

Summer School on Dark Matter and New Physics

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## Lectures on QCD

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Outline:

- 1) The origin of QCD
- 2) Lagrangian of QCD and quantization
- 3) Renormalization
- 4) QCD at colliders

## I. The Origin of QCD.

L2

QCD is the sector of the SM of particle physics that describes the strong interactions of quarks and gluons.

Basic tools to study the structure of the Nucleon and the physics at hadron colliders.

QCD is a quantum gauge field theory.

- A brief history of QFT.

QFT is a subject which has evolved considerably over the years and continues to do so. It has found applications in many other branches of science, in particular condensed matter physics and also biology and economics.

- 19<sup>th</sup>.C. Maxwell's equations : a classical field theory for electromagnetism.

- 1900 : Planck hypothesises the photon as the quantum of radiation.

- 1920s/30s : development of particle quantum mechanics.  
relativistic version has problems (negative energy).

- 1930s/40s realized that relativity + quantum mechanics  $\Rightarrow$  particles are the quanta of a quantised classical field.
- 1940s : formulation of the calculation rules for QED.  
Feynman diagrams , path integrals.
- 1950s : the understanding of how to deal with the divergences of Feynman diagrams , through renormalization.
- 1960s : How can it apply to weak + strong interactions
- 1970s : renormalization of non-Abelian gauge theories , the RG and asymptotic freedom , the formulation of the SM  
non-perturbative methods , Lattice gauge theory application to critical behavior
- 1980s : string theory , quantum gravity , conformal field theory  
All quantum field theories are just effective over some length .  
Effective field theories.  
Factorization especially for the phenomenon at colliders.
- 1990s : holography , strong / weak duality .

QCD is the fundamental theory of the strong interactions, which binds protons and neutrons inside the nucleus. It is important to form heavy elements and thus life.

1932. discovery of neutron. by Chadwick

hadrons : Greek hadros . bulky  
leptos small

1935. The first theory of strong interactions constructed by Yukawa

	$n \rightarrow \pi$
attractive and repulsive	only attractive
massless	massive $(1\text{ fm})^{-1} \sim 200 \text{ MeV.}$
weak	strong.

1947.  $\pi$  meson was discovered in cosmic rays.  
 $\pi^+, \pi^0, \pi^-$ : isovector

It correctly describes some features of nuclear interactions, but it cannot be the final truth.

### Exp. problem:

Many more mesons and baryons were discovered after

the discovery of the  $\pi$  mesons, both in cosmic rays  
and accelerator experiments. (The power of  $E=mc^2$ .)

How to describe their interactions?

A theoretical problem is the coupling constant  $g$  is

$$\frac{g^2}{4\pi} \sim 14 \text{ which is } 10^3 \text{ larger than } \frac{e^2}{4\pi} \sim \frac{1}{137}$$

At that time, only perturbation <sup>method</sup> (theory) is known. Successful in QED. How to use that to make predictions. and to compare with experiments?

1950s - 1970s: The prevailing attitude was that the entire Lagrangian method of field theory should be discarded.

The idea of the bootstrap was popular. There are no fundamental particles and no fundamental fields.

Everything depends on itself in some self-consistent way.

Now develop: The theory of the S-matrix was being developed intensely.

• twistor string theory The purely kinematical requirements of unitarity,

• soft theorem Causality, space-time and internal symmetries can be

• double copy restrictive enough to determine the form of the S-matrix.

• color/kinematic duality

Regge theory was developed in this period. It is a good

• geometric view of scattering amplitudes phenomenological effective theory for hadron scattering at

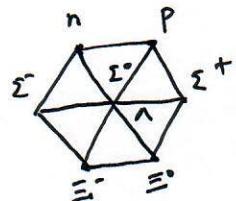
• automatic loop calculation high energies and small transverse momenta.

The main job ~~is~~ to handle the rapidly growing zoo of new "elementary" particles and resonances.  $\Rightarrow$

Build a "Mendeleev Table" of hadrons.

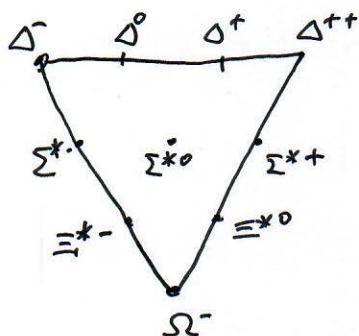
1961. It was found that all known hadrons can be grouped into some octets and decuplets representing multiplets of  $SU(3)$ , mainly by Gell-Mann.

↑  
flavor.



weight diagrams of  $SU(3)$  group

eight-fold.



$\Omega^-$  and its mass were predicted and confirmed by experiment.

1969 Nobel prize.

Question: Where are the triplets and anti-triplets, the particles belonging to the fundamental representation of  $SU(3)$ ?

Gell-Mann was asked after the seminar ~~in~~ in Columbia.

1964. Gell-Mann and Zweig made the bold suggestion that the hadrons were not simple structures but were instead built from three basic particles.

"Quarks" vs. "Aces"

$$3 \otimes \bar{3} = 8 + 1 = \text{Mesons}$$

$$3 \otimes 3 \otimes \bar{3} = 10 + 8 + \bar{8} + 1 : \text{baryons.}$$

Questions: Is a quark a fermion or a boson?

What are the charges of quarks?

Quantum numbers of observed hadrons indicate

Quarks are fermions.

$$e_u = \frac{2}{3}, \quad e_d = e_s = -\frac{1}{3}.$$

	u	d	s
iso-spin	$\frac{1}{2}$	$-\frac{1}{2}$	0
strangeness	0	0	-1
charges	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Questions: What forces bind the quarks together?

Can we observe directly quarks?

How to understand the structure of

$$\Delta^{++} (\text{uuu . S-wave , } \uparrow\uparrow\uparrow) ?$$

Pauli principle forbids it if they are real.  
spin-statistics theorem.

$(x_{12}^2)(x_{23}^2)(x_{31}^2)$   
zeros in the ground  
wavefunction leads to  
zeros in the proton  
form factors.

1965 Bogoliubov, Struminsky, Tavkhelidze proposed that the quarks should possess an additional quantum number.

1964-65. Greenberg, Nambu and Hori introduced the quantum number 'color'.

Greek Chroma

$$\epsilon^{ijk} q^i q^j q^k$$

Google Chrome

Color Confinement: Only color-singlet states are allowed to exist in Nature. No theoretical explanation yet.

Some/many theorists did not believe in the existence of quarks. They are mathematical.

Evidences: baryon spectroscopy, the magnetic moment calculation, meson-baryon/baryon-baryon  $\sigma_t$ , Zweig forbidding decays  $\phi \rightarrow p\pi$

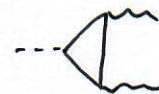
1969.  $e + N \rightarrow e + X$  at SLAC.  $\rightarrow$  Inspired by Feynman, Bjorken and Paschos discussed an intuitive but powerful model, in which the nucleon is built of 'fundamental pointlike constituents'.

'partons' called by Feynman.

It soon became clear that everything works wonderfully well if one assumes that these partons have the quantum numbers of quarks.

$$\pi^0 \rightarrow \gamma\gamma$$

$$\sigma \propto N_c^2$$



$$e^+ e^- \rightarrow \text{hadrons}$$

$$\sigma \propto N_c$$



Questions:

Yukawa model indicates that there is a strong force.

Parton model implies that the interaction between quarks is weak. How to understand these two different behaviors?

(Where ~~are~~ are gluons?). Is  $SU(3)_c$  octet representation real, just as the fundamental representation?

Field theory was dead? Not really.

1954 : Yang-Mills theory  $\mathcal{L}_Y = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \bar{\psi} i\gamma^\mu (\partial^\mu - igT^a b^a) \psi$

Non-Abelian gauge theory.

Massless gauge bosons!

→ Unrealistic.

but interesting

Many played with it. Feynman

Quantization of it is far from easy.

quantum gravity.

Violation of unitarity!

1967 : Faddeev and Popov developed their ghost method.

The theory became self-consistent and could be used to calculate scattering amplitude perturbatively.

Some tried to  
Apply Yang-Mills theory to Weak interactions.

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Fermi theory : nonrenormalizable and nonunitary

It can be considered as an EFT of a gauge theory,  
in which the gauge boson is massive.

1964. Englert - Brout - Higgs - Guralnik - Hagen - Kibble mechanism.

1967. Weinberg and Salam constructed the correct gauge theory for weak interactions, based on the model of Glashow in 1961.

1971.  
<sub>72</sub>tHooft and Veltman proved that the renormalizability and unitarity.

1973. The strong interaction can be described by Yang-Mills theory. Pati and Salam; Fritzsch, Gell-Mann and Leutwyler, and Weinberg. They suggested that the quarks interact with each other by the exchange of gluons, the octet representation of  $SU(3)_c$ .

Charge renormalization by Gross, Wilczek and Politzer.

"Asymptotic freedom" different from QED. 2004 Nobel Prize.

Dramatic history.

1969. Khriplovich calculated  $T_{\mu\nu}$  in Coulomb gauge.

the only source of the charge renormalization.

But he did not understand the meaning of his result.  
and even thought about applying Yang-Mills theory to  
strong interaction.

1972. 't Hooft calculated it, but did not realize the  
overwhelming significance of the result, so did not  
publish it.

In 1974, the dust settled. The picture became clear. QCD  
is A self-consistent theory of strong interaction.

At high energies. QCD explained SLAC deep inelastic data  
very well. "Scaling violations"

At low energies, the coupling becomes large. Many  
non-perturbative methods were developed.

Lattice QCD simulation, QCD Sum rule, chiral  
perturbation theory, Hamiltonian EFT, ...

Later, precision experimental tests of QCD, gluon jets.

## II. Lagrangian of QCD and its quantization.

In the construction of theories for the fundamental interactions forces

in Nature, the concept of symmetries (in particular, Lorentz invariance) the action principle, and quantized fields is essential.

The action  $S(\phi)$  is a functional of fields configuration, and invariant under the postulated symmetries.  $\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} = 0$

The dynamics follows from  $\delta S / \delta \phi = 0$  (Euler-Lagrange eq)

The particle content follows from the poles of Green functions of the fields.

Especially for spin-1 massless particles, the concept of fields. symmetries and particles are tied together.

Recall the case of QED.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (\not{\partial}_\mu - ie \not{A}_\mu) \psi \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu + \bar{\psi} (i\not{\partial} - m) \psi \end{aligned}$$

$\mathcal{L}$  is invariant under the gauge symmetry.

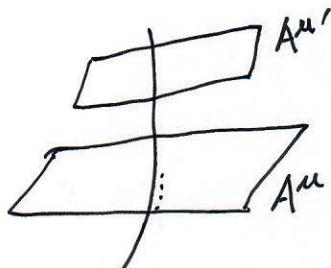
$$A^\mu(x) \rightarrow A^\mu(x) + \frac{1}{e} \partial^\mu \alpha(x)$$

$\uparrow$  local.

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x).$$

$\uparrow$  phase. we accept this in QM, where  $|\phi|^2$  is measurable.

$$\text{But. } A^\mu \rightarrow e^{i\alpha(x)} A^\mu$$



They are equivalent, which means we have used more degrees of freedom in fields than in particles.

E. Wigner.

Weinberg "Quantum theory of fields" vol. 1.

Lorentz invariance requires that all fields are constructed as direct sums of the irreducible fields that transform according to a general irreducible representation of the homogeneous Lorentz group.

(A, B). two labels.

(0, 0). scalar.

(\frac{1}{2}, 0). left-handed spinor

(0, \frac{1}{2}). right-handed spinor

$\vec{p} \xrightarrow{k} \vec{p}' \xrightarrow{R} \vec{p}$   $\int \vec{E} \cdot d\vec{p} = 0$  14  
 $(\frac{1}{2}, \frac{1}{2})$  is not a four-vector because. Coulomb gauge.  
 $\delta_{\mu\nu}^{\text{supa}}$

$$\begin{aligned}
 \stackrel{\Delta}{\triangle} \quad U(\lambda) A^\mu(x) U^{-1}(\lambda) &= (\lambda^{-1})^\mu_\nu A^\nu(\lambda x) + \partial^\mu \alpha(\lambda x) \\
 U(\lambda) A_\mu(x) U^{-1}(\lambda) &= \lambda^\nu_\mu A_\nu(\lambda x) + \partial_\mu \alpha(\lambda x) \quad \boxed{\text{Homework}}
 \end{aligned}$$

But we have no other choice, and therefore

Add constraint on  $L$ .  $L$  must be invariant under the gauge transformation above.  $\Rightarrow A^\mu$  couples to a conserved current

$$(\partial_\mu \alpha) \cdot J^\mu = \partial_\mu (\alpha J^\mu) - \alpha \underbrace{\partial_\mu J^\mu}_{=0} \quad F^{\mu\nu} F_{\mu\nu} \text{ is already invariant.}$$

One may propose to use  $F_{\mu\nu}$  rather than  $A^\mu$  to describe the spin-1 particle.

$$\begin{pmatrix} 0 & E' & E^2 & E^3 \\ 0 & B^3 & -B^2 & 0 \\ 0 & B^1 & 0 & 0 \end{pmatrix}$$

Aharanov-Bohm

• AB effect.  $A^\mu$  can have some measurable effect

•  $F^{\mu\nu}$  if  $\delta_{\mu\nu} \psi$ . due to the derivative in  $F_{\mu\nu}$ , the force drops more quickly than  $\frac{1}{r}$ . Homework

Due to the same reason, we should use  $\eta_{\mu\nu}$  to describe spin-2 massless particles.  $\eta_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ . not  $R_{\mu\nu\rho\sigma}$

We need to require  $h_{\mu\nu}$  couples to a conserved current

$$h_{\mu\nu} T^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

Since we do not know any other conservation current.

there is no spin-( $\frac{1}{2}$ ) massless particles as a mediator of long-range force.

### T.1 Non-Abelian gauge symmetry and invariant Lagrangian

Given ~~that~~ a internal symmetry with group  $G$ ,

The "matter fields" transform as

$$\psi_{\alpha a}(x) \xrightarrow{g \in G} \psi'_{\alpha a}(x) \equiv D_{ab}(g^{-1})(x) \psi_{\alpha b}(x).$$

$$D(g) = e^{-i T^A g^A} \quad A=1, \dots, \dim G.$$

$$[T^A, T^B] = i f^{ABC} T^C.$$

QED.  $f^{ABC} = 0$

$$\delta \psi_a(x) = i \epsilon_{(x)}^A T^A_{ab} \psi_b(x)$$

$\uparrow$   
local

$\bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$  is invariant under global transformations  
but not local transformations.

$$\delta(\partial_\mu \psi) = i \varepsilon^A T^A \partial_\mu \psi + i \underbrace{(\partial_\mu \varepsilon^A) T^A \psi}$$

To cancel the extra term,

$$D_{\mu,ab} = \delta_{ab} \partial_\mu - i A^A(x) \cdot T^A_{ab}$$

We require

$$\delta(D_\mu \psi) = i \varepsilon^A T^A (D_\mu \psi)$$

$$\Rightarrow \delta A_\mu^A T^A = i \varepsilon^A [T^A, T^B] A_\mu^B + \partial_\mu \varepsilon^A T^A$$

$$\text{or } \delta A_\mu^A = f^{ABC} \varepsilon^C A_\mu^B + \partial_\mu \varepsilon^A$$

Homework

3

For finite transformations,  $U = D(g^{-1}) = e^{i \varepsilon^A T^A}$

$$\psi \rightarrow U \psi$$

$$A_\mu = A_\mu^A T^A \rightarrow U A_\mu U^+ + i U [\partial_\mu U^+]$$

$$D_\mu \rightarrow U D_\mu U^+$$

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Field strength tensor

$$\text{QED. } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = i [D_\mu, D_\nu]$$

This is gauge invariant.

$$\begin{aligned} \text{QCD. } G_{\mu\nu} &\equiv G_{\mu\nu}^A T^A = i [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] \\ &= (\partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f^{ABC} A_\mu^B A_\nu^C) T^A \\ &\equiv G_{\mu\nu}^A T^A \end{aligned}$$

$$S G_{\mu\nu} = f^{ABC} \epsilon^c G_{\mu\nu}^B.$$

$$G_{\mu\nu} \rightarrow U G_{\mu\nu} U^\dagger$$

$\text{Tr}[G_{\mu\nu} G^{\mu\nu}]$  or  $G_{\mu\nu}^A G^{AMN}$  is gauge invariant

Building blocks :  $\psi, D_\mu, G_{\mu\nu}$ .

Mass-dim :  $\frac{3}{2} \quad 1 \quad 2$

$$\mathcal{L} = -\frac{1}{4} g^{AB} G_{\mu\nu}^A G^{B\mu\nu} - \frac{1}{2} \partial_{AB} \epsilon_{\mu\nu\rho\sigma} G^{AM\mu} G^{B\rho\sigma}$$

$$+ \bar{\psi} (i\cancel{D} - m) \psi.$$



For QCD. parity is not violated.

So  $\partial_{AB} = 0$ . A total derivative.

$$-\frac{1}{4} g^{AB} G_{\mu\nu}^A G^{B\mu\nu}.$$

Some facts:  $g^{AB} = g^{BA} > 0$

$$g_{AB}^{AB} f^{BCD} + g_{CB} f^{BAD} = 0$$

Homework 4

Further proof shows that

$$g_{AB} = \begin{pmatrix} g_1^{-2} & & & \\ & g_1^{-2} & & \\ & & g_1^{-2} & \\ & & & g_2^{-2} \\ & & & & \ddots & \end{pmatrix} \quad \text{for } G = G_1 \otimes \dots \otimes G_m.$$

Thus, we can use

$$-\frac{1}{4} \frac{1}{g^2} G_{\mu\nu}^A G^{A\mu\nu} \quad \text{for } SU(3)$$

Now, redefine  $A^\mu \rightarrow g A^\mu$

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu} + \bar{\psi} (i\cancel{D} - m) \psi$$

$$D_\mu = \partial_\mu - ig A_\mu^A T^A$$

$$A_\mu \rightarrow U A_\mu U^+ + \frac{i}{g} U [\partial_\mu U^+]$$

$$G_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = (\partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C) T^A$$

$g$  is the coupling.

$$\cancel{\partial}_\mu \quad \cancel{\partial}_\nu$$

all quarks are the same.

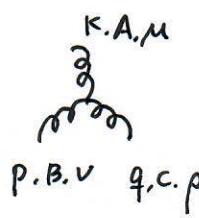
$$\mathcal{L} = -\frac{1}{2} \left( \partial_\mu A_\nu^A \partial^\mu A^{A\nu} - \partial_\mu A_\nu^A \partial^\nu A^{A\mu} \right) + \bar{\psi} (i\cancel{D} - m) \psi$$

$$+ g \bar{\psi} \gamma^\mu T^A \psi \cdot \cancel{A}_\mu^A - g f^{ABC} (\partial_\mu A_\nu^A) A^{B\mu} A^{C\nu}$$

$$- \frac{1}{4} g^2 f^{ABC} f^{ADE} A_\mu^B A_\nu^C A^{D\mu} A^{E\nu}$$

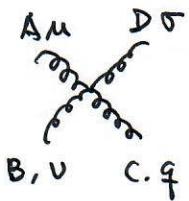


$$ig T^A \gamma^\mu$$



$$ig V_{\mu\nu\rho}^{ABC} (k, p, q)$$

$$\hookrightarrow -if^{ABC} \times [ (k-p)_\rho g_{\mu\nu} + (p-q)_\mu g_{\nu\rho} + (q-k)_\nu g_{\rho\mu} ]$$



$$-ig^2 \times [ f^{ABE} f^{CDE} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$

$$+ f^{ACE} f^{BDE} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho})$$

$$+ f^{ADE} f^{BCE} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) ]$$

$$a \xrightarrow[p]{} b \quad \frac{i(\cancel{D} + m)}{p^2 - m^2 + i\epsilon} \delta_{ab}$$

~~~~~

The usual procedure of inverting the differential operator of the quadratic terms fails, because the inverse does not exist.

## II. 2

Quantization of gauge theories with Faddeev-Popov method

$\mathcal{L} \rightarrow$  identify the canonical variables and momenta  $\rightarrow \partial\mathcal{L} \rightarrow$  Hamilton path integral

$\rightarrow$  reintroduce  $A^A_0$  as auxiliary integration variable  $\rightarrow$  integrate over canonical momenta  $\rightarrow$  Lagrange path integral.

## II. 2. 1

Path integral in axial gauge

identify the canonical variables

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu}$$

$$\Pi_\mu^A = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^A)} = G_{\mu 0}^A \Rightarrow \Pi_0^A = 0 \Rightarrow A_0^A \text{ is not a canonical variable}$$

Euler-Lagrange eq for  $A_0^A$

We see this before from gauge Lorentz representation.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_0^A)} = \frac{\partial \mathcal{L}}{\partial A_0^A} \Rightarrow \underbrace{\partial^0 \Pi^{A0} - \partial^i \Pi^{Ai}}_{=0} + g \Pi^{Bi} f^{ABC} A^{Ci} = 0$$

$$\vec{\nabla} \cdot \vec{\Pi}^A + g f^{ABC} \vec{\Pi}^B \cdot \vec{A}^C = 0 \quad \vec{\nabla} \cdot \vec{E} = 0 \text{ for QED}$$

- Gauge-fixing

$A^\mu(x)$  or  $+ \partial^\mu \alpha(x)$   
 $A^\nu(y)$

This unphysical freedom should not propagate!  
degree of freedom  
Fix the gauge!

Coulomb gauge  $A^0 = 0$   
axial gauge  $A^3 = 0$

Choose  $A^3 = 0$

Can be done always!

Homework 5.

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The canonical coordinate fields are

$A^{A\mu}(x), \mu = 1, 2$ . [ corresponding to the  $\pm 1$  helicity state, physical gauge ]

$$\pi^{A3} = G^{A30} = \partial^3 A^{A0}$$

$$-(\partial^3)^2 A^{A0} + -\partial^j \pi^{Aj} + g f^{ABC} \pi^{Bj} A^{Cj} = 0, j=1, 2$$

$A^{A0}$  can be solved in terms of the canonical fields.

Now the Hamilton path integral formula is

$$\begin{aligned} & \langle S | T [O_a(x_a) O_b(x_b) \dots] | S \rangle \\ &= |N|^2 \int D[A^{Aj}] D[\pi^{Aj}] e^{i \int d^4x (-\pi^{Aj} \dot{A}^j - \mathcal{H})} \\ & \quad O_a(x_a) O_b(x_b) \dots \\ & \mathcal{H} = \frac{1}{2} \pi^{Aj} \pi^{Aj} - \pi^{Aj} (\partial^j A^{A0} - g f^{ABC} A^{B0} A^{Cj}) \\ & \quad + \frac{1}{2} G^{Ai} G^{Aj} - \frac{1}{2} \partial^3 A^{A0} \partial^3 A^{A0}. \quad A^{A0} \text{ is fixed} \end{aligned}$$

Consider

$$M = \int D[A^{Aj}] D[A^{A0}] D(\pi^{Aj}) e^{i \int d^4x (-\pi^{Aj} \dot{A}^j - \mathcal{H}')}$$

Here  $\mathcal{H}' = \mathcal{H}$  with  $A^{A0}$  is not fixed.  $\mathcal{H}'$  is quadratic in  $A^{A0}$

$$\int_{-\infty}^{+\infty} dx e^{-x^2 + 2x} = \int dx e^{-(x-1)^2 + 1} = \left[ \int dx e^{-(x-1)^2} \right] \cdot e^1.$$

$$1 = -\frac{1}{2} + 2 \cdot 1$$

$$\text{So } \frac{\partial H'}{\partial A^{A_0}} = (\partial^2)^2 A^{A_0} + \partial^j \pi^{Aj} - g f^{ABC} \pi^{Bj} A^{Cj} = 0 \quad 122$$

~~→~~ ⇒ the same  $A^{A_0}$  as before.

$$\text{So } \langle S | T [O_a(x_a) O_b(x_b) \dots] | S \rangle = M$$

Similarly,  $H'$  is quadratic in  $\pi^{Aj}$

$$\frac{\partial H'}{\partial \pi^{Aj}} = -\pi^{Aj} + G^{Ai0}$$

We can change the exponent to  $- \frac{1}{4} G^{Am} G_{\mu\nu}^A \Big|_{A_3^A=0}$

Hence,

$$\begin{aligned} & \langle S | T [O_a(x_a) O_b(x_b) \dots] | S \rangle \\ &= |N|^2 \cdot \int D[A^{\mu\nu}] \delta(A^{A_3}) \cdot e^{i \int d^4x \cdot L} O_a(x_a) O_b(x_b) \dots \end{aligned}$$

This is the standard form  $\int D[\dots] e^{iS}$  except for the gauge-fixing  $\delta$ -function.

- Faddeev-Popov method and ghosts.  $\rightarrow \delta(A_\epsilon^{A_3})$

We generalize the above procedure.  $\rightarrow \delta[f^A - F^A] \det M$

$$\int D[A^{A_3}] \delta(A^{A_3}) \dots$$

$$\rightarrow \int D[f^A] \delta(f^A - F^A) \dots \xrightarrow{\text{a fixed function.}}$$

↳ is a function of  $A^A$

$$\rightarrow \int D[\epsilon^A] \cdot \delta(f^A[A_\epsilon] - F^A) \cdot \det \left. \frac{\delta f^A[A_\epsilon]}{\delta \epsilon^B} \right|_{\epsilon=0} \dots$$

↑  
Drop.  
M<sup>AB</sup>[A]

$$\langle \Omega | T(O_a(x_a) O_b(x_b) \dots) | \Omega \rangle$$

$$= |N|^2 \int D[A^{AB}] \delta(f^A[A_\epsilon] - F^A) \det M[A] e^{i \int d^4x L} O_a O_b \dots$$

$$\int D[F] \cdot GCF \delta(f^A - F^A) = G[f[A_\epsilon]]$$

$$\text{Choose } G[f] = e^{i \int d^4x (-\frac{1}{2\xi} f^A(x) f^A(x))}.$$

$$\langle \Omega | T(O_a(x_a) O_b(x_b) \dots) | \Omega \rangle$$

$$= |N|^2 \cdot \int D[A^{AB}] \det M[A] \cdot e^{i \int d^4x (L - \frac{1}{2\xi} f^A f^A)} O_a O_b \dots$$

The gauge-fixing term can be interpreted as an additional term in the Lagrangian

$$L_{GF} = -\frac{1}{2\xi} f^A f^A$$

For det.

$$\det \left. \frac{\delta f^A(x)}{\delta \epsilon^B(y)} \right|_{\epsilon=0} = \int D[c^A, \bar{c}^A] e^{\int d^4x d^4y \bar{c}^A(x) \frac{\delta f^A(x)}{\delta \epsilon^B(y)}} c^B(y)$$

↑      ↑  
ghost    anti-ghost  
real. Grassmann fields, anticommuting.

$$L_{FP} = \bar{c}^A (-i M^{AB}) c^B$$

Covariant gauge.

$$f^A = \partial_\mu A^{A\mu}$$

$$\frac{\delta f^A[A_\xi](x)}{\delta \varepsilon^B(y)}|_{\xi=0} = (\partial_\mu \partial^\mu \delta^{AB} + g f^{ACB} \partial_\mu A^{C\mu}) \int \delta^{(4)}(x-y)$$

Define.  ~~$\bar{c}^A$~~   $\bar{c}^A \rightarrow c^A$

$$\mathcal{L}_{FP} = \partial_\mu \bar{c}^A \partial^\mu c^A - g f^{ABC} (\partial_\mu \bar{c}^A) c^B A^{C\mu}$$

$$\tilde{\mathcal{L}} = -\frac{1}{2} G_{\mu\nu}^A G^{A\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{A\mu})(\partial^\nu A^{A\nu}) + \mathcal{L}_{FP}$$

$$A_\mu \xrightarrow[k]{\sim} B_\nu \quad \delta^{AB} \frac{i}{k^2 + i\varepsilon} \cdot \left( -g_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right)$$

$$A \xrightarrow[k]{\sim} B \quad \delta^{AB} \frac{i}{k^2 + i\varepsilon}$$

$$B \xrightarrow[k]{\sim} A \quad g f^{ABC} \rho^\mu$$

### III. Renormalization in Field theory

Renormalization is a procedure that deal with the UV divergences appearing in higher-order <sup>perturbative</sup> calculations.

It is a formal manipulation, part of the definition of the QFT, which allows one to calculate finite, testable expectation values (cross sections).

From the view point of EFT, renormalization is a procedure that allows one to calculate the physical observables at low energies, independent of how it is corrected at high energies.

Making use of the Feynman rules of QCD, one can calculate the amplitudes/cross sections of any scattering process.

$$M = M_0 + \frac{ds}{4\pi} M_1 + \left(\frac{ds}{4\pi}\right)^2 M_2 + \dots$$

$\downarrow$        $\downarrow$        $\downarrow$   
 LO      NLO      NNLO

loop integrals . see LLY's lecture.

UV divergences when  $k \rightarrow \pm\infty$ , a feature of the local field theory

To recover the prediction ability of QCD, renormalization of the theory is needed.

A theory is renormalizable if all divergences can be removed by renormalization of a finite number of couplings in the Lagrangian.

In QCD, any observable  $O$  is a function of  $m_j, g_s$ .  $O(m_j, g_s)$ . Since it is measurable,  $O(m_j, g_s)$  is finite.

Then, it is required that  $m_j, g_s$  are also infinite large<sup>so</sup> that they cancel the divergences in the loop integrals.

It seems weird to use infinite large parameters in the calculations. However, if the divergence is universal, the structure of infinities in the loop integrals is fixed, then QCD is still a predictive theory of power, after redefinitions of a finite number of parameters. origin of UV divergences:

All fundamental particles are point particles and the interactions are local. If a electron has a finite<sup>but small</sup> radius, the electric potential energy inside the electron would be so large that it becomes unstable. As a result, we assume require that the electron is point-like. As such, from the uncertainty principle, the energy inside the electron, reflected by the mass, is infinite.

## Two ways to perform renormalization

- bare parameter renormalization

Use bare Lagrangian and its Feynman rules to calculate a observable  $O_1(m_j, g_s)$ . The loop integral is divergent.

One can use some "regularization" techniques to represent such divergences, such as  $\ln^n\left(\frac{\Lambda}{m_j}\right)$  or  $\zeta_n$ .

$\uparrow$   $\uparrow$   
 cut off reg. dimensional.  
 $\alpha = 4 - 2\epsilon$

Two other observables  $O_2(m_j, g_s)$  and  $O_3(m_j, g_s)$  are supposed to be calculated before  $O_1(m_j, g_s)$  in order to extract the renormalized (physical)  $m_j^R$  and  $g_s^R$

$$m_j^R = m_j + C + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right) + \dots$$

$$g_s^R = g_s + C + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right) + \dots$$

Then replacing  $m_j, g_s$  in  $O_1(m_j, g_s)$  with  $m_j^R, g_s^R$ .

One finds all divergences cancel out and gets a finite  $O_1^R(m_j^R, g_s^R)$

The cancellation of divergences takes place order by order in  $\alpha_s$ .

- Bogoliubov - Parasiuk - Hepp - Zimmermann renormalization scheme.

Since the observables are finite, it is more natural that the parameters they depend on are finite. Therefore, one can use the renormalized parameters such as  $m_j^R$ ,  $g_s^R$  in the Lagrangian directly and any observable is just a function of the renormalized parameters.

However, this can be achieved at the cost of adding more interaction terms in the Lagrangian. Explicitly, the fields in the bare Lagrangian should be redefined as

$$q_j = z_{2,j}^{1/2} q_{j,R}$$

$$A^\mu = z_3^{1/2} A_R^\mu$$

$$\psi^a = z_2^{c/2} \psi_R^a$$

The original Lagrangian

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^R + \mathcal{L}_{\text{QCD}}^{\text{C.T.}}$$

with  $\mathcal{L}_{\text{QCD}}^R = \mathcal{L}_{\text{QCD}} (m_j \rightarrow m_j^R, g_s \rightarrow g_s^R)$ . The 'R' in new fields is dropped

$$\begin{aligned} \mathcal{L}_{QCD}^{C.T.} = & -\frac{1}{4} \delta_3 (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)^2 + \sum_j \bar{q}_j (i \delta_2^j \not{\partial} - \delta_m^j) q_j \\ & - \delta_2^c \bar{c}^a \partial^2 c^a \\ & + \sum_j g_s^R \delta_1^j A_\mu^\alpha \bar{q}_j \gamma^\mu q_j - g_s^R \delta_1^{3g} f^{abc} (\partial_\mu A_\nu^\alpha) A_\mu^b A_\nu^c \\ & - \frac{1}{4} g_s^{R^2} \delta_1^{4g} (f^{eab} A_\mu^a A_\nu^b) (f^{ecd} A_\mu^c A_\nu^d) - g_s^R \delta_1^F f^{abc} \bar{c}^a \partial^\mu A_\mu^b c^c \end{aligned}$$

$$\delta_2^j = z_{2,j}^{-1}, \quad \delta_3 = z_3^{-1}, \quad \delta_2^c = z_2^c - 1, \quad \delta_m^j = z_{2,j} m_j - m_j^R$$

$$\delta_1^j = \frac{g_s}{g_s^R} z_{2,j} z_3^{1/2} - 1, \quad \delta_1^{3g} = \frac{g_s}{g_s^R} z_3^{3/2} - 1$$

$$\delta_1^{4g} = \frac{g_s^2}{g_s^{R^2}} z_3^2 - 1, \quad \delta_1^F = \frac{g_s}{g_s^R} z_2^c z_3^{1/2} - 1$$

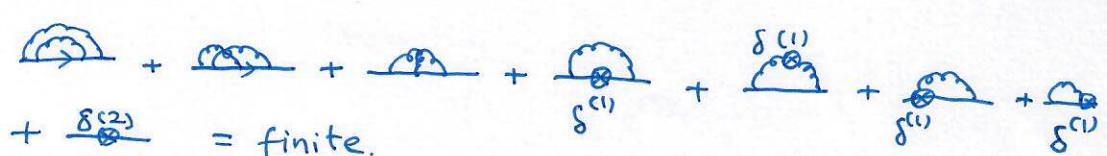
All the divergences in  $\mathcal{L}_{QCD}$  are implicitly incorporated in  $\mathcal{L}_{QCD}^{C.T.}$

Specifically,  $\delta_2^j, \delta_2^c, \delta_m^j, \delta_3$  cancel the divergences in the propagators of quarks, ghosts, gluons.  $\rightarrow + \text{cloud} + \rightarrow \otimes \rightarrow = \text{finite}$

$\delta_1^j, \delta_1^{3g}, \delta_1^{4g}, \delta_1^F$  cancel the divergences in vertices.



At 2-loops,



$$\overbrace{\text{---}} + \overbrace{\text{---}} + \overbrace{\text{---}} + \overbrace{\text{---}} = [\overbrace{\text{---}} + \overbrace{\text{---}}][\overbrace{\text{---}} + \overbrace{\text{---}}]$$

$$\rightarrow + \text{loop} + \text{---} = \text{finite.}$$

↑

Different schemes.  $m_j^R, g_s^R$  different  
Renormalization schemes.

Note that. the same scheme should be used when comparing  
the predictions on two observables.

The absolute value of a single observable is meaningless.

It's the relation among observables that is predictable and physical.

The building is 100 meter high.

Voltage 10 V.

The most used are the modified minimal subtraction ( $\overline{\text{MS}}$ )

and on-shell renormalization schemes.

The relation between different schemes are universal. It's  
easy to transform the result from one scheme to another.

Perkin Chap. 16.

After calculating the counterterms in QCD, (see LLY's Lecture)

the running behaviors of the renormalized parameters  $m_j^R, g_s^R$   
as a function of the scale are known.

The content of a hadron is different when measured  
by particles with different energy.

$$\beta(g_s^R) \equiv \frac{d g_s^R}{d \ln \mu} = g_s^R \frac{d}{d \ln \mu} \left[ -\ln(1+\delta_1^j) + \ln(1+\delta_2^j) + \frac{1}{2} \ln(1+\delta_3) \right]$$

The scale  $\mu$  is a result of choosing  $\overline{\text{MS}}$  scheme.

$$g_s^R \mu^\epsilon \quad g_s^R \mu^{2\epsilon} \cdot d \quad \alpha = 4 - 2\epsilon. \quad g_s^R \text{ still massless.}$$

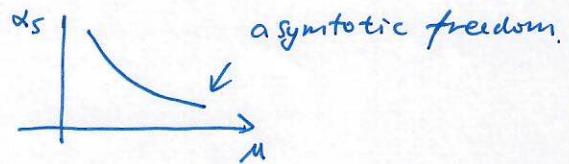
At one-loop,

$$\begin{aligned} \beta(g_s^R) &= -\frac{g_s^{R3}}{(4\pi)^2} \left( \frac{11}{3} C_A - \frac{4}{3} n_f T_F \right) = -\frac{g_s^{R3}}{3(4\pi)^2} \cdot (33 - 2n_f) \\ &= -\frac{g_s^R \alpha_s}{4\pi} \beta_0. \end{aligned}$$

$C_A = 3$ . the Casimir operator of the adjoint representation in  $SU(3)_c$ .

$n_f$  is the number of active quarks.

$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3} n_f. \\ &= 11 - \frac{2}{3} \times 6 > 0 \end{aligned}$$



Solve the RGE. of  $g_s$ .

$$\begin{aligned} \alpha_s(\mu) &= \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{4\pi} \beta \cdot \ln \frac{\mu^2}{\mu_0^2}} \quad \mu \rightarrow \infty, \quad \alpha_s(\mu) \rightarrow 0 \\ & \quad \mu \rightarrow \Lambda_{QCD}, \quad \alpha_s(\mu) \rightarrow \infty \end{aligned}$$

Perturbative QCD is not applicable anymore. below  $\Lambda_{QCD}$ .

## IV QCD at colliders.

When two high energetic particles A and B collide, a large number of final-state particles can be produced. The differential cross section for the specific process  $A(p_A) + B(p_B) \rightarrow X_1(p_1), X_2(p_2), \dots, X_n(p_n)$  is given by

$$(2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f) M(p_A, p_B \rightarrow p_1, p_2, \dots, p_n)$$

$$= \lim_{t_0 \rightarrow \infty(1-i\epsilon)} \left( S P_A P_B - P_B | T \left( \exp \left[ -i \int_{-t_0}^{t_0} dt H_I(t) \right] \right) | P_A P_B >_0 \right)_{C.A.}$$

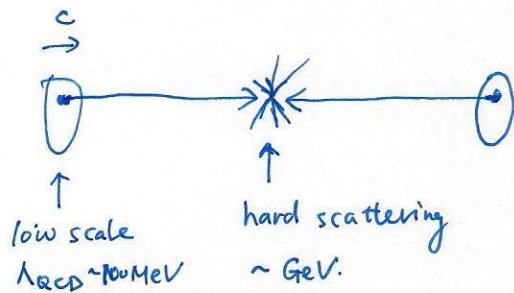
↓

$$1 - i \int_{-t_0}^{t_0} dt H_I(t) + \dots$$

Given that the QCD coupling is small.

### IV.1 Factorization.

At hadron colliders, the initial states ~~is~~<sup>are</sup> protons. The parton model tells us the main contribution comes from the parton (quarks gluons) interactions in the hard scattering process.



The initial state quarks/gluons are confined inside the proton.

We need a function to describe the state.

Take the Drell-Yan process as an example

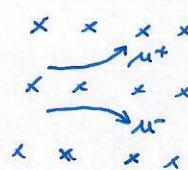
$$p(P_A) + \bar{p}(P_B) \rightarrow \mu^+(p_1) + \mu^-(p_2) + X$$

The four momenta of the final-state  $\mu^+, \mu^-$  can be measured

$$Q^2 = (p_1 + p_2)^2$$

rapidity  $y = \frac{1}{2} \ln \frac{(p_1 + p_2) \cdot P_A}{(p_1 + p_2) \cdot P_B}$

$$v \propto \frac{v}{B}$$



$$\frac{d\sigma}{dQ^2 dy} = \sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B f_{a/p}(\xi_A, \mu) f_{b/p}(\xi_B, \mu) \times H_{ab}\left(\frac{\xi_A}{x_A}, \frac{\xi_B}{x_B}, Q, \mu, \alpha_S(\mu)\right) + O\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$$

$$x_A = e^y \sqrt{\frac{Q^2}{S}}, \quad x_B = e^{-y} \sqrt{\frac{Q^2}{S}}, \quad \sqrt{S} \text{ collider energy.}$$

$f_{a/p}(\xi, \mu)$  denotes the probability to find a parton 'a' with a momentum fraction  $\xi$  in the proton, parton distribution function (PDF)

$\mu$  is the factorization scale / renormalization scale.

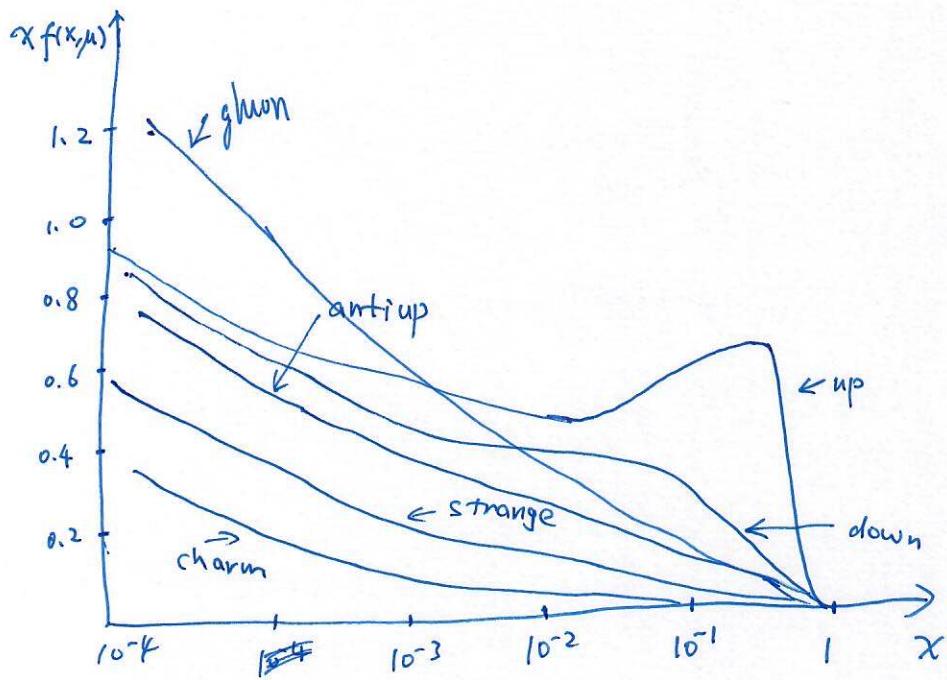
$$f_{q/p}(\xi, \mu) \equiv \frac{1}{4\pi} \int dx^- e^{-i\xi p^+ x^-} \langle P | \bar{\psi}(0, x^-; \omega_L) \gamma^+ G \psi(0, 0, \omega_L) | P \rangle$$

$$f_{q/p}(\xi, \mu) = \frac{1}{2\pi \xi p^+} \int dx^- e^{-i\xi p^+ x^-} \langle P | F_a(0, x^-, \omega_L)^{+\nu} G_{ab} F_b(0, 0, \omega_L)_\nu^+ | P \rangle$$

$$P^\pm = (P^0 \pm P^3)/\sqrt{2}, \quad G = P^0 \exp\left[ig \int_0^{x^-} dy^- A_c^+(0, y^-, \omega_L) t_c\right]$$

The PDF satisfies the RGE

$$\frac{d}{d \ln \mu} f_{q/p}(\xi, \mu) = \underbrace{\sum_b \int_\xi^1 \frac{d\xi}{\xi} P_{a/b}(\xi, \alpha_S) f_{b/p}\left(\frac{\xi}{\xi}, \mu\right)}_{\int d\xi \int dx^- P_{a/b}(\xi, \alpha_S) f_{b/p}(x, \mu) \delta(\xi x - \xi)}$$



obtained by global fitting the existing experimental data.

#### IV. 2. Infrared safety.

Higher order QCD corrections  $\left\{ \begin{array}{l} \text{real} \\ \text{virtual.} \end{array} \right. \begin{array}{l} \text{IR} \\ \text{UV + IR} \end{array}$  divergences

The UV divergences are canceled after renormalization.

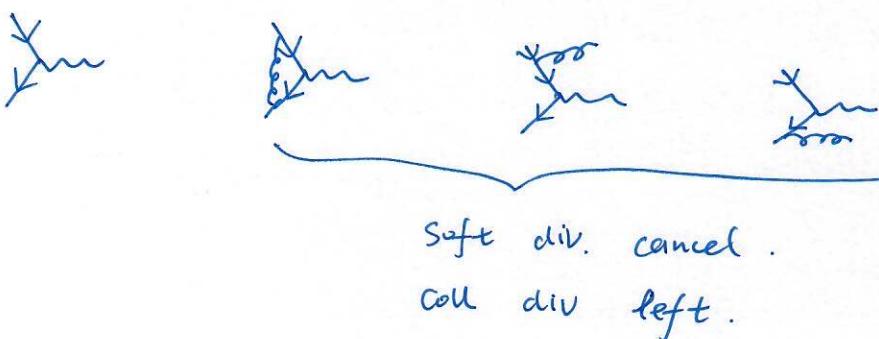
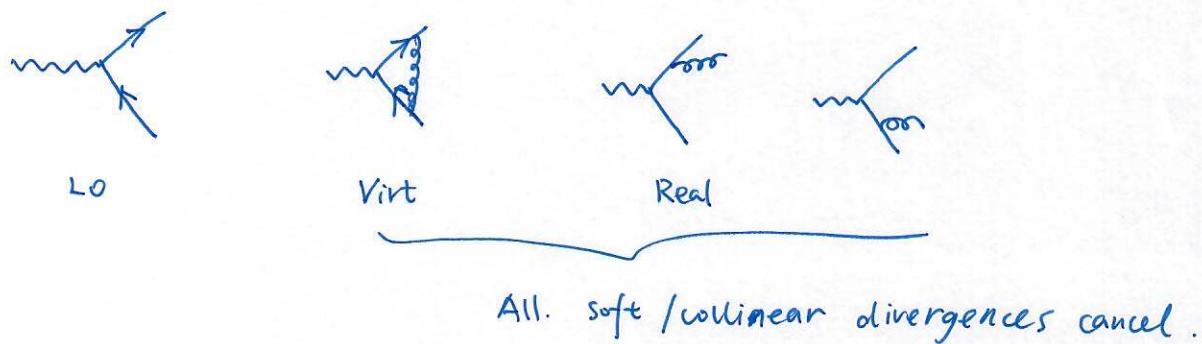
In QED, the electron is massive. The IR divergences come from soft gluons. Actually, a single electron and  $|e\gamma_s\rangle_{le}$  can not be distinguished, they form degenerate states of the Hamiltonian. The sum of all degenerate states contains no soft divergences.

In QCD, the situation is more complicated because of additional collinear divergences.  $|q\rangle$  and  $|q g_c\rangle$  also consist a degenerate state of the Hamiltonian.

## Kinoshita - Lee - Nauenberg (KLN) Theorem:

All the IR divergences cancel out in the sum of all degenerate initial and final states.

In practice, for processes at hadron colliders, the collinear divergences of the virtual and real corrections do not cancel completely since the momenta of the initial partons are constrained. However, the left collinear divergence is universal and can be absorbed by renormalization of the PDF.



Any infrared soft observable should not depend on the number of the soft and collinear particles, insensitive to the emission of soft and collinear particles.

$$(1) \quad F_J^{n+1} (p_1, \dots, p_{j-1}, p_j = \lambda q, p_{j+1}, \dots, p_{n+1})$$

$$\rightarrow F_J^n (p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}) \quad \text{if } \lambda \rightarrow 0.$$

$$(2) \quad F_J^{n+1} (p_1, \dots, p_i, \dots, p_j, \dots, p_{n+1}) \rightarrow$$

$$\rightarrow F_J^n (p_1, \dots, p_i, \dots, p_{n+1}) \quad \text{if } p_i \rightarrow z p, p_j \rightarrow (1-z)p$$

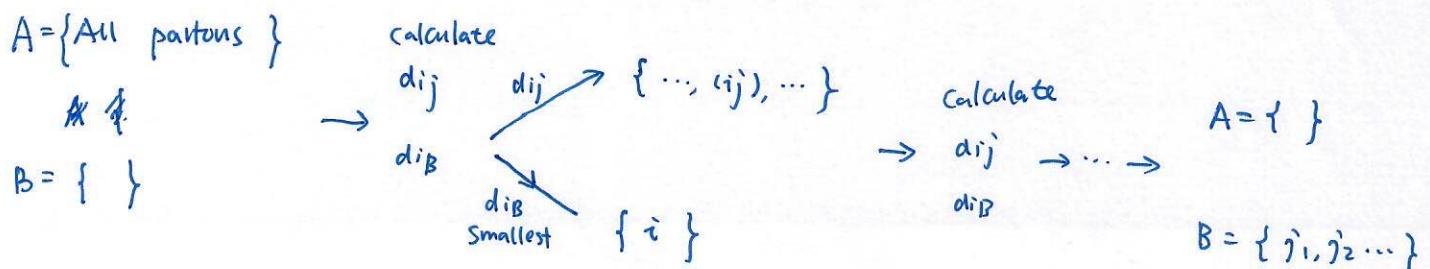
$$(3) \quad F_J^n (p_1, \dots, p_n) \rightarrow 0 \quad \text{if } p_i \cdot p_j \rightarrow 0.$$

An example,  $e^+e^- \rightarrow \text{hadrons}$ . total cross section.

Jet algorithm for jet processes.

$$\left\{ \begin{array}{l} d_{ij} = \min(k_{T,i}^2, k_{T,j}^2) \cdot \frac{\Delta_{ij}^2}{R^2}, \quad \Delta_{ij}^2 = (x_i - x_j)^2 + (\phi_i - \phi_j)^2 \\ d_{iB} = k_{T,i}^2 \end{array} \right. , \quad R = 0.4, 1.$$

$$p = \begin{cases} 1 & k_T \\ 0 & \text{Cambridge/Aachen} \\ -1 & \text{anti-}k_T. \end{cases} \quad \text{all IR safe.}$$



$i, j \rightarrow (ij)$

- $P_i, P_j \rightarrow P_{ij}^{\mu} = P_i^{\mu} + P_j^{\mu}$  massive jet
- $P_i, P_j \rightarrow E_{t,j} = E_{ti} + E_{tj}$   
 $\eta = \frac{E_{ti}}{E_t} \cdot \eta_i + \frac{E_{tj}}{E_t} \cdot \eta_j$  massless jet  
 $\phi = \frac{E_{ti}}{E_t} \phi_i + \frac{E_{tj}}{E_t} \phi_j$

At the end of the jet clustering, apply a  $P_T^j \geq 30 \text{ GeV}$  cut.

$$|\eta_j| < 4.$$

Very soft jet is excluded.

Fastjet package can be used in practice.