{c1}](sn_+)(1+\gamma_5) h_v \right]$$

decouple already

General procedure



- **NO LP interactions between collinear-2 and collinear-1 fields.**

$$\langle M_2|(\bar{\chi}W_{c2})(tn_-)\frac{\not{n}_-}{2}(1-\gamma_5)(W_{c2}^\dagger\chi)|0\rangle = \frac{if_{M_2}m_B}{2}\int_0^1 du\, e^{iu\hat{t}}\,\phi_{M_2}(u)$$

$$\langle M_1|(\bar{\xi}W_{c1})\frac{\not{n}_+}{2}[W_{c1}^\dagger i\not{D}_{\perp c}W_{c1}](sn_+)(1+\gamma_5)h_v|\bar{B}\rangle = -m_b m_B\int_0^1 d\tau\, e^{i\tau\hat{s}}\,\Xi_{M_1}(\tau)$$

SCET$_\mathrm{I}$ form factor!

$$\langle M_1M_2|Q|\bar{B}\rangle = im_B^2\left\{f_+^{BM_1}(0)\int_0^1 du\, T^{\mathrm{I}}(u)f_{M_2}\phi_{M_2}(u) - \frac{1}{2}\int_0^1 dudz\, H^{\mathrm{II}}(u,z)\,\Xi_{M_1}(1-z)f_{M_2}\phi_{M_2}(u)\right\}$$

- **Hard kernels T^{II} by SCET$_\mathrm{I}$ → SCET$_\mathrm{II}$ matching;** [Beneke, Jager '05]

$$\Xi_{M_1}(\tau) = \frac{m_B}{4m_b}\int_0^\infty d\omega\int_0^1 dv\, J_\parallel(\tau;v,\omega)\hat{f}_B\phi_{B+}(\omega)f_{M_1}\phi_{M_1}(v)$$

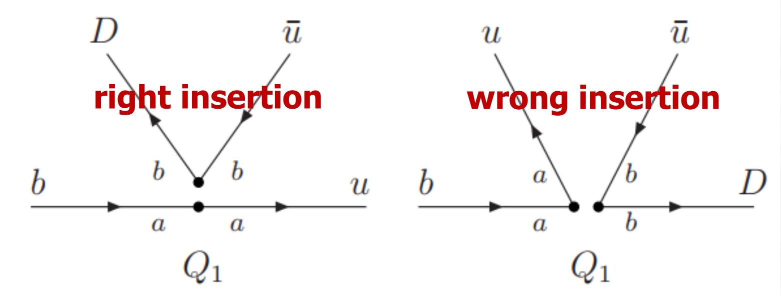
$$T^{\mathrm{II}}(\omega,u,v) = -\frac{m_B}{8m_b}\int_0^1 dz\, H^{\mathrm{II}}(u,z)J_\parallel(1-z;v,\omega)$$

- **Final results for hard kernels $T^{I,II}$ at leading power:**

$$\langle M_1M_2|Q|\bar{B}\rangle = T^{\mathrm{I}}(\mu_h)*\phi_\pi(\mu_h)\,f_+^{B\pi}(0) + \overbrace{H^{\mathrm{II}}(\mu_h)*U_\parallel(\mu_h,\mu_{hc})*J(\mu_{hc})}^{T^{\mathrm{II}}}*\phi_\pi(\mu_h)*\phi_\pi(\mu_{hc})*\phi_{B+}(\mu_{hc})$$

Hard kernel T^I at NNLO

□ **Note:**



$$O_{\rm QCD} \equiv [\bar{q}\,\frac{\not{n}_-}{2}(1-\gamma_5)q]\,[\bar{q}\,\not{n}_+(1-\gamma_5)b] = C_{FF}\, C_{\bar{q}q}\, O_1$$

factorized QCD operator, with $\langle O_{QCD}\rangle$ =FF*LCDA

□ QCD → SCETI matching calculation:

- For "right insertion":

$$\langle Q_i\rangle = T_i\,\langle O_{\rm QCD}\rangle + \sum_{a>1} H_{ia}\langle O_a\rangle$$

- For "wrong insertion":

$$\langle Q_i\rangle = \tilde{T}_i\,\langle O_{\rm QCD}\rangle + \tilde{H}_{i1}\langle \tilde{O}_1 - O_1\rangle + \sum_{a>1} \tilde{H}_{ia}\langle \tilde{O}_a\rangle$$

□ Complete SCET operator basis;

- Nonlocal SCET operator basis for RI:

$$O_1 = [\bar{\chi}\,\frac{\not{n}_-}{2}P_L\chi]\,[\bar{\xi}\,\not{n}_+P_L h_v]\,,$$

only physical operator and factorizes into FF*LCDA.

$$O_2 = [\bar{\chi}\,\frac{\not{n}_-}{2}P_L\gamma_\perp^\alpha\gamma_\perp^\beta\chi]\,[\bar{\xi}\,\not{n}_+P_L\gamma^\perp_\beta\gamma^\perp_\alpha h_v]\,,$$

$$O_3 = [\bar{\chi}\,\frac{\not{n}_-}{2}P_L\gamma_\perp^\alpha\gamma_\perp^\beta\gamma_\perp^\gamma\gamma_\perp^\delta\chi]\,[\bar{\xi}\,\not{n}_+P_L\gamma^\perp_\delta\gamma^\perp_\gamma\gamma^\perp_\beta\gamma^\perp_\alpha h_v]$$

- Nonlocal SCET operator basis for WI:

$$\tilde{O}_1 = [\bar{\xi}\,\gamma_\perp^\alpha P_L\chi]\,[\bar{\chi}P_R\gamma^\perp_\alpha h_v]\,,$$

evanescent operators must be renormalized to zero.

$$\tilde{O}_2 = [\bar{\xi}\,\gamma_\perp^\alpha\gamma_\perp^\beta\gamma_\perp^\gamma P_L\chi]\,[\bar{\chi}P_L\gamma^\perp_\alpha\gamma^\perp_\gamma\gamma^\perp_\beta h_v]\,,$$

$$\tilde{O}_3 = [\bar{\xi}\,\gamma_\perp^\alpha\gamma_\perp^\beta\gamma_\perp^\gamma\gamma_\perp^\delta\gamma_\perp^\epsilon P_L\chi]\,[\bar{\chi}P_R\gamma^\perp_\alpha\gamma^\perp_\epsilon\gamma^\perp_\delta\gamma^\perp_\gamma\gamma^\perp_\beta h_v]$$

□ On-shell matrix elements at NNLO: full QCD side

$$\langle Q_i\rangle = \Big\{ A^{(0)}_{ia} + \frac{\alpha_s}{4\pi}\Big[A^{(1)}_{ia} + Z^{(1)}_{ext}\,A^{(0)}_{ia} + Z^{(1)}_{ij}A^{(0)}_{ja}\Big] + \left(\frac{\alpha_s}{4\pi}\right)^2\Big[A^{(2)}_{ia} + Z^{(1)}_{ij}A^{(1)}_{ja} + Z^{(2)}_{ij}A^{(0)}_{ja} + Z^{(1)}_{ext}\,A^{(1)}_{ia} + Z^{(2)}_{ext}\,A^{(0)}_{ia} + Z^{(1)}_{ext}\,Z^{(1)}_{ij}A^{(0)}_{ja} + Z^{(1)}_{\alpha}A^{(1)}_{ia} + (-i)\,\delta m^{(1)}\,A'^{(1)}_{ia}\Big] + \mathcal{O}(\alpha_s^3)\Big\}\,\langle O_a\rangle^{(0)}$$

□ On-shell matrix elements at NNLO: SCET side

$$\langle O_a\rangle = \Big\{ \delta_{ab} + \frac{\hat{\alpha}_s}{4\pi}\Big[M^{(1)}_{ab} + Y^{(1)}_{ext}\,\delta_{ab} + Y^{(1)}_{ab}\Big] + \left(\frac{\hat{\alpha}_s}{4\pi}\right)^2\Big[M^{(2)}_{ab} + Y^{(1)}_{ac}M^{(1)}_{cb} + Y^{(2)}_{ab} + Y^{(1)}_{ext}\,M^{(1)}_{ab} + Y^{(2)}_{ext}\,\delta_{ab} + Y^{(1)}_{ext}\,Y^{(1)}_{ab} + \hat{Z}^{(1)}_{\alpha}M^{(1)}_{ab}\Big] + \mathcal{O}(\alpha_s^3)\Big\}\,\langle O_b\rangle^{(0)}$$

Hard kernel T^I at NNLO

□ Master formula for T^I: right insertion

$$T_i^{(0)} = A_{i1}^{(0)}\,,$$

$$T_i^{(1)} = A_{i1}^{(1)\mathrm{nf}} + Z_{ij}^{(1)}\,A_{j1}^{(0)}\,,$$

$$T_i^{(2)} = \underline{A_{i1}^{(2)\mathrm{nf}} + Z_{ij}^{(1)}\,A_{j1}^{(1)} + Z_{ij}^{(2)}\,A_{j1}^{(0)} + Z_{\alpha}^{(1)}\,A_{i1}^{(1)\mathrm{nf}} + (-i)\,\delta m^{(1)}\,A_{i1}^{\prime(1)\mathrm{nf}}} - T_i^{(1)}\left[C_{FF}^{(1)} + Y_{11}^{(1)} - Z_{ext}^{(1)}\right] - \sum_{b>1} \underline{H_{ib}^{(1)}\,Y_{b1}^{(1)}}\,.$$

renormalized 2-loop QCD matrix element of Q_i.

from SCET evanescent operators $O_{2,3}$.

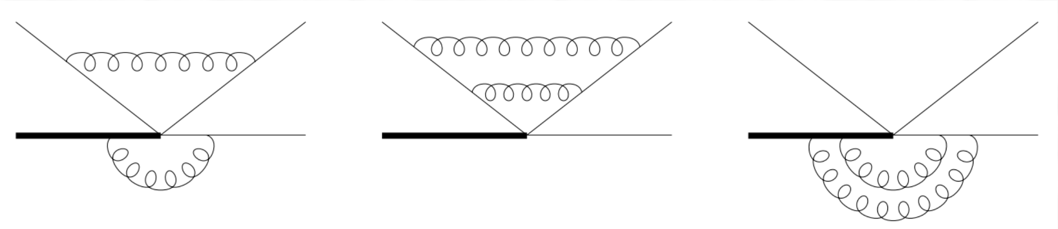
□ Master formula for T^I: wrong insertion

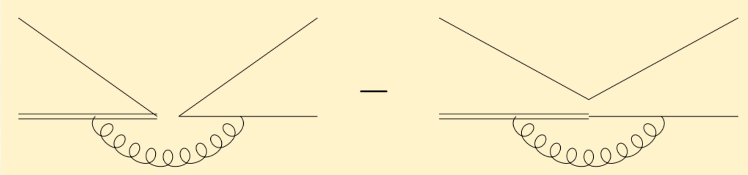
$$\widetilde{T}_i^{(0)} = \widetilde{A}_{i1}^{(0)}\,,$$

$$\widetilde{T}_i^{(1)} = \widetilde{A}_{i1}^{(1)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(0)} + \underbrace{\widetilde{A}_{i1}^{(1)\mathrm{f}} - A_{21}^{(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}}_{\mathcal{O}(\epsilon)} - \underbrace{\left[\widetilde{Y}_{11}^{(1)} - Y_{11}^{(1)}\right]\widetilde{A}_{i1}^{(0)}}_{\mathcal{O}(\epsilon)}\,,$$

$$\begin{aligned}\widetilde{T}_i^{(2)} = {} & \underline{\widetilde{A}_{i1}^{(2)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(1)} + Z_{ij}^{(2)}\,\widetilde{A}_{j1}^{(0)} + Z_{\alpha}^{(1)}\,\widetilde{A}_{i1}^{(1)\mathrm{nf}}} \\ & \underline{+ (-i)\,\delta m^{(1)}\,\widetilde{A}_{i1}^{\prime(1)\mathrm{nf}} + Z_{ext}^{(1)}\left[\widetilde{A}_{i1}^{(1)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(0)}\right]} \\ & - \widetilde{T}_i^{(1)}\left[C_{FF}^{(1)} + \widetilde{Y}_{11}^{(1)}\right] - \sum_{b>1} \underline{\widetilde{H}_{ib}^{(1)}\,\widetilde{Y}_{b1}^{(1)}} \\ & + \left[\widetilde{A}_{i1}^{(2)\mathrm{f}} - A_{21}^{(2)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}\right] + (-i)\,\delta m^{(1)}\left[\widetilde{A}_{i1}^{\prime(1)\mathrm{f}} - A_{21}^{\prime(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}\right] \\ & + \left(Z_{\alpha}^{(1)} + Z_{ext}^{(1)}\right)\left[\widetilde{A}_{i1}^{(1)\mathrm{f}} - A_{21}^{(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}\right] \\ & - \left[\widetilde{M}_{11}^{(2)} - M_{11}^{(2)}\right]\widetilde{A}_{i1}^{(0)} \\ & - \left(C_{FF}^{(1)} - \xi_{45}^{(1)}\right)\left[\widetilde{Y}_{11}^{(1)} - Y_{11}^{(1)}\right]\widetilde{A}_{i1}^{(0)} - \left[\widetilde{Y}_{11}^{(2)} - Y_{11}^{(2)}\right]\widetilde{A}_{i1}^{(0)}\,.\end{aligned}$$

from Fierz difference of factorizable QCD diagrams

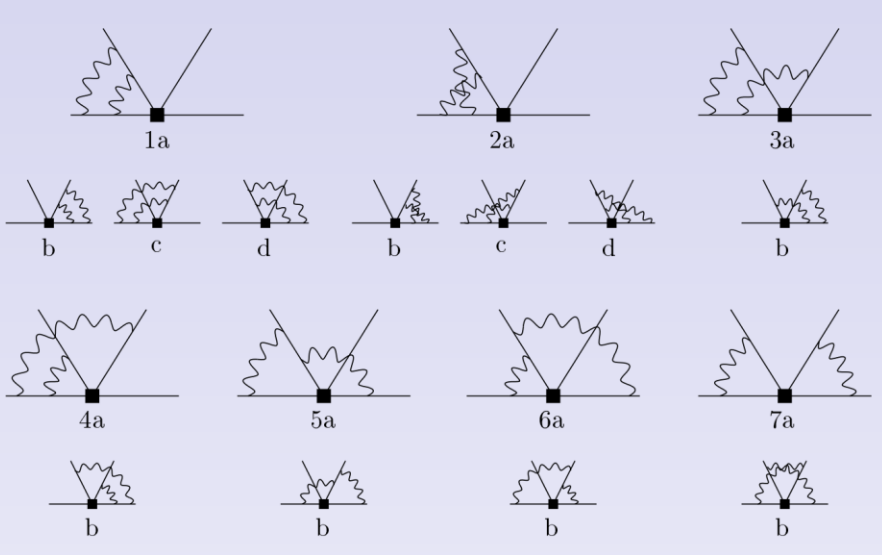


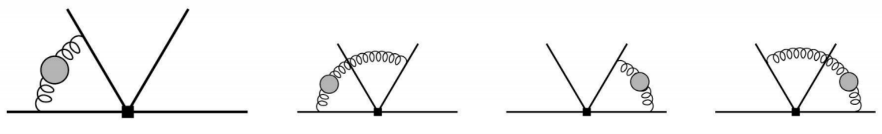


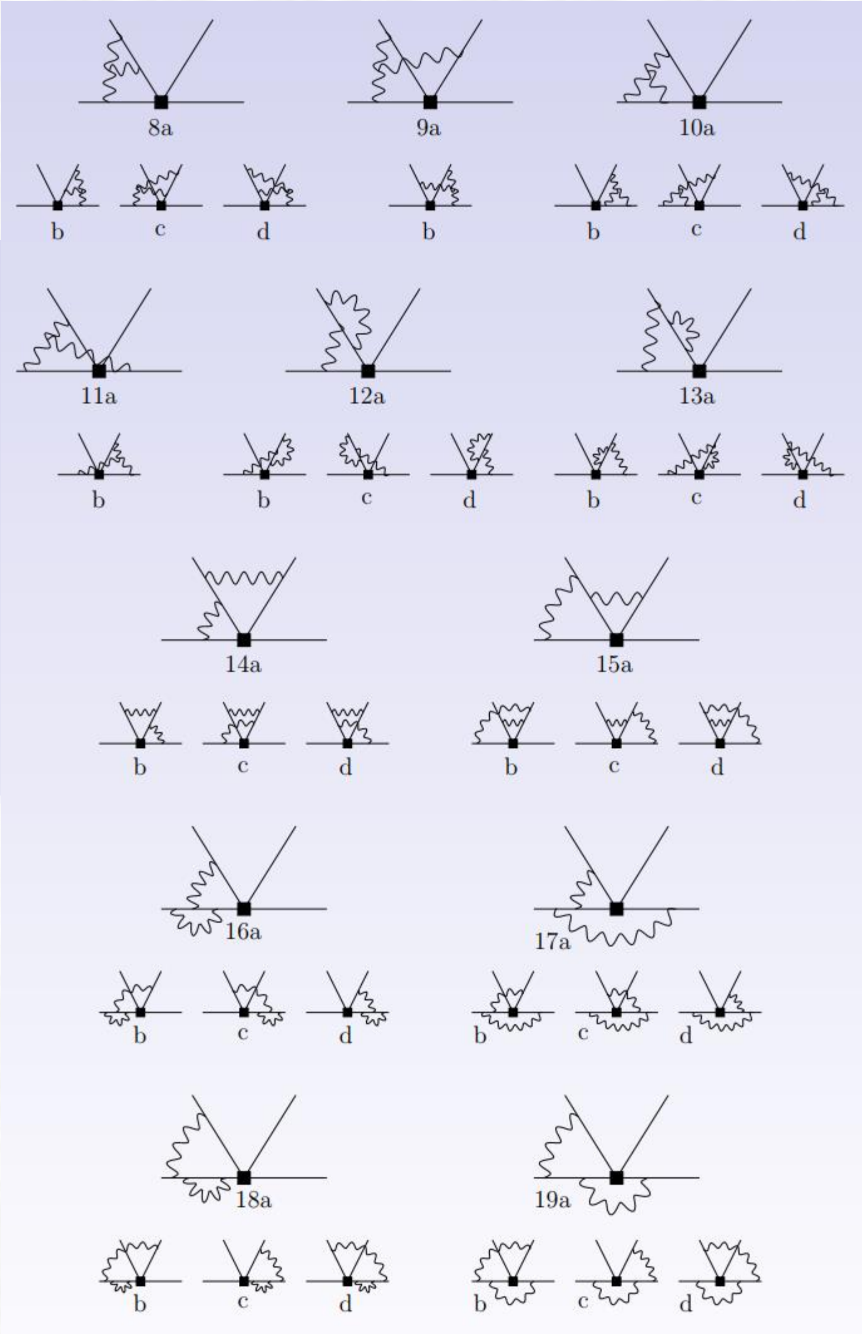
From renormalizing the 1- and 2-loop SCET matrix element of $\widetilde{O}_1 - O_1$ to zero.

Two-loop QCD diagrams

- **Relevant Feynman diagrams in QCD:**
  - totally 70 “non-fact.” diagrams;
  - one-loop counter-term insertions;
  - evanescent operators at 1-loop; $Z_{ij}^{(1)} A_{j1}^{(1)}$







$$
\begin{aligned}
E_1^{(1)} &= \bar{u}\gamma^\mu\gamma^\nu\gamma^\rho T^a(1-\gamma_5)b\bar{d}\gamma_\mu\gamma_\nu\gamma_\rho T^a(1-\gamma_5)u - 16Q_1,\\
E_2^{(1)} &= \bar{u}\gamma^\mu\gamma^\nu\gamma^\rho(1-\gamma_5)b\bar{d}\gamma_\mu\gamma_\nu\gamma_\rho(1-\gamma_5)u - 16Q_2,\\
E_1^{(2)} &= \bar{u}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda T^a(1-\gamma_5)b\bar{d}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda T^a(1-\gamma_5)u - 20E_1^{(1)} - 256Q_1,\\
E_2^{(2)} &= \bar{u}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda(1-\gamma_5)b\bar{d}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\lambda(1-\gamma_5)u - 20E_2^{(1)} - 256Q_2.
\end{aligned}
$$

Penguin topologies and various insertions

□ Effective Hamiltonian including penguin operators: [BBL '96; CMM '98]

$$\mathcal{H}_{\rm eff} = \frac{4G_F}{\sqrt{2}} \sum_{p=u,c} V_{pD}^* V_{pb} \left(C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{10} C_i Q_i + C_{7\gamma} Q_{7\gamma} + C_{8g} Q_{8g} \right) + \text{h.c.}$$

$$Q_1^p = (\bar{p}_L \gamma^\mu T^A b_L)\, (\bar{D}_L \gamma_\mu T^A p_L),$$
$$Q_2^p = (\bar{p}_L \gamma^\mu b_L)\, (\bar{D}_L \gamma_\mu p_L),$$

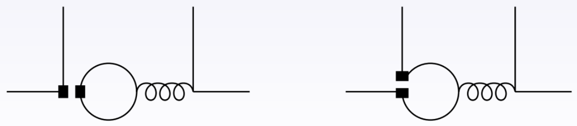
current-current operators

$$Q_3 = (\bar{D}_L \gamma^\mu b_L) \textstyle\sum_q (\bar{q} \gamma_\mu q),$$
$$Q_4 = (\bar{D}_L \gamma^\mu T^A b_L) \textstyle\sum_q (\bar{q} \gamma_\mu T^A q),$$
$$Q_5 = (\bar{D}_L \gamma^\mu \gamma^\nu \gamma^\rho b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho q),$$
$$Q_6 = (\bar{D}_L \gamma^\mu \gamma^\nu \gamma^\rho T^A b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho T^A q)\,.$$

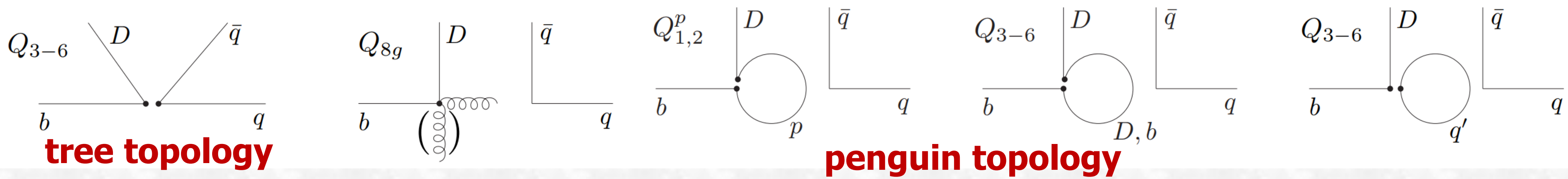
QCD penguin operators

$$Q_{8g} = \frac{-g_s}{32\pi^2} \overline{m}_b \, \bar{D} \sigma_{\mu\nu} (1+\gamma_5) G^{\mu\nu} b,$$

chromo-magnetic dipole operators

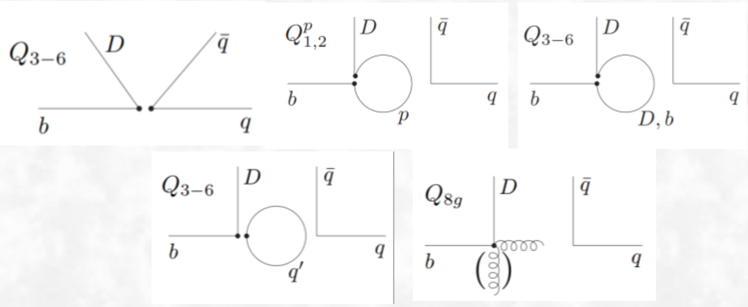


□ Various operator insertions but all featured by **wrong insertion:**



(i) Dirac structure of Q_i (ii) color structure of Q_i (iii) types of contraction, and (iv) quark mass in the fermion loop;

Hard kernel T^I at NNLO



- **QCD → SCETI matching calculation:**

$$\langle Q_i \rangle = \sum_a \widetilde{H}_{ia} \, \langle \widetilde{O}_a \rangle$$

- **Note: always wrong insertion!**

$$O_1 = \sum_{q=u,d,s} \left[\bar{\chi}_D \frac{\not{n}_-}{2}(1-\gamma_5)\chi_q\right] \; \left[\bar{\xi}_q \not{n}_+ (1-\gamma_5) h_v\right] ,$$

the only physical operator and factorizes into FF*LCDA.

- **Complete SCET operator basis:**

$$\widetilde{O}_n = \sum_{q=u,d,s} \left[\bar{\xi}_q \, \gamma_\perp^\alpha \gamma_\perp^{\mu_1} \gamma_\perp^{\mu_2} \cdots \gamma_\perp^{\mu_{2n-2}} \chi_q\right] \; \left[\bar{\chi}_q (1+\gamma_5) \gamma_{\perp\alpha} \gamma_{\perp\mu_{2n-2}} \gamma_{\perp\mu_{2n-3}} \cdots \gamma_{\perp\mu_1} h_v\right] ,$$

n now up to 4, with 7 gamma matrices

$\widetilde{O}_1 - O_1/2$ is another evanescent operator

- **On-shell matrix elements at NNLO: on the full QCD side**

$$\begin{aligned}\langle Q_i \rangle = \Big\{ & \widetilde{A}^{(0)}_{ia} + \frac{\alpha_s}{4\pi} \left[\widetilde{A}^{(1)}_{ia} + Z^{(1)}_{\text{ext}} \widetilde{A}^{(0)}_{ia} + Z^{(1)}_{ij} \widetilde{A}^{(0)}_{ja}\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\widetilde{A}^{(2)}_{ia} + Z^{(1)}_{ij} \widetilde{A}^{(1)}_{ja} + Z^{(2)}_{ij} \widetilde{A}^{(0)}_{ja} + Z^{(1)}_{\text{ext}} \widetilde{A}^{(1)}_{ia} + Z^{(2)}_{\text{ext}} \widetilde{A}^{(0)}_{ia} \right. \\ & \left. + Z^{(1)}_{\text{ext}} Z^{(1)}_{ij} \widetilde{A}^{(0)}_{ja} + Z^{(1)}_{\alpha} \widetilde{A}^{(1)}_{ia} + (-i)\, \delta m^{(1)} \, \widetilde{A}'^{(1)}_{ia}\right] + \mathcal{O}(\alpha_s^3) \Big\} \langle \widetilde{O}_a \rangle^{(0)}\end{aligned}$$

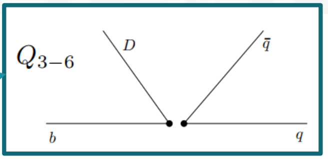
- **On-shell matrix elements at NNLO: SCET side**

$$\begin{aligned}\langle O_a \rangle = \Big\{ & \delta_{ab} + \frac{\hat{\alpha}_s}{4\pi} \left[M^{(1)}_{ab} + Y^{(1)}_{\text{ext}} \, \delta_{ab} + Y^{(1)}_{ab}\right] + \left(\frac{\hat{\alpha}_s}{4\pi}\right)^2 \left[M^{(2)}_{ab} + Y^{(1)}_{ac} M^{(1)}_{cb} \right. \\ & \left. + Y^{(2)}_{ab} + Y^{(1)}_{ext} \, M^{(1)}_{ab} + Y^{(2)}_{ext} \, \delta_{ab} + Y^{(1)}_{ext} \, Y^{(1)}_{ab} + \hat{Z}^{(1)}_{\alpha} M^{(1)}_{ab}\right] + \mathcal{O}(\alpha_s^3) \Big\} \; \langle O_b \rangle^{(0)}\end{aligned}$$

T^I at NNLO

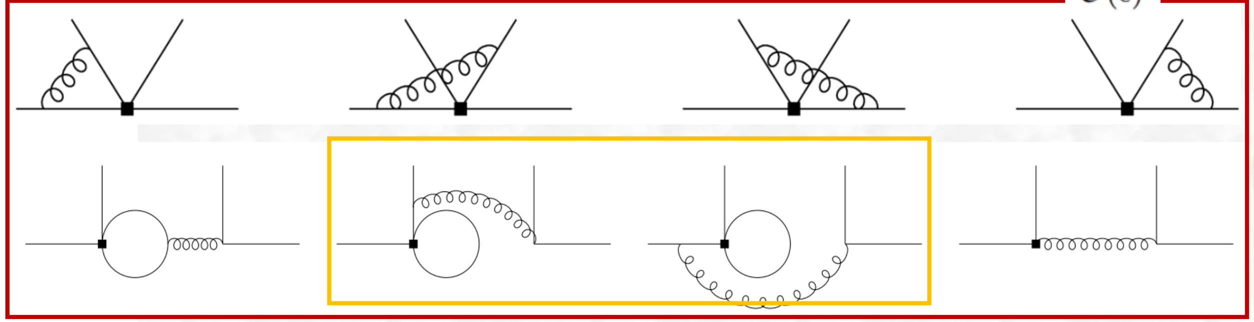
- **Master formula for T^I:**

$$\frac{1}{2}\widetilde{T}_i^{(0)} = \widetilde{A}_{i1}^{(0)}\,,$$

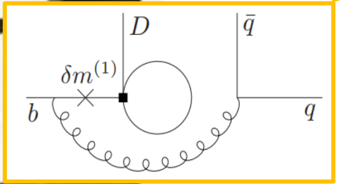


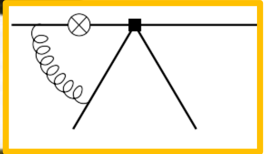
tree-level matching of Q_i involves already evanescent SCET operators

$$\frac{1}{2}\widetilde{T}_i^{(1)} = \widetilde{A}_{i1}^{(1)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(0)} + \underbrace{\widetilde{A}_{i1}^{(1)\mathrm{f}} - A_{31}^{(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}}_{\mathcal{O}(\epsilon)} - \underbrace{[\widetilde{Y}_{11}^{(1)} - Y_{11}^{(1)}]\,\widetilde{A}_{i1}^{(0)}}_{\mathcal{O}(\epsilon)} - \underbrace{\sum_{b>1}\widetilde{A}_{ib}^{(0)}\,\widetilde{Y}_{b1}^{(1)}}_{\mathcal{O}(\epsilon)}$$

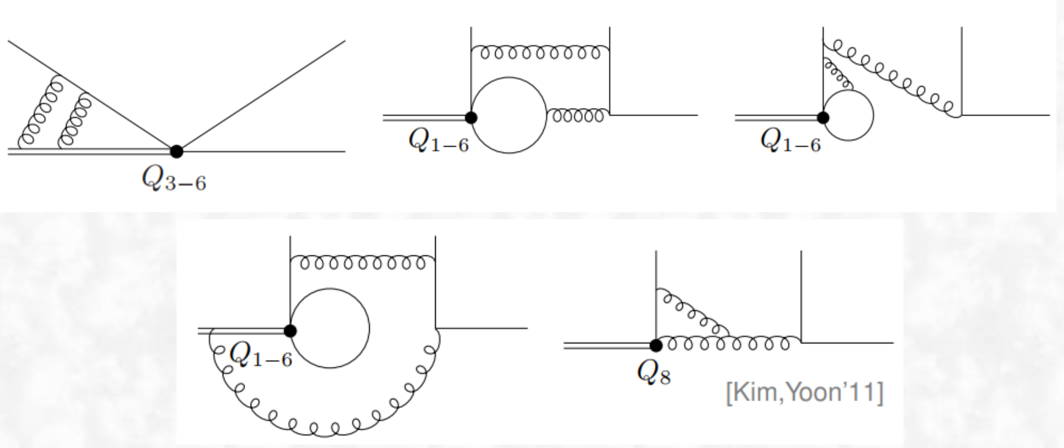


$$\begin{aligned}\frac{1}{2}\widetilde{T}_i^{(2)} = {} & \widetilde{A}_{i1}^{(2)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(1)} + Z_{ij}^{(2)}\,\widetilde{A}_{j1}^{(0)} + Z_{\alpha}^{(1)}\,\widetilde{A}_{i1}^{(1)\mathrm{nf}} \\ & + (-i)\,\delta m^{(1)}\,\widetilde{A}_{i1}^{\prime(1)\mathrm{nf}} + Z_{\mathrm{ext}}^{(1)}\,[\widetilde{A}_{i1}^{(1)\mathrm{nf}} + Z_{ij}^{(1)}\,\widetilde{A}_{j1}^{(0)}] \\ & - \frac{1}{2}\widetilde{T}_i^{(1)}\,[C_{FF}^{(1)} + \widetilde{Y}_{11}^{(1)}] - \sum_{b>1}\widetilde{H}_{ib}^{(1)}\,\widetilde{Y}_{b1}^{(1)} \\ & + [\widetilde{A}_{i1}^{(2)\mathrm{f}} - A_{31}^{(2)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}] + (-i)\,\delta m^{(1)}\,[\widetilde{A}_{i1}^{\prime(1)\mathrm{f}} - A_{31}^{\prime(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}] \\ & + (Z_{\alpha}^{(1)} + Z_{\mathrm{ext}}^{(1)})\,[\widetilde{A}_{i1}^{(1)\mathrm{f}} - A_{31}^{(1)\mathrm{f}}\,\widetilde{A}_{i1}^{(0)}] \\ & - [\widetilde{M}_{11}^{(2)} - M_{11}^{(2)}]\,\widetilde{A}_{i1}^{(0)} \\ & - (C_{FF}^{(1)} - \xi_{45}^{(1)})\,[\widetilde{Y}_{11}^{(1)} - Y_{11}^{(1)}]\,\widetilde{A}_{i1}^{(0)} - [\widetilde{Y}_{11}^{(2)} - Y_{11}^{(2)}]\,\widetilde{A}_{i1}^{(0)} \\ & - \sum_{b>1}\widetilde{A}_{ib}^{(0)}\,\widetilde{M}_{b1}^{(2)} - \sum_{b>1}\widetilde{A}_{ib}^{(0)}\,\widetilde{Y}_{b1}^{(2)}.\end{aligned}$$





about 100 Feynman diagrams



Some ingredients

- Matching relation for O_{QCD}:

$$O_{\rm QCD} \equiv \left[\bar{q}\,\frac{\not{n}_-}{2}(1-\gamma_5)q\right]\left[\bar{q}\,\not{n}_+(1-\gamma_5)b\right] = C_{FF}\,C_{\bar{q}q}\,O_1$$

$$\bar{q}\,\frac{\not{n}_-}{2}(1-\gamma_5)q = C_{\bar{q}q}\,\bar{\chi}\,\frac{\not{n}_-}{2}(1-\gamma_5)\chi\,,$$

$$\bar{q}\,\not{n}_+(1-\gamma_5)b = C_{FF}\,\bar{\xi}\,\not{n}_+(1-\gamma_5)h_v\,.$$

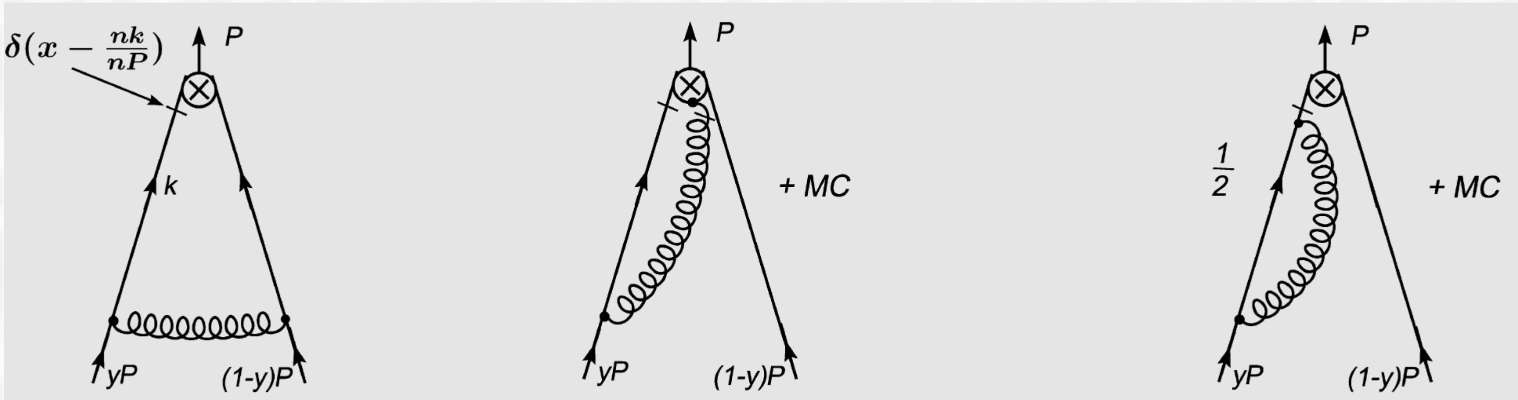
- SCET renormalization kernel $Y_{11}^{(1)}$:

$$Y_{11}^{(1)}(u,u') = Z_J^{(1)}\,\delta(u-u') + Z_{BL}^{(1)}(u,u')$$

$$Z_J^{(1)} = C_F\left\{-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[-L-\frac{5}{2}\right]\right\}$$

$$Z_Q(v,w) = \delta(v-w) + Z_Q^{(1)}(v,w) + \ldots\,,$$

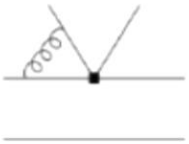
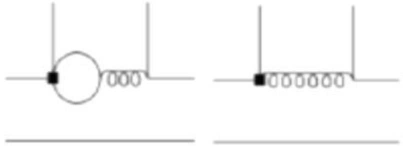
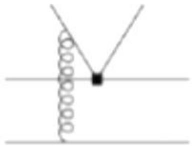
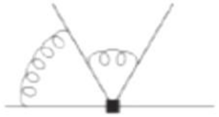
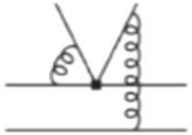
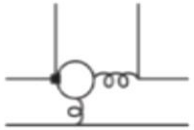
$$Z_Q^{(1)}(v,w) = -\frac{\alpha_s C_F}{2\pi\epsilon}\left\{\frac{1}{w\bar{w}}\left[v\bar{w}\frac{\theta(w-v)}{w-v} + \bar{v}w\frac{\theta(v-w)}{v-w}\right]_+ - \frac{1}{2}\delta(v-w) + \Delta\left(\frac{v}{w}\theta(w-v) + \frac{\bar{v}}{\bar{w}}\theta(v-w)\right)\right\},$$



Status of the NNLO calculation of T^I & T^{II}

- For each Q_i insertion, both **tree** & **penguin** topologies, and contribute to both T^I & T^{II}.

$$\langle M_1 M_2 | Q_i | B \rangle \simeq F^{BM_1}\, T_i^I \otimes \phi_{M_2} + T_i^{II} \otimes \phi_B \otimes \phi_{M_1} \otimes \phi_{M_2}$$

	T^I, tree	T^I, penguin	T^{II}, tree	T^{II}, penguin
LO: $\mathcal{O}(1)$		$T^I = 1 + \mathcal{O}(\alpha_s) + \cdots$		$T^{II} = \mathcal{O}(\alpha_s) + \cdots$
NLO: $\mathcal{O}(\alpha_s)$ BBNS '99-'04				
NNLO: $\mathcal{O}(\alpha_s^2)$	 Bell '07,'09 Beneke, Huber, Li '09 Huber, Krankl, Li '16	 Kim, Yoon '11, Bell Beneke, Huber, Li '15 Bell, Beneke, Huber, Li '20	 Beneke, Jager '05 Kivel '06, Pilipp '07	 Beneke, Jager '06 Jain, Rothstein, Stewart '07

Status of the NNLO calculation of T^{I} & T^{II}

- Complete NNLO calculation for T^{I} & T^{II} at leading power in QCDF/SCET now complete;
- **Soft-collinear factorization** at **2-loop** established via explicit calculations;
- For **tree amplitudes**, cancellation between T^{I} & T^{II};

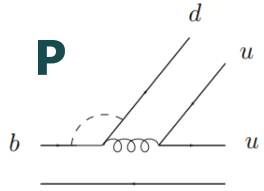
colour-allowed tree α_1 colour-suppressed tree α_2

$$\langle M_1 M_2 | Q_i | B \rangle \simeq F^{BM_1}\, T_i^{I} \otimes \phi_{M_2} + T_i^{II} \otimes \phi_B \otimes \phi_{M_1} \otimes \phi_{M_2}$$

$$\begin{aligned}\alpha_1(\pi\pi) &= 1.009 + [0.023 + 0.010\,i]_{\rm NLO} + [0.026 + 0.028\,i]_{\rm NNLO} \\ &\quad - \left[\frac{r_{\rm sp}}{0.445}\right]\left\{[0.014]_{\rm LOsp} + [0.034 + 0.027i\,]_{\rm NLOsp} + [0.008]_{\rm tw3}\right\} \\ &= 1.000^{+0.029}_{-0.069} + (0.011^{+0.023}_{-0.050})i\end{aligned}$$

$$\begin{aligned}\alpha_2(\pi\pi) &= 0.220 - [0.179 + 0.077\,i]_{\rm NLO} - [0.031 + 0.050\,i]_{\rm NNLO} \\ &\quad + \left[\frac{r_{\rm sp}}{0.445}\right]\left\{[0.114]_{\rm LOsp} + [0.049 + 0.051i\,]_{\rm NLOsp} + [0.067]_{\rm tw3}\right\} \\ &= 0.240^{+0.217}_{-0.125} + (-0.077^{+0.115}_{-0.078})i\end{aligned}$$

- For **leading-power QCD penguin amplitudes**, cancellation between $Q^{p}_{1,2}$ & $Q_{3-6,8g}$



QCD penguins α_4

$$\begin{aligned}a_4^u(\pi\bar{K})/10^{-2} &= -2.87 - [0.09 + 0.09i]_{\rm V_1} + [0.49 - 1.32i]_{\rm P_1} - [0.32 + 0.71i]_{\rm P_2,\,Q_{1,2}} + [0.33 + 0.38i]_{\rm P_2,\,Q_{3-6,8}} \\ &\quad + \left[\frac{r_{\rm sp}}{0.434}\right]\left\{[0.13]_{\rm LO} + [0.14 + 0.12i]_{\rm HV} - [0.01 - 0.05i]_{\rm HP} + [0.07]_{\rm tw3}\right\} \\ &= (-2.12^{+0.48}_{-0.29}) + (-1.56^{+0.29}_{-0.15})i\,,\end{aligned}$$

Summary

- **Matching of heavy-to-light currents from QCD onto SCET plays an important role in B-decay processes; NNLO matching coefficients now complete for various Dirac structures.**
- **NNLO QCD corrections to color-allowed, color-suppressed tree & leading-power penguin amplitudes now complete.**
- **Individual contributions sizeable, but tend to cancel each other among them, and thus NNLO shift in amplitudes is rather small.**
- **Updated phen. analyses including these NNLO corrections and progress from other sides are work in progress.**

Thank You for your attention!