Physics in the Early Universe

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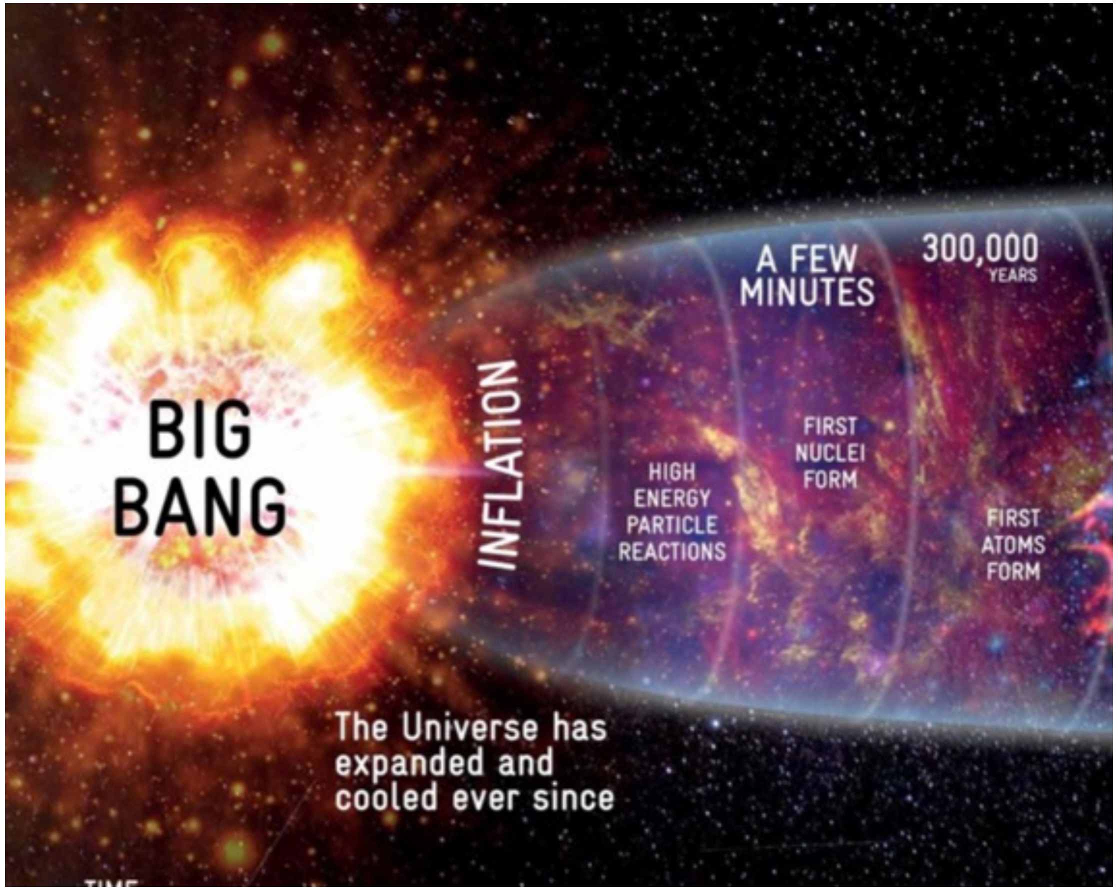
Outline

- The Smooth, Expanding Universe
  - General relativity
  - Freedmann Equations
  - Evolution of Energy
- Beyond Equilibrium
  - Boltzmann Equation
  - Big Bang Nucleosynthesis
  - Recombination
- Inflation
  - Motivations
  - Quantum fluctuations



References

- “Modern Cosmology”, Scott Dodelson
- “Tasi Lectures on Inflation”, Daniel Baumann, 0907.5424

The Smooth, Expanding Universe

- Einstein Equation: $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Ricci tensor   Ricci scalar   Energy-momentum tensor

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

Christoffel symbol

$$\Gamma^{\mu}_{\alpha\beta} = -\frac{1}{2}\left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right)$$

metric

What is metric?

- The metric defines the distance.

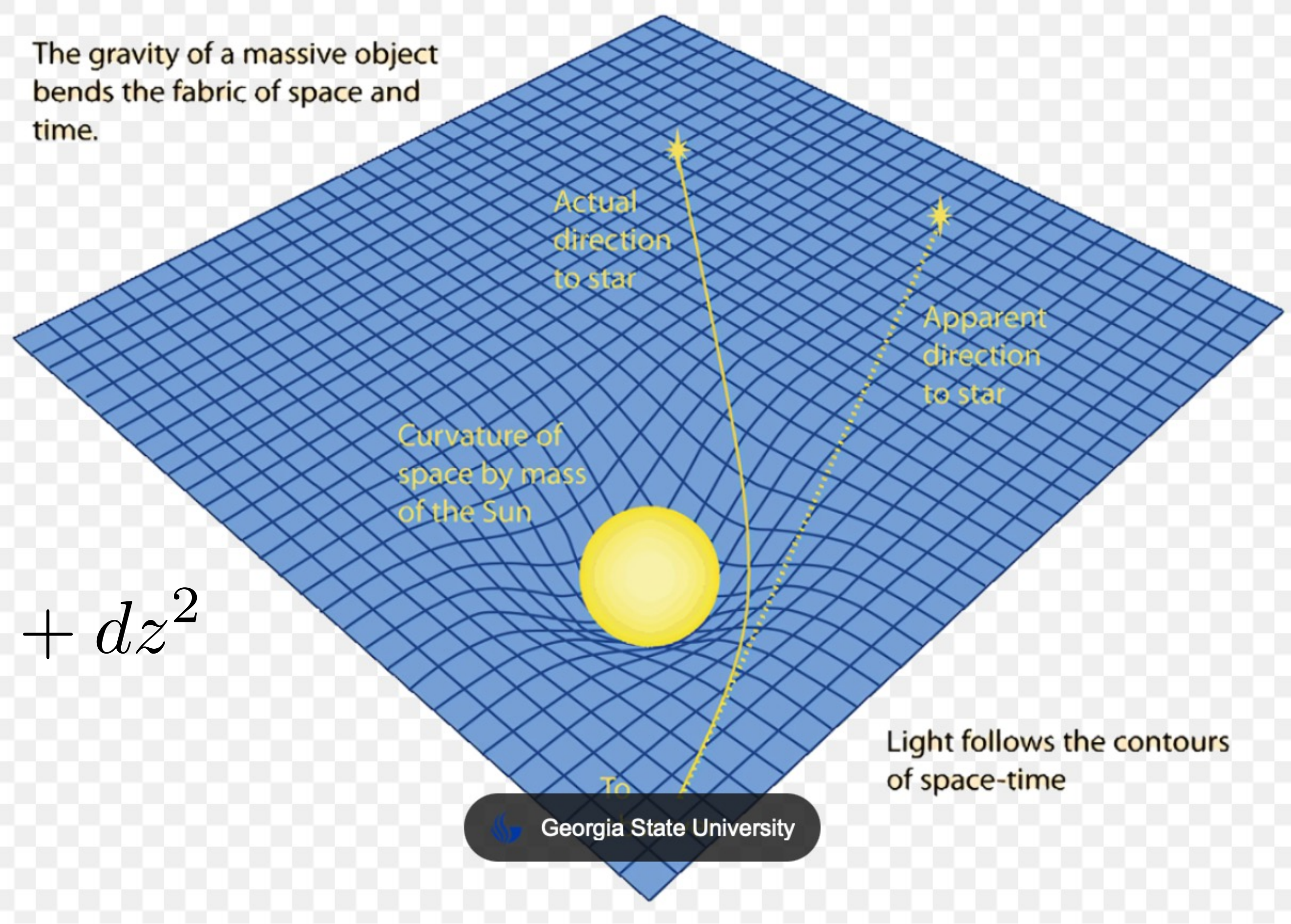
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- In Minkowski

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

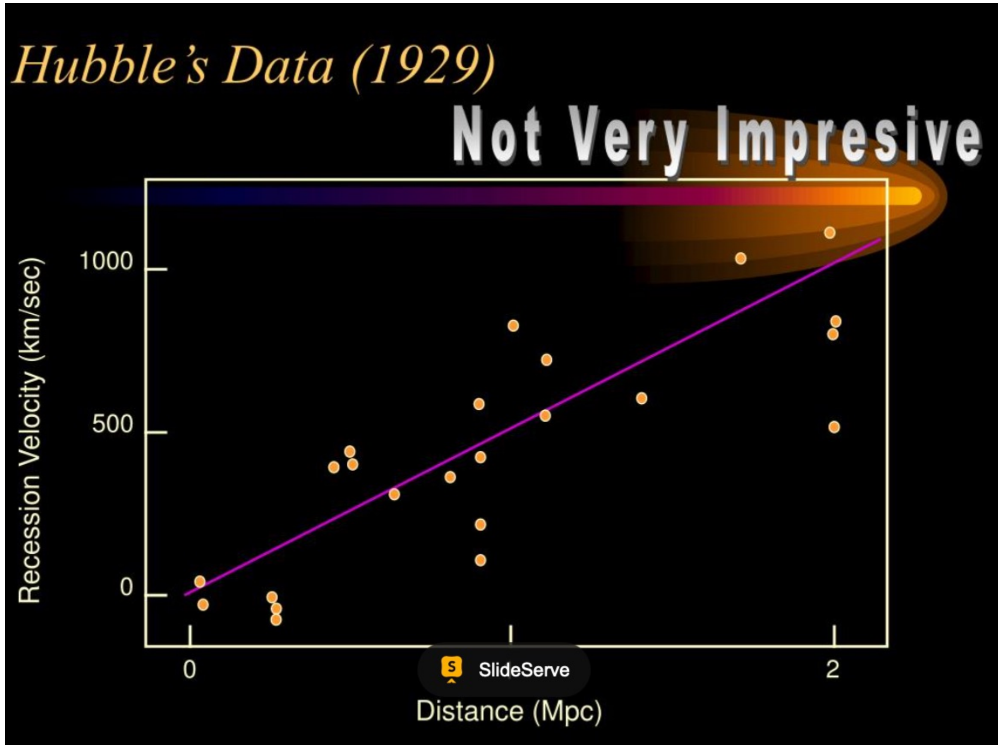
- The geodesic line

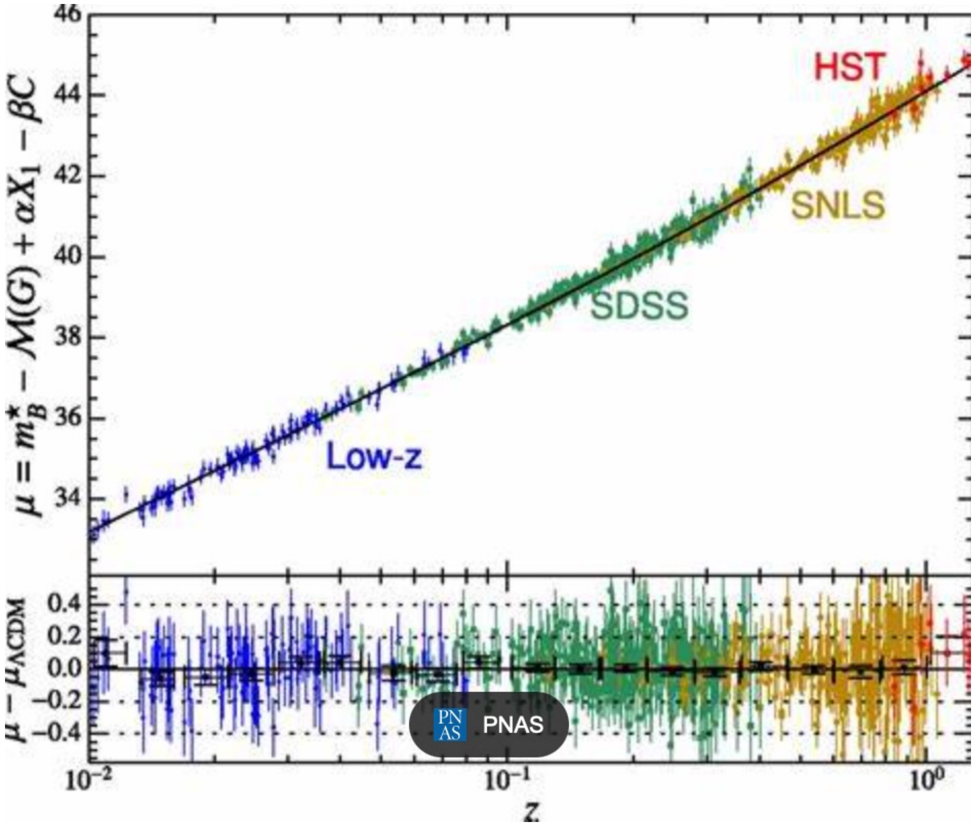
$$\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$



Hubble’s Law

$$v = Hd$$





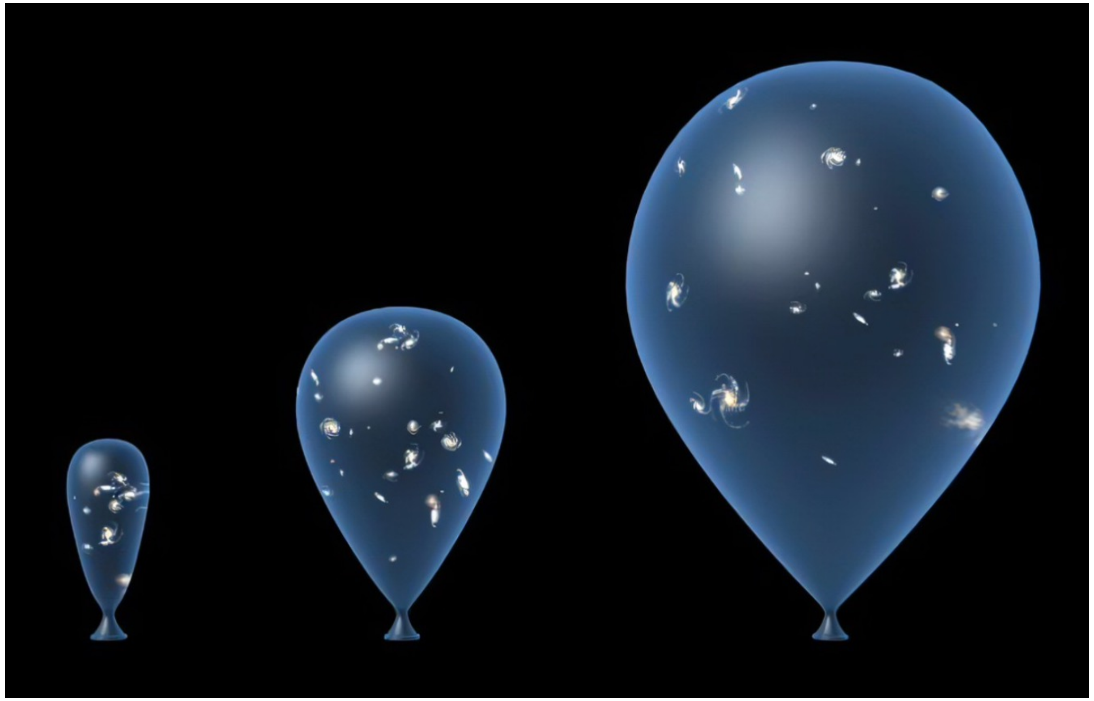
Homogeneous and Isotropic, expanding Universe

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

$$H = \frac{\dot{a}}{a}$$

$$k = \begin{cases} 1 & \text{Closed} \\ 0 & \text{Flat} \\ -1 & \text{Open} \end{cases}$$



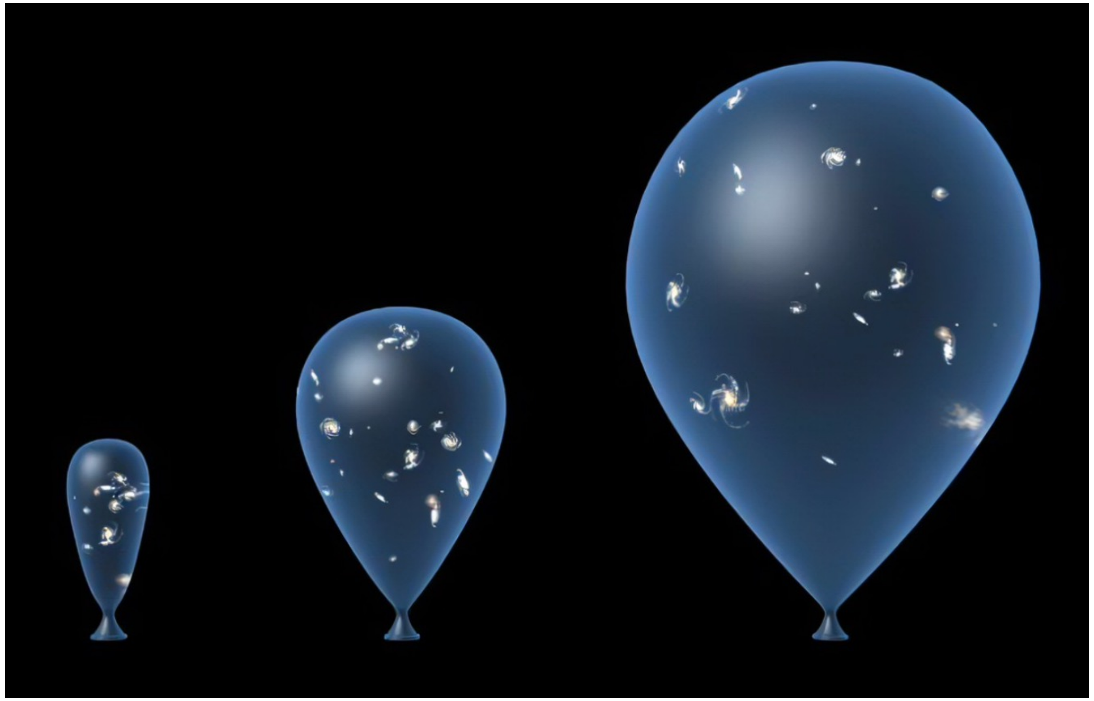
Homogeneous and Isotropic, expanding Universe

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

FRW metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

$$k = \begin{cases} 1 & \text{Closed} \\ 0 & \text{Flat} \\ -1 & \text{Open} \end{cases}$$



Friedmann Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu}$$

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

Ideal gas model

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

00:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

Evolution of Energy

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

Energy momentum conservation:

$$T^\mu_{\nu;\mu} = 0 \qquad T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0$$

$\nu = 0$ component:

$$\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}\left[3\rho + 3\mathcal{P}\right] = 0$$

The Evolution of the Universe

- We want to calculate: $a(t), \rho(t), P(t)$
- The equations we have:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad \frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}\left[3\rho + 3\mathcal{P}\right] = 0$$

- We need one more equation.

  The relation between ρ and P.

Cosmic Inventory

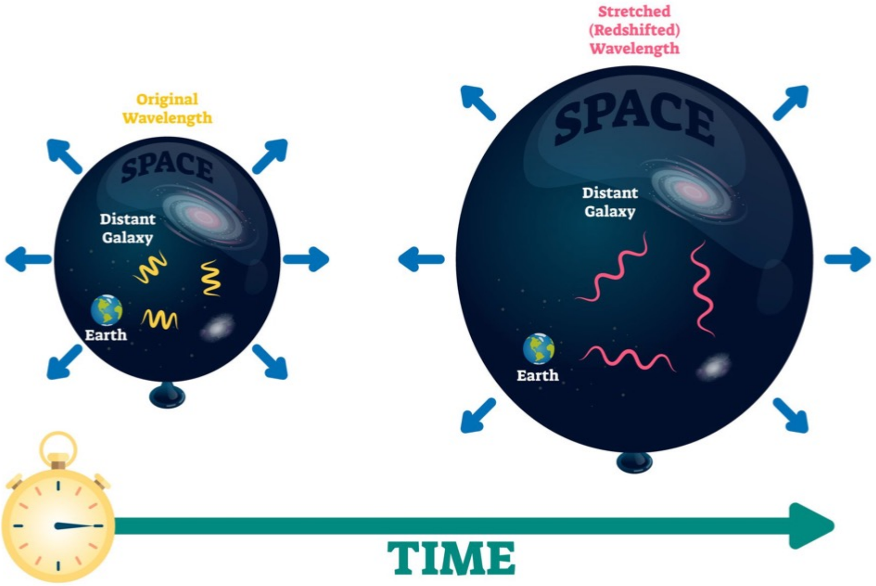
- Relativistic particles (Radiation)
  - Photons & neutrinos

$$\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}\left[3\rho + 3\mathcal{P}\right] = 0$$

$$\mathcal{P} = \frac{1}{3}\rho \quad \Longrightarrow \quad \frac{\dot{\rho}}{\rho} = -\frac{4\dot{a}}{a} = -4H \quad \Longrightarrow \quad \rho \sim a^{-4}$$

-3 for particle number conservation,
-1 for red shift.

$$E_\gamma = h\nu = \frac{h}{\lambda}$$



Cosmic Inventory

- Non-relativistic particles (Matter)
  - Baryon & Dark matter

$$\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}\left[3\rho + 3\mathcal{P}\right] = 0$$

$$\mathcal{P} = 0 \quad \Longrightarrow \quad \frac{\dot{\rho}}{\rho} = -\frac{3\dot{a}}{a} = -3H \quad \Longrightarrow \quad \rho \sim a^{-3}$$

-3 for particle number conservation

Cosmic Inventory

$$\frac{\partial\rho}{\partial t}+\frac{\dot{a}}{a}\left[3\rho+3\mathcal{P}\right]=0$$

- Vacuum energy
  - Dark energy &

    The potential energy during inflation

$$\mathcal{P}=-\rho \quad \Longrightarrow \quad \rho\sim\text{const.}$$

$$T^{\mu}_{\nu}=\begin{pmatrix}-\rho & 0 & 0 & 0\\ 0 & \mathcal{P} & 0 & 0\\ 0 & 0 & \mathcal{P} & 0\\ 0 & 0 & 0 & \mathcal{P}\end{pmatrix}$$

- Ex: Scalar field $\mathcal{L}=-\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-V(\phi)$

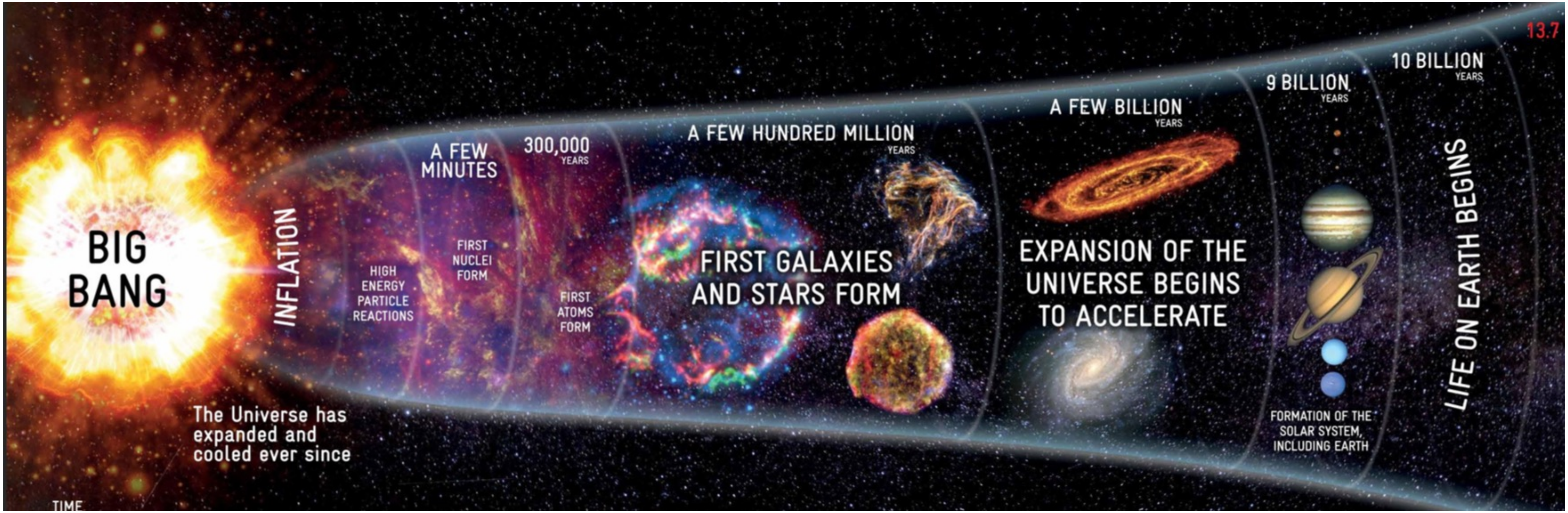
$$T^{\mu}_{\nu}=-\frac{1}{2}\partial^{\mu}\phi\partial_{\nu}\phi+g^{\mu}_{\nu}\mathcal{L} \quad \Longrightarrow \quad -g^{\mu}_{\nu}V(\phi)$$

$$\Longrightarrow \quad \rho=-\mathcal{P}=V(\phi)$$

The larger the Universe is, the more potential energy it has.
The expansion is doing negative work.

The Thermal History of the Universe

$\rho_\gamma \sim a^{-4}$ $\rho_{\text{Matter}} \sim a^{-3}$ $\rho_{\text{vac}} \sim a^0$



Radiation Domination Era

- Evolution of the scale factor

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad\qquad \rho = \rho_0\left(\frac{a}{a_0}\right)^{-4}$$

$$\Longrightarrow \qquad a(t) = \left[2\left(\frac{8\pi G}{3}\right)^{1/2}\right]\rho_0^{1/4} a_0 t^{1/2}$$

$$H = \frac{1}{2t}$$

Radiation Domination Era

- Bosons and Fermions

$$\rho = g \int \frac{d^3k}{(2\pi)^3} E(\mathbf{k}) f(\mathbf{k}) \qquad f(\mathbf{k}) = \frac{1}{e^{E(\mathbf{k})/T} \mp 1}$$

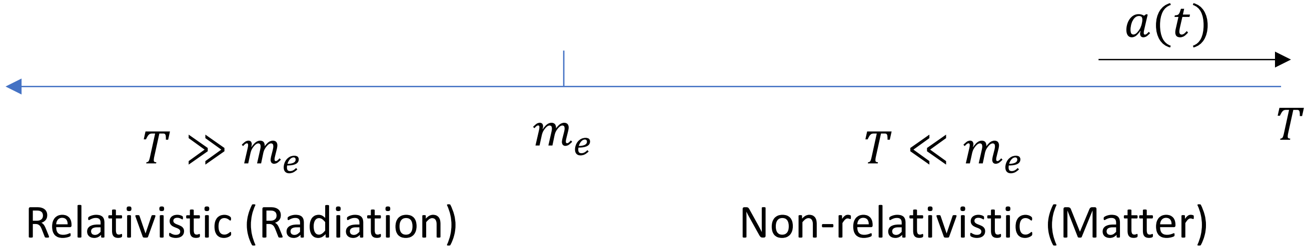
for bosons and fermions

$$\rho = \begin{cases} (\pi^2/30) g T^4 & \text{(boson)} \\ (7/8)(\pi^2/30) g T^4 & \text{(fermion)} \end{cases}$$

- Evolution of temperature $T \sim a^{-1}$

Radiation Domination Era

- Threshold Effect (Freeze out of heavy particles)



$a(t)$

$T \gg m_e$ m_e $T \ll m_e$ T

Relativistic (Radiation) Non-relativistic (Matter)

- $$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$ $$H \approx \frac{\rho^{1/2}}{M_{\text{pl}}} \sim \frac{m_e^2}{M_{\text{pl}}} \approx 10^{-22} m_e$$
- Relaxation rate $R \sim \alpha_{\text{EM}}^n m_e$
- $R \gg H$, the evolution is quasi-static.

Entropy

- First Law of Thermal Dynamics

$$\left.\begin{array}{l} dU = -\mathcal{P}dV + TdS \\ U = \rho(T)V \quad \Rightarrow \quad dU = V\dfrac{d\rho}{dT}dT + \rho dV \end{array}\right\}$$

$$\Rightarrow \qquad dS = \frac{1}{T}\frac{d\rho}{dT}VdT + \left(\frac{\rho+\mathcal{P}}{T}\right)dV$$

$$\Rightarrow \qquad s = \left(\frac{\partial S}{\partial V}\right)_T = \frac{\rho+\mathcal{P}}{T}$$

Entropy

- The entropy density

$$\rho = \frac{\pi^2}{30} g_* T^4 \qquad \mathcal{P} = \frac{1}{3}\rho$$

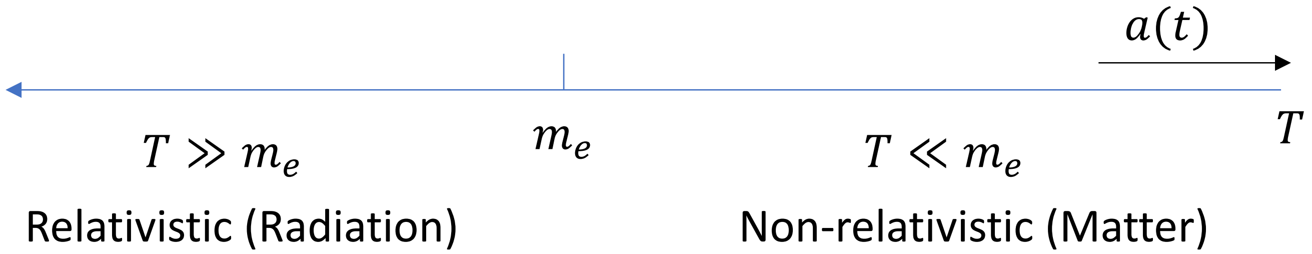
$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

$$s = \frac{\rho + \mathcal{P}}{T} = \frac{2\pi^2}{45} g_{*S} T^3$$

$$g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

Entropy

- Match the entropy before and after integrating out heavy particles



$a(t)$

T

m_e

$T \gg m_e$

Relativistic (Radiation)

$T \ll m_e$

Non-relativistic (Matter)

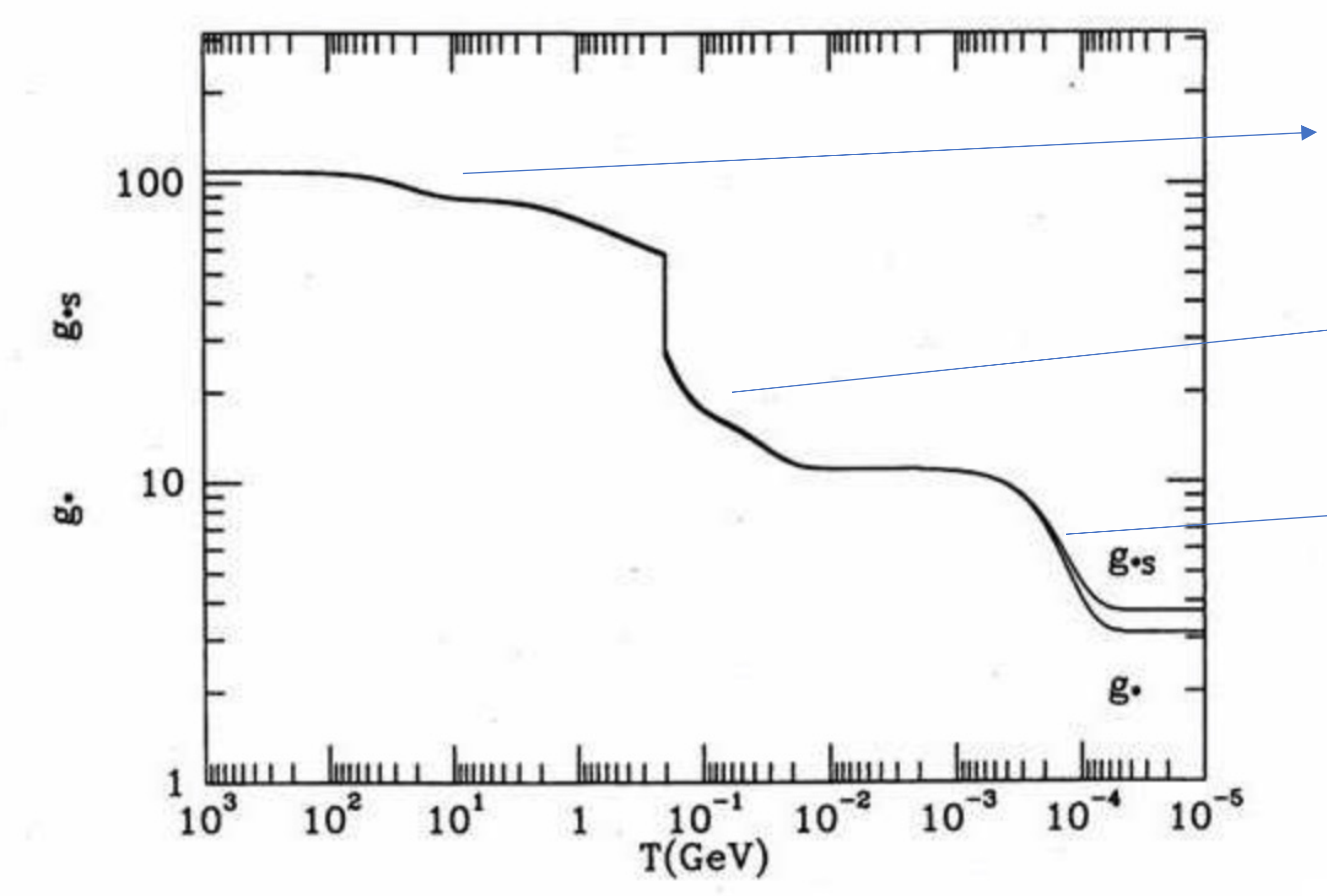
Neutrinos are already decoupled. Consider only the gamma and electron

$\gamma \times 2, e^{\pm} \times 4$ $\quad\quad$ $\gamma \times 2$

$$g_* = g_{*S} = \frac{11}{2} \qquad g_* = g_{*S} = 2$$

$$\Rightarrow \quad T_\gamma^{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_\gamma^{\text{before}}$$

Evolution of Relativistic Degrees of Freedom



EW phase transition

QCD phase transition

Electron freeze-out

Kolb & Turner “The Early Universe”

Matter Domination Era

- Evolution of the scale factor

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad\qquad \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3}$$

$$\Longrightarrow \qquad a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3} a_0$$

$$H = \frac{2}{3t}$$

Vacuum Energy Domination Era

- Evolution of the scale factor

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\rho = \rho_0 = \mathrm{const.}$$

$$H = H_0$$

$$\Rightarrow \quad a(t) = a_0 e^{H(t-t_0)}$$

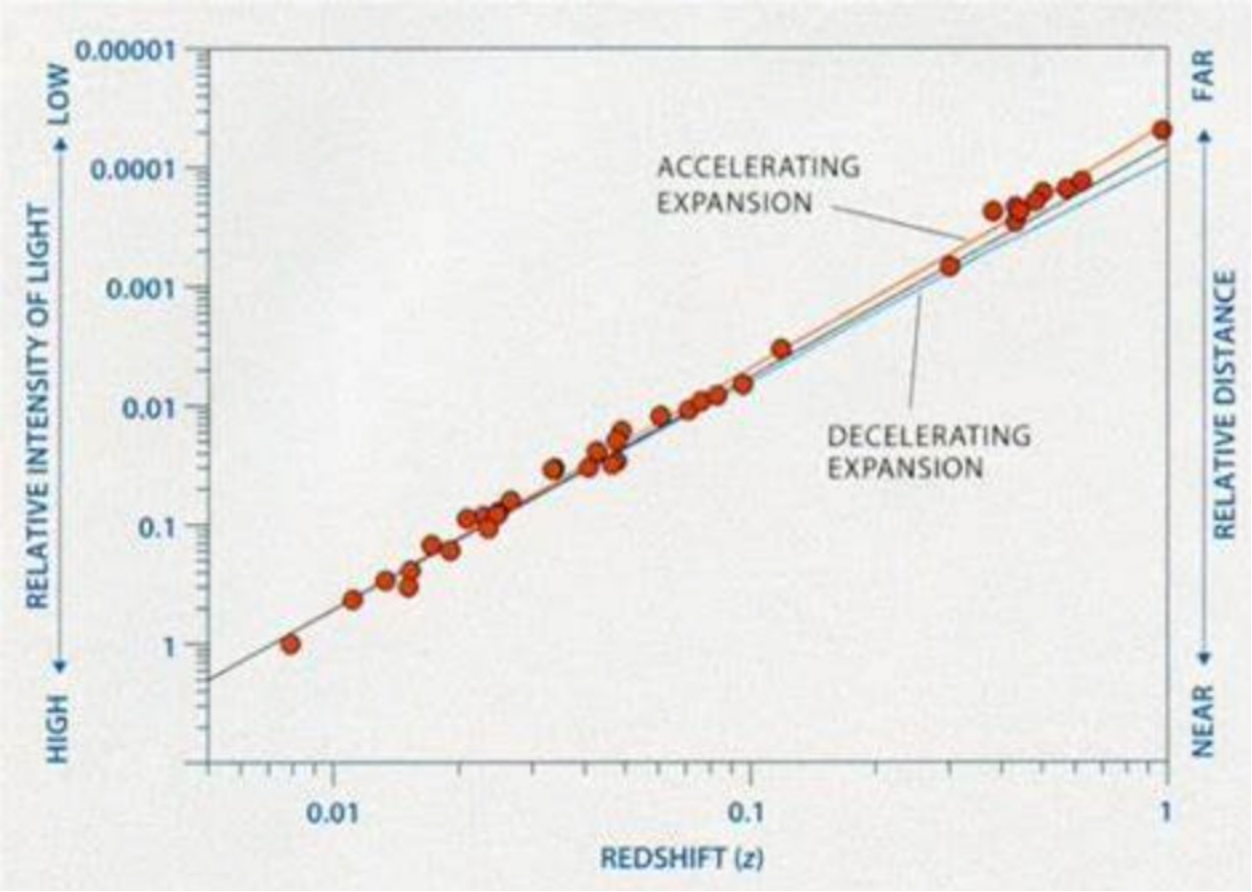
A Little Summary

- Evolution of scale factor

$$a(t) = \begin{cases} a_0(2H_0t)^{1/2} & \text{RD} \\ a_0(3H_0t/2)^{2/3} & \text{MD} \\ a_0e^{H_0(t-t_0)} & \text{CC} \end{cases}$$

- Acceleration

$$\begin{array}{ll} \ddot{a} < 0 & \text{RD} \\ \ddot{a} < 0 & \text{DM} \\ \ddot{a} > 0 & \text{CC} \end{array}$$



Today's Universe

Hubble parameter today

$$H^2 = \frac{8\pi G}{3}\rho \Longrightarrow \underbrace{\frac{3H_0^2}{8\pi G}}_{\rho_{\text{crit}}} = \sum_i \rho_i \quad \text{+ curvature term}$$

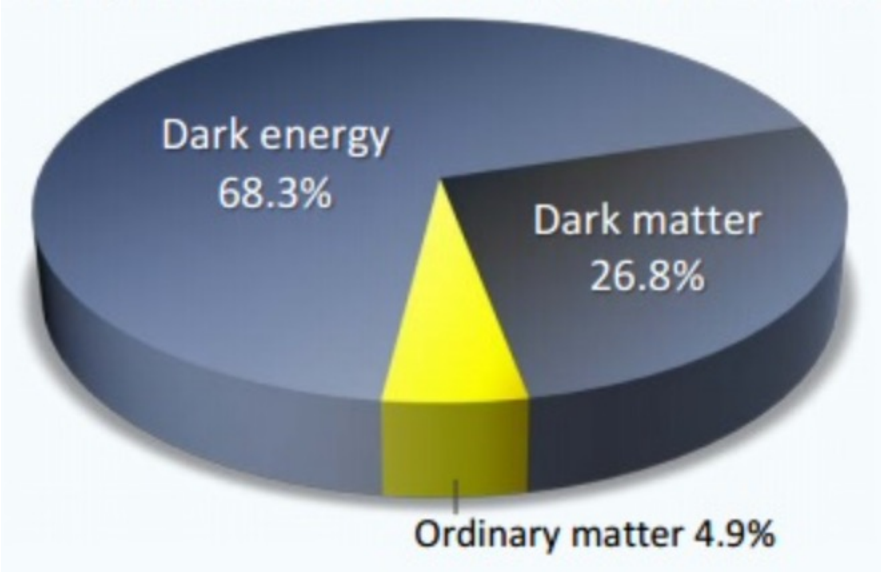
$$\Omega_i = \frac{\rho_i^{(0)}}{\rho_{\text{crit}}}$$

$$\Omega_\gamma = 5.38\times10^{-5}$$

$$\Omega_b = 0.0493$$

$$\Omega_{DM} = 0.265$$

$$\Omega_\Lambda = 0.685$$



Matter-Radiation Equality

- g_* today

$$g_* = 2 + 6 \times \frac{7}{8} \times \left(\frac{T_\nu}{T_\gamma}\right)^4 \approx 3.36$$

$$2 \times \frac{\pi^2}{30} g_* T_{\rm CMB}^4 / \rho_{\rm crit} = \Omega_\gamma = 5.38 \times 10^{-5}$$

$$\Omega_M = \Omega_{DM} + \Omega_b \approx 0.27$$

- Matter-Radiation Equality:

$$\Omega_M \left(\frac{a_{eq}}{a_0}\right)^{-3} = \frac{3.36}{2} \Omega_\gamma \left(\frac{a_{eq}}{a_0}\right)^{-4} \quad \Longrightarrow \quad 1 + z_{eq} \equiv \frac{a_0}{a_{eq}} \approx 3000$$

How large the vacuum energy is?

- $\Omega_\Lambda \approx 0.685$

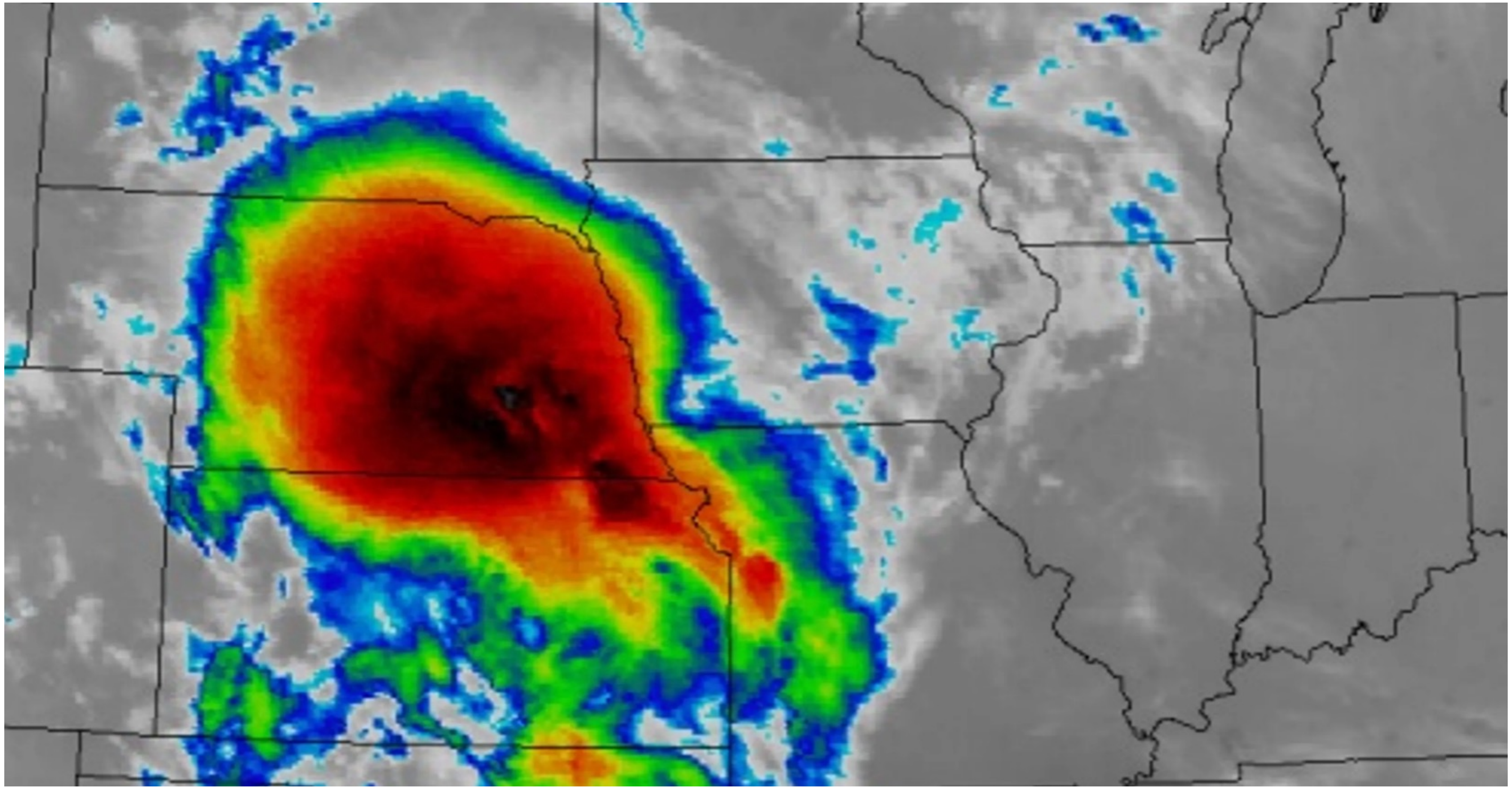
  $$\rho_\Lambda = \Omega_\Lambda \times \rho_{\rm crit} = \Omega_\Lambda \times \frac{3H_0^2}{8\pi G}$$

  $$H_0 = 67.4 \ {\rm km/sec/Mpc} = 1.44 \times 10^{-42} \ {\rm GeV}$$

  $$\rho_\Lambda = 2.52 \times 10^{-47} \ {\rm GeV}^4$$

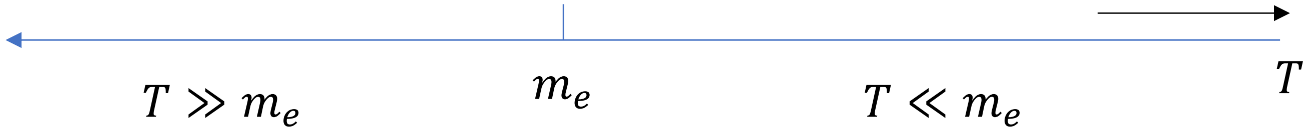
- Quantum corrections from the Planck scale $\sim G^{-2} \sim 10^{76}\ \mathrm{GeV}^4$
- 120 orders of magnitude fine-tuning. The Cosmological Constant problem

Beyond the Equilibrium



Beyond the Equilibrium

- We have been cheating in the case of $e^{\pm}$ annihilation.



$T \gg m_e$ m_e $T \ll m_e$ T

Relativistic (Radiation)

Non-relativistic (Matter)

$$\Gamma_{e^{\pm}\to\gamma\gamma} \propto n_e \sim e^{-m_e/T}$$

- At some point, $\Gamma_{e^{\pm}\to\gamma\gamma} < H$

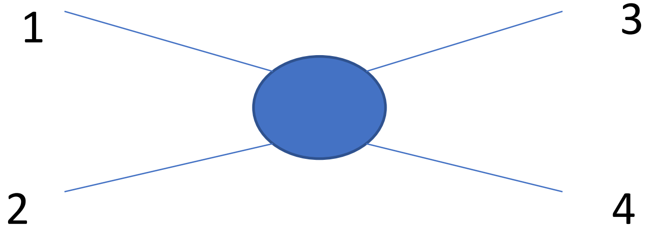
- Nonequilibrium processes always happen during the expansion of the Universe.

The Boltzmann Equation

- What if no interactions?
  - Particle number must be conserved.
  - n_i: physical number density of some kind of particle.

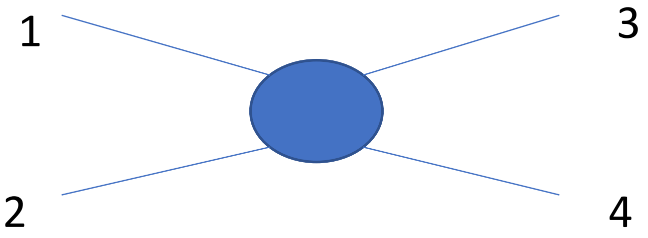
$$\frac{d(a^3 n_i)}{dt} = 0 \quad \Longrightarrow \quad \frac{dn_i}{dt} + 3Hn_i = 0$$

- With interactions.
  - Particle 1 can disappear through 12 → 34.
  - Particle 1 can reappear through 34 → 12 .



The Boltzmann Equation

- Number density $n_1 = \int \frac{d^3p_1}{(2\pi)^3} f_1(\mathbf{p}_1)$.



$$a^{-3}\frac{dn_1a^3}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\int \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\times(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}(12\to 34)|^2$$

$$\times\{f_3f_4[1\pm f_1][1\pm f_2]-f_1f_2[1\pm f_3][1-\pm f_4]\}$$

34 → 12 12 → 34

The Boltzmann Equation

$$a^{-3}\frac{dn_1a^3}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\int \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\times(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}(12\to 34)|^2$$

$$\times\{f_3f_4[1\pm f_1][1\pm f_2]-f_1f_2[1\pm f_3][1-\pm f_4]\}$$

$1+f$ for bosonic final state, Bosonic condensation.
$1-f$ for fermionic final state, Pauli exclusion principle.

The Boltzmann Equation

$$a^{-3}\frac{dn_1a^3}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)|\mathcal{M}(12 \to 34)|^2$$

$$\times \{f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 - \pm f_4]\}$$

For homogeneous distributions: f_i can only be functions of E_i.

Thermal equilibrium means Collision term = 0.

$$f_i^{eq} = \frac{1}{e^{(E_i - \mu_i)/T} \mp 1}$$

Non-equilibrium

- Non-equilibrium usually happens when $H > \Gamma$.
  - H is suppressed by Planck scale.
  - We usually need Γ to be suppressed exponentially.
  - Non-equilibrium usually happens when particle number is exponentially suppressed.

$$f_i = \frac{1}{e^{(E_i-\mu_i)/T} \mp 1} \approx e^{\mu_i/T} e^{-E_i/T} \ll 1$$

$$n_i = g_i e^{\mu_i/T} \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} = e^{\mu_i/T} n_i^{(0)}$$

Non-equilibrium

$$a^{-3}\frac{dn_1a^3}{dt} = n_1^{(0)}n_2^{(0)}\langle\sigma v\rangle\left\{\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}} - \frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right\}$$

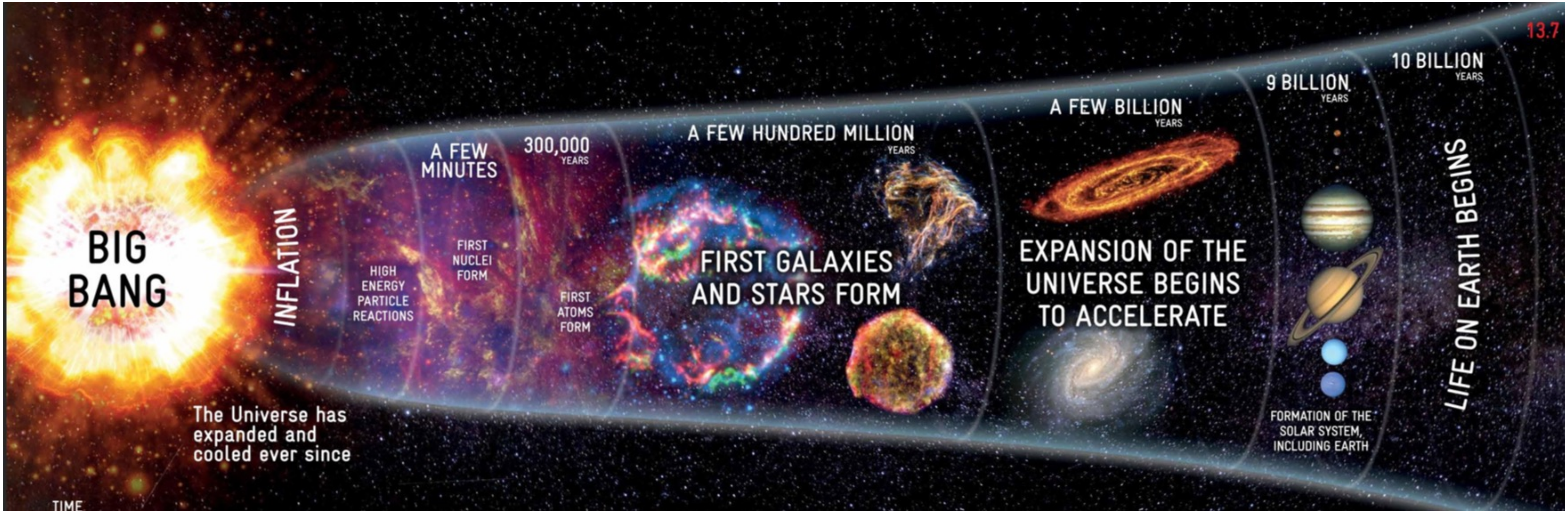
$$\langle\sigma v\rangle = \frac{1}{n_1^{(0)}n_2^{(0)}}\int\frac{d^3p_1}{(2\pi)^3 2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\int\frac{d^3p_3}{(2\pi)^3 2E_3}\int\frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\times e^{-(E_1+E_2)/T}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)|\mathcal{M}|^2$$

In case of thermal equilibrium: $$\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}} = \frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}$$

The Thermal History of the Universe

$\rho_\gamma \sim a^{-4}$ $\rho_{\text{Matter}} \sim a^{-3}$ $\rho_{\text{vac}} \sim a^0$



Big Bang Nucleosynthesis

- Basic facts:
  - It started when $T \sim 1\ \mathrm{MeV}$.
  - Tightly coupled relativistic particles: $\gamma, e^{\pm}$.
  - Tightly coupled non-relativistic particles: baryons (p, n).
  - Decoupled relativistic particles: neutrinos.
  - $\eta \equiv n_b / n_\gamma \approx 6.2 \times 10^{-10}$.
  - Nuclear binding energy, $B \equiv Z m_p + (A - Z) m_n - m$
  - Neutrons and protons can interconvert via weak interactions:

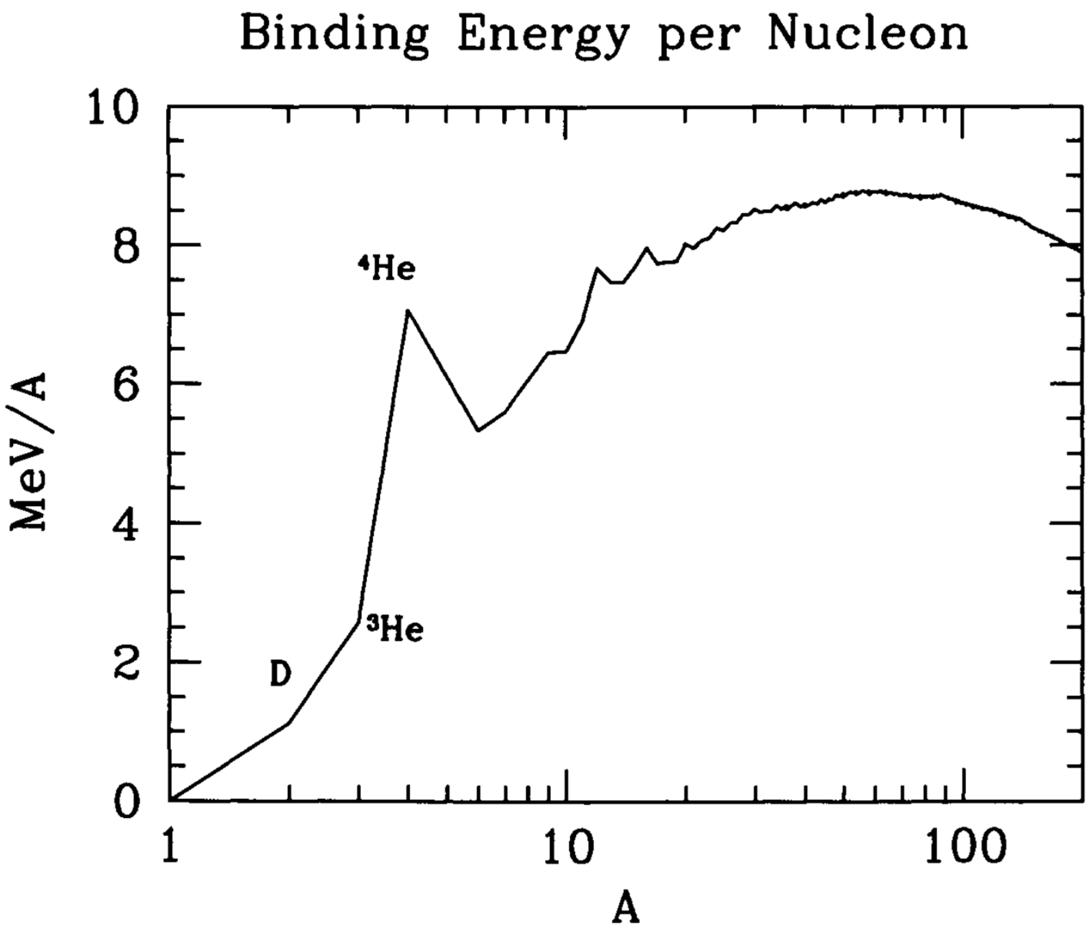
$$p + \bar{\nu} \leftrightarrow n + e^{+} \qquad p + e^{-} \leftrightarrow n + \nu \qquad n \leftrightarrow p + e^{-} + \bar{\nu}$$

  - Light elements are produced via electromagnetic interactions:

$$p + n \rightarrow D + \gamma \qquad D + D \rightarrow n + {}^{3}\mathrm{He} \qquad {}^{3}\mathrm{He} + D \rightarrow p + {}^{4}\mathrm{He}$$

Big Bang Nucleosynthesis (Simplified version)

- We only consider hydrogen and helium and their isotopes. (Deuterium, tritium, and ^{3}He)
- For $T > 0.1\ \mathrm{MeV}$, we assume no light nuclei are formed, we can consider only proton. and neutrons.
- We first calculate the neutron proton ratio, and then use it to calculate the abundances of other isotopes.



Big Bang Nucleosynthesis (Simplified version)

- Consider $n + p \leftrightarrow D + \gamma$

- Equilibrium condition: $$\frac{n_D n_\gamma}{n_D^{(0)} n_\gamma^{(0)}} = \frac{n_n n_p}{n_n^{(0)} n_p^{(0)}} \xrightarrow{n_\gamma = n_\gamma^{(0)}} \frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}}$$

- For NR particles: $$n^{(0)} = \int \frac{d^3p}{(2\pi)^3} e^{-m/T - p^2/2mT} = \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

$$\Rightarrow \quad \frac{n_D}{n_p n_n} = \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T}$$

Big Bang Nucleosynthesis (Simplified version)

$B_D = 2.2$ MeV

$$\left.\begin{aligned} &\frac{n_D}{n_p n_n} = \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T} \\ &n_p \sim n_b \quad , n_b = \eta_b n_\gamma \approx \eta_b T^3 \end{aligned}\right\} \Longrightarrow \frac{n_D}{n_n} \sim \eta_b \underbrace{\left(\frac{T}{m_p}\right)^{3/2}}_{10^{-15}} e^{B_D/T}$$

$\Longrightarrow$ n_D/n_n << 1 as long as $T > 0.1$ MeV.

Neutron proton ratio

- Leptons (electron, positron and neutrinos are in thermal equilibrium)

$$n_e = n_\nu = n_e^{(0)} = n_\nu^{(0)} = n_l^{(0)}$$

- Boltzmann Equation

$$a^{-3}\frac{dn_n a^3}{dt} = n_n^{(0)} n_l^{(0)} \langle \sigma v \rangle \left\{ \frac{n_p n_l}{n_p^{(0)} n_l^{(0)}} - \frac{n_n n_l}{n_n^{(0)} n_l^{(0)}} \right\}$$

$$\Longrightarrow \quad a^{-3}\frac{dn_n a^3}{dt} = n_l^{(0)} \langle \sigma v \rangle \left\{ \frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right\}$$

Neutron proton ratio

- Baryon number is conserved $n_p + n_n \sim a^{-3}$ $X_n \equiv \frac{n_n}{n_n + n_p}$

$$a^{-3}\frac{dn_n a^3}{dt} = n_l^{(0)}\langle\sigma v\rangle\left\{\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n\right\}$$

$$\Downarrow$$

$$\frac{dX_n}{dt} = \Gamma_{np}\left\{(1 - X_n)e^{-Q/T} - X_n\right\}$$

$$\langle\sigma v\rangle n_l^{(0)} \qquad Q = m_n - m_p$$

Neutron proton ratio

- Convert t to T $\quad T \sim a^{-1} \Longrightarrow \dfrac{dT}{Tdt} = -\dfrac{da}{adt} = -H \qquad x \equiv \dfrac{Q}{T}$
- In RD, $H \sim \rho^{\frac{1}{2}} \sim T^2 \sim x^{-2} \Longrightarrow H = x^{-2} H(x=1)$
- $\dfrac{dX_n}{dx} = \dfrac{x\Gamma_{np}}{H(x=1)} \left\{ e^{-x} - X_n(1+e^{-x}) \right\}$

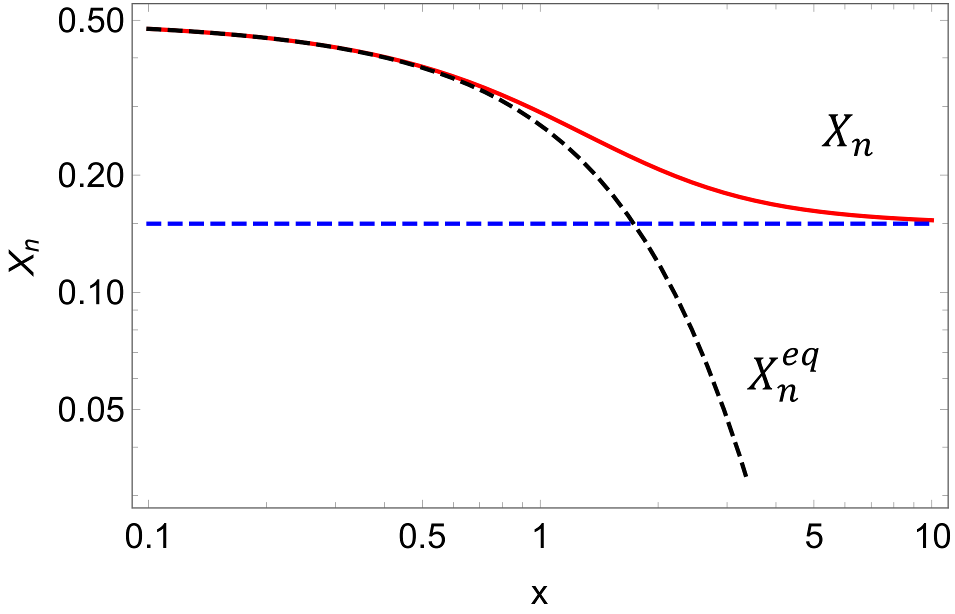
$$\Gamma_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2)$$

$$\tau_n = 886.7 \text{ sec} \qquad H(x=1) = 1.13 \text{ sec}^{-1}$$

Neutron proton ratio

$$\Gamma_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2)$$

- $$\frac{dX_n}{dx} = \frac{x\Gamma_{np}}{H(x=1)}\left\{e^{-x} - X_n(1+e^{-x})\right\}$$
- At $x \ll 1$, the system is in thermal equilibrium, $X_n = 1/2$.
- X_n freezes at 0.15 at $T \approx 0.5$ MeV
- Neutrons decay after freezing-out, until they are inside stable nucleus.



Light Element Abundances

- Deuterium revisit $$\frac{n_D}{n_n} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}$$
- $n_D \approx n_n$: $$\log(\eta_b) + \frac{3}{2}\log(T_{\text{nuc}}/m_p) \sim -\frac{B_D}{T_{\text{nuc}}}$$

$$\Longrightarrow \qquad T_{\text{nuc}} \approx 0.07\ \text{MeV}$$

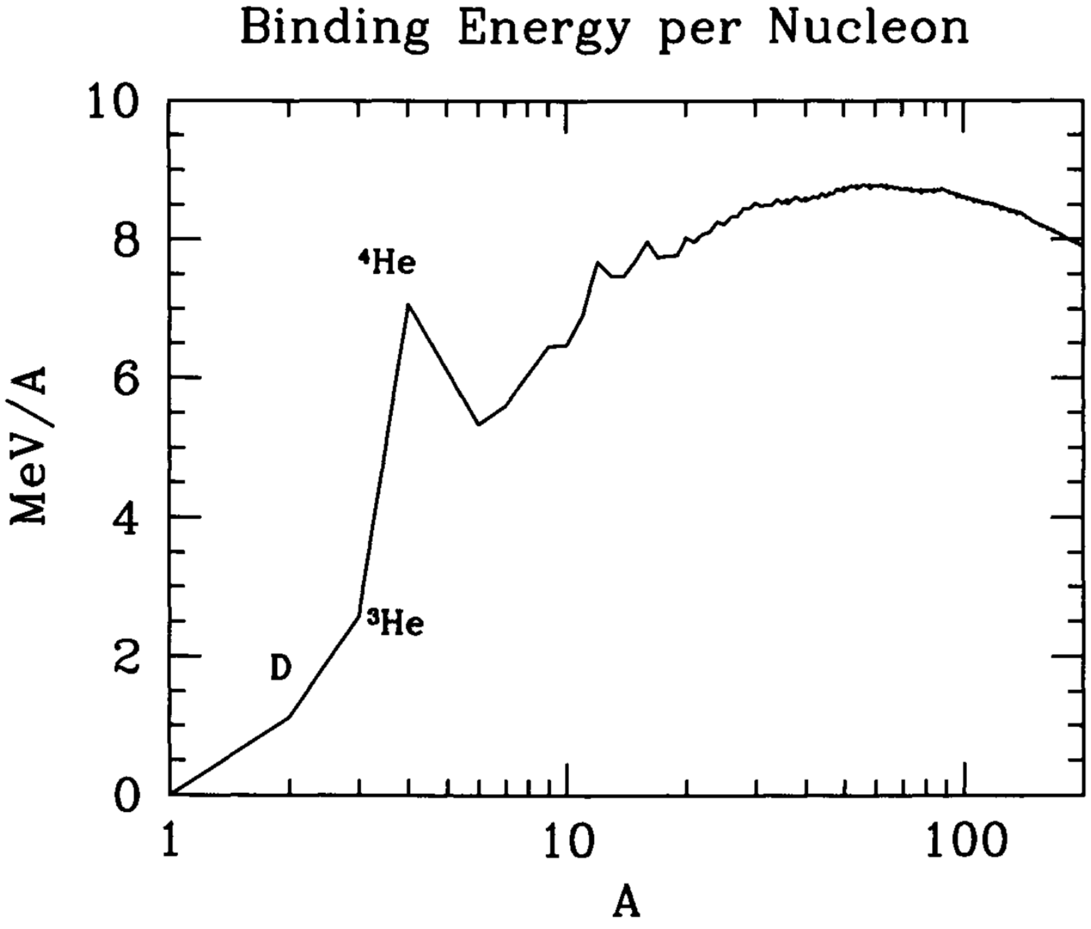
- Neutron decay from $T_{freeze} = 0.5$ MeV to $T_{nuc} \approx 0.07$ MeV.
- X_n from 0.15 to 0.11.

Light Element Abundances

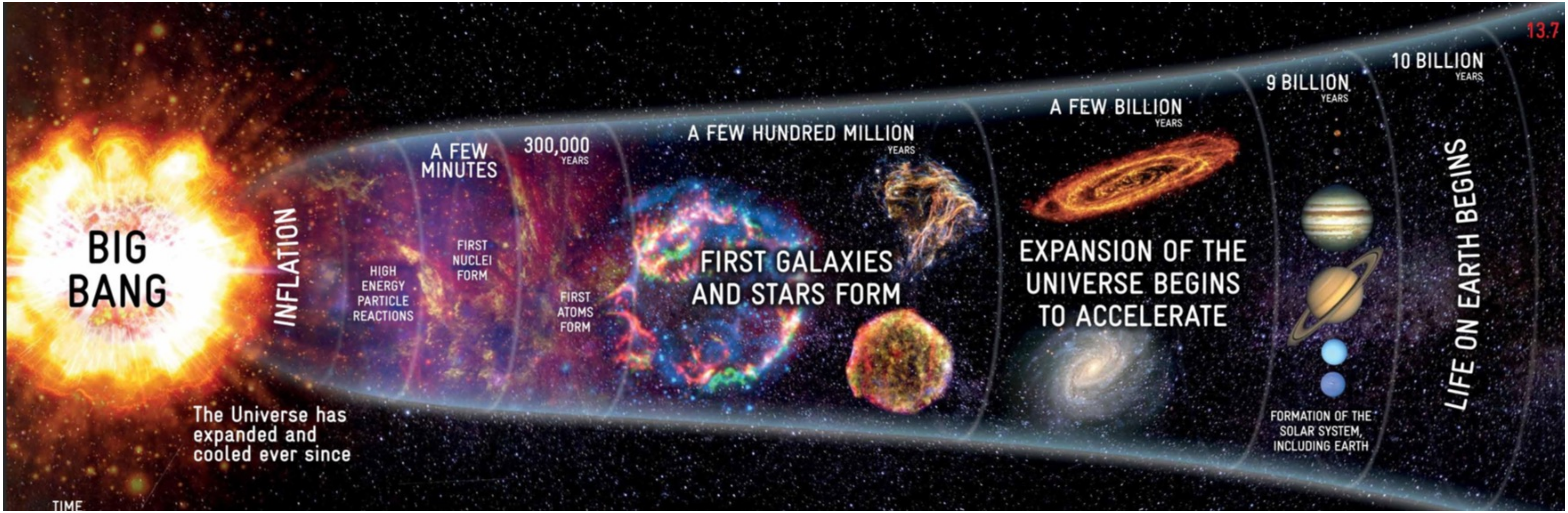
- ^{4}He has the largest binding energy per nucleon.
- All the neutrons will go into ^{4}He.

- Mass fraction:

$$X_4 \equiv \frac{4n_{^4\mathrm{He}}}{n_b} = 2X_n(T_{\mathrm{nuc}}) \approx 0.22$$



Recombination



Recombination

- $p + e \rightarrow H + \gamma$

All non-relativistic

$$\frac{n_p n_e}{n_p^{(0)} n_e^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}} \xRightarrow{n_\gamma = n_\gamma^{(0)}} \frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

$$X_e \equiv \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$$

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left[\left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-(m_e + m_p - m_H)/T} \right]$$

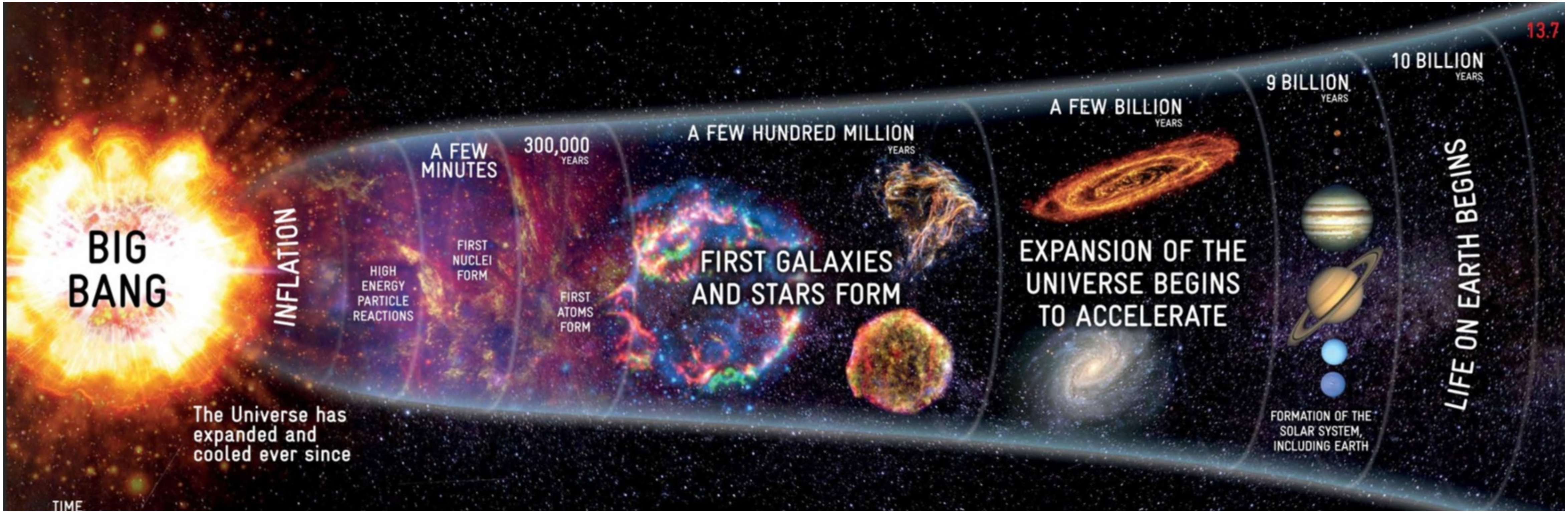
Recombination

- $$\frac{X_e^2}{1-X_e} = \frac{1}{n_e+n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-(m_e+m_p-m_H)/T}\right]$$
- At $T \sim 10$ eV: $$\frac{X_e^2}{1-X_e} \approx \eta_b^{-1}\left(\frac{m_e}{T}\right)^{3/2} \sim 10^{18} \gg 1$$

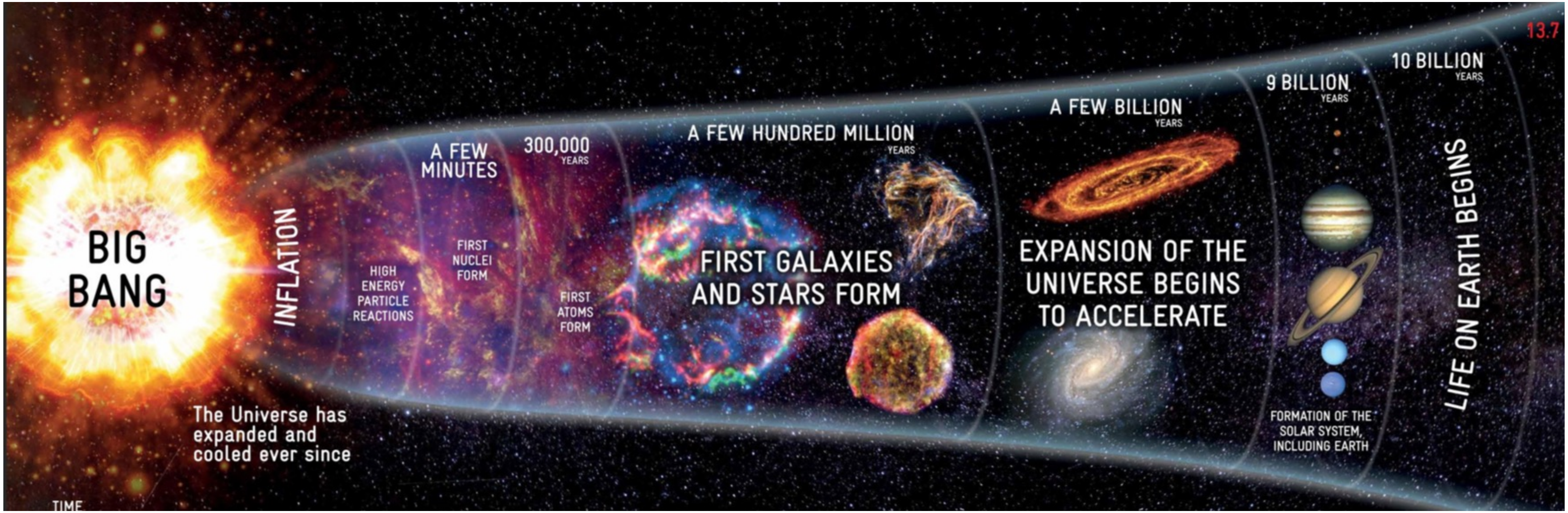
$\Rightarrow$ $X_e \approx 1$ Recombination not start yet.

- $\frac{X_e^2}{1-X_e} \sim 1$ when $\frac{m_e+m_p-m_H}{T} \sim \log 10^{18}$ $\Rightarrow$ $T_{\rm rec} \approx 0.33\ \mathrm{eV}$

Summary


BIG BANG
INFLATION
HIGH ENERGY PARTICLE REACTIONS
A FEW MINUTES
FIRST NUCLEI FORM
300,000 YEARS
FIRST ATOMS FORM
A FEW HUNDRED MILLION YEARS
FIRST GALAXIES AND STARS FORM
A FEW BILLION YEARS
EXPANSION OF THE UNIVERSE BEGINS TO ACCELERATE
9 BILLION YEARS
FORMATION OF THE SOLAR SYSTEM, INCLUDING EARTH
10 BILLION YEARS
LIFE ON EARTH BEGINS
13.7
The Universe has expanded and cooled ever since
TIME

Basics about Inflationary Universe



Problems with the Thermal Big Bang Universe

- Horizon Problem

- Flatness Problem

The FRW metric of the Homogeneous and Isotropic Universe

- FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right)$$

- Choose a new coordinate

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\chi^2 + \Phi_k(\chi^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right)$$

$$r^2 = \Phi_k(\chi^2) \equiv \begin{cases} \sinh^2\chi & k=-1 \\ \chi^2 & k=0 \\ \sin^2\chi & k=+1 \end{cases}$$

$$H \equiv \frac{\dot{a}}{a}$$

Conformal Time and Horizons

- Conformal time: $\tau = \int \frac{\mathrm{d}t}{a(t)}$

$$\mathrm{d}s^2 = a(\tau)^2 \left[-\mathrm{d}\tau^2 + \left(\mathrm{d}\chi^2 + \Phi_k(\chi^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right)\right]$$

Only consider radial motion

$$\mathrm{d}s^2 = a(\tau)^2 \left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2\right]$$

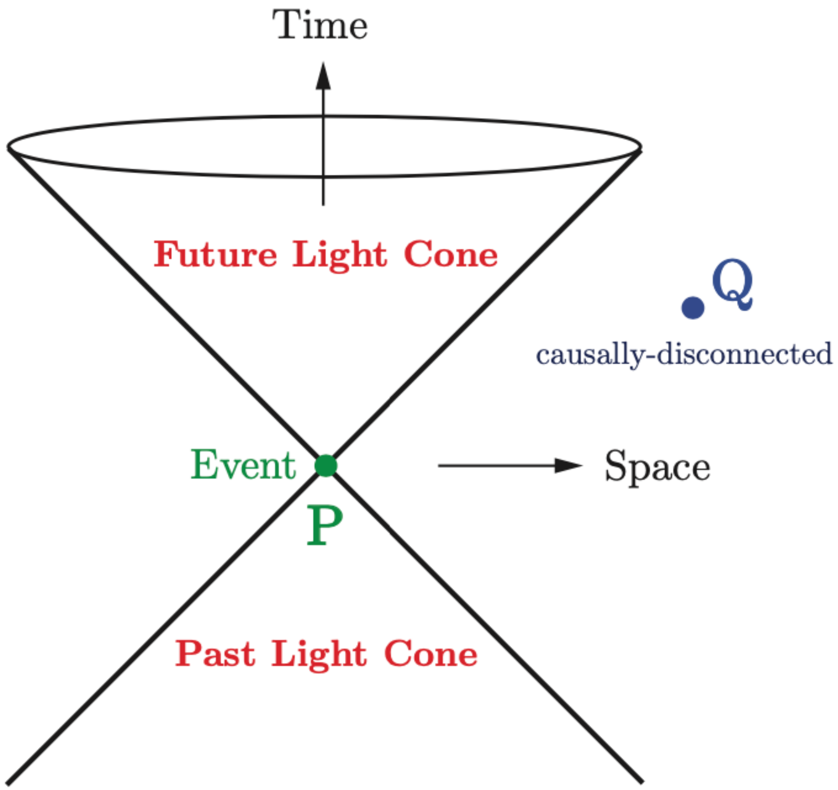
Conformal Time and Horizons

- Radial null geodesics of light

$$\mathrm{d}s^2 = a(\tau)^2 \left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2\right]$$

$$\chi(\tau) = \pm\tau + \text{const.}$$

P and Q are causially disconnected



Particle Horizon

- The maximum distance light can propagate between an initial time t_i and some later time t:

$$\chi_p(\tau) = \tau - \tau_{\rm i} = \int_{t_{\rm i}}^{t} \frac{{\rm d}t}{a(t)}$$

- Physical size: $d_p(t) = a(t)\chi_p$

Event Horizon

- A event horizon defines the set of points from which signals sent at a given moment τ will never be received by an observer in the future.

$$\chi \ > \ \chi_e = \int_{\tau}^{\tau_{\max}} \mathrm{d}\tau = \tau_{\max} - \tau$$

- Physical size of event horizon

$$d_e(t) = a(t)\chi_e$$

The Horizon Problem

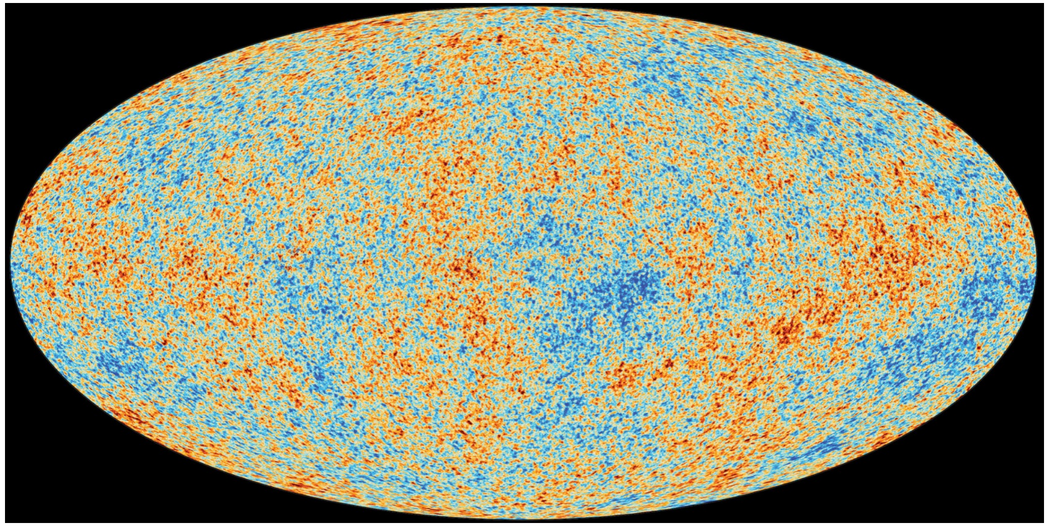
- In thermal big bang, $a(t) \sim t^p$,
- $p = 1/2$ for RD, $2/3$ for MD. $p < 1$.

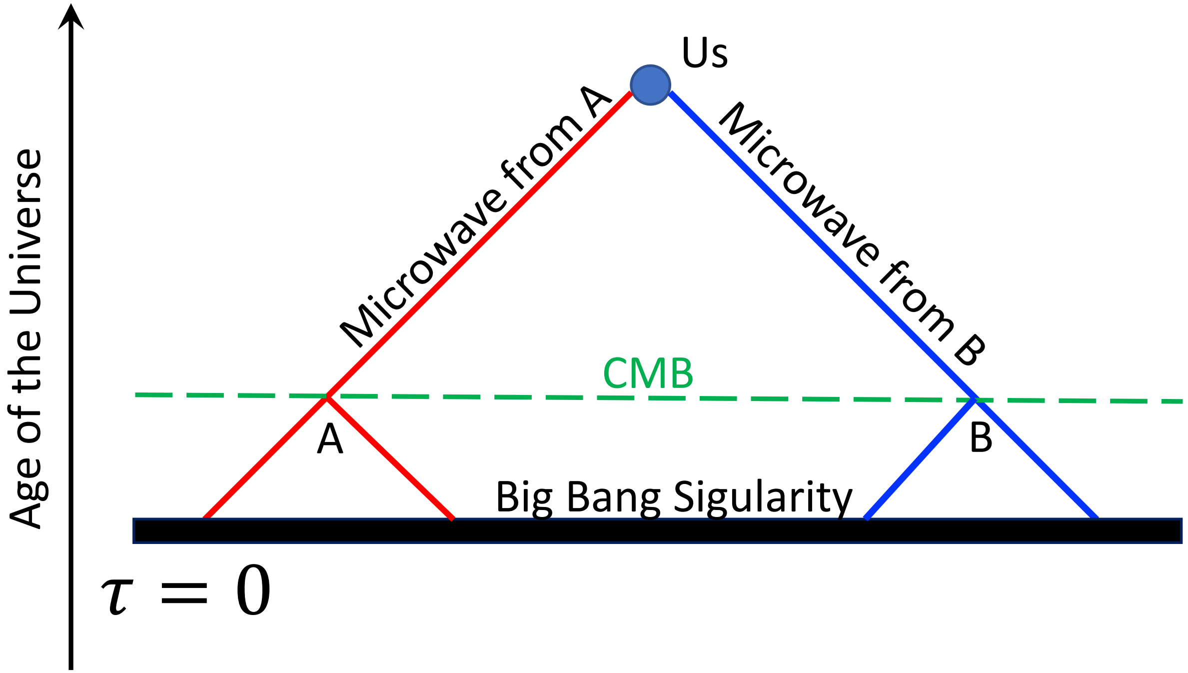
$$d\tau = t^{-p}dt = \frac{1}{1-p}dt^{1-p} \Longrightarrow \tau = \frac{1}{1-p}t^{1-p} \qquad a \sim \tau^{\frac{p}{1-p}}$$

$$\Longrightarrow \tau \in (0, \infty)$$

- The event horizon becomes larger and larger.
- We can see farther and farther.
- The comoving horizon today much larger than the horizon at recombinnation
- The conformal time has a finite beginning.

The Horizon Problem





The past lightcones of A and B had never met.

Flatness problem

a^{-4} in radiation domination

$$\Omega_k \equiv \Omega - 1 = \frac{\rho - \rho_{\text{crit}}}{\rho_{\text{crit}}}$$

$$H^2 = \frac{1}{3}\rho(a) - \frac{k}{a^2}$$

Define how flat the Universe is

$$1 - \Omega(a) = \frac{-k}{(aH)^2} \sim a^2$$

$$\begin{aligned} |\Omega(a_{\text{BBN}}) - 1| &\leq \mathcal{O}(10^{-16}) \\ |\Omega(a_{\text{GUT}}) - 1| &\leq \mathcal{O}(10^{-55}) \\ |\Omega(a_{\text{pl}}) - 1| &\leq \mathcal{O}(10^{-61}) \end{aligned}$$

The Universe is flatter at earlier time during radiation domination.

A first look of Inflation

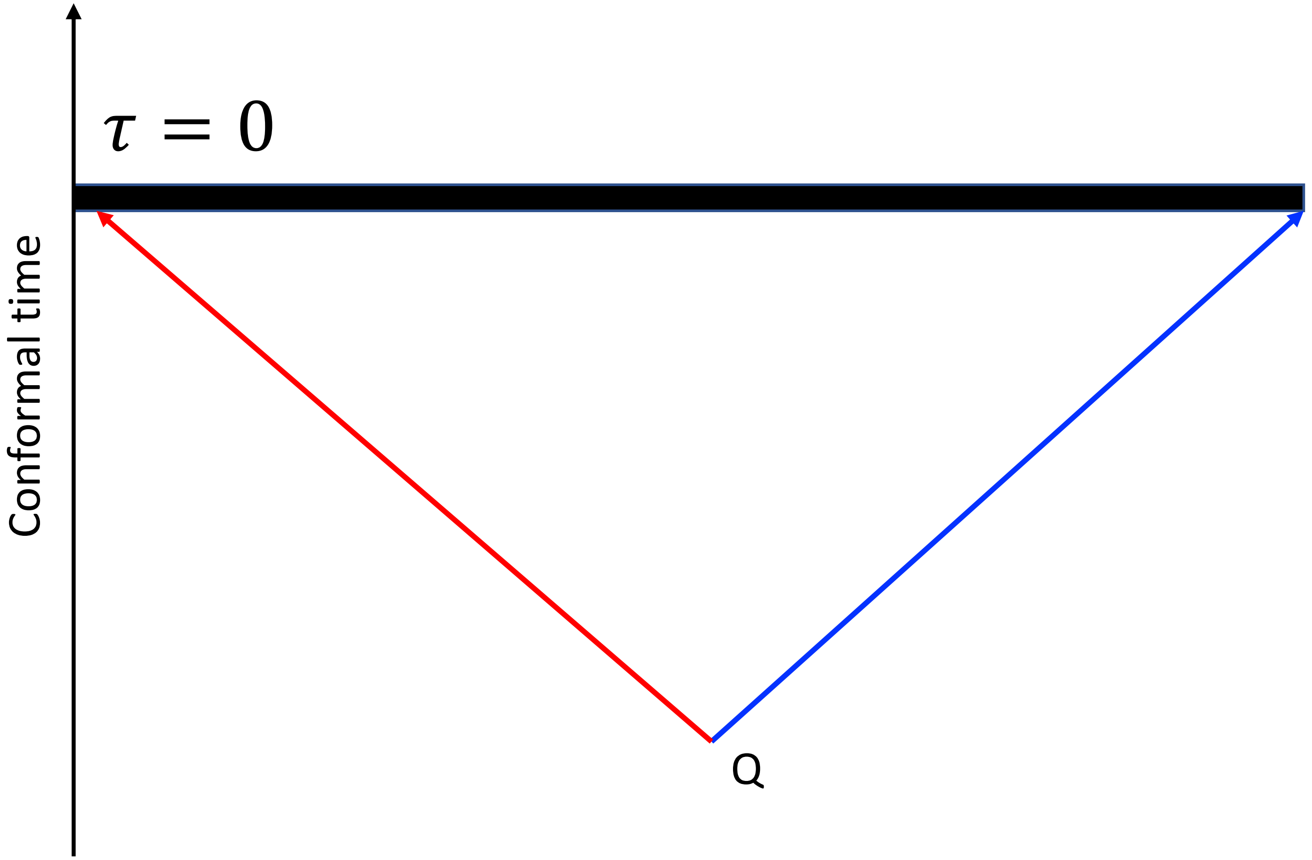
- To solve the problems, we need accelerating expansion.
- Parameterize the scale factor as $a(t) \sim t^p$, we need $p > 1$.

$$d\tau = t^{-p} dt = \frac{1}{1-p} dt^{1-p} \quad \Longrightarrow \quad \tau = \frac{1}{1-p} t^{1-p}$$

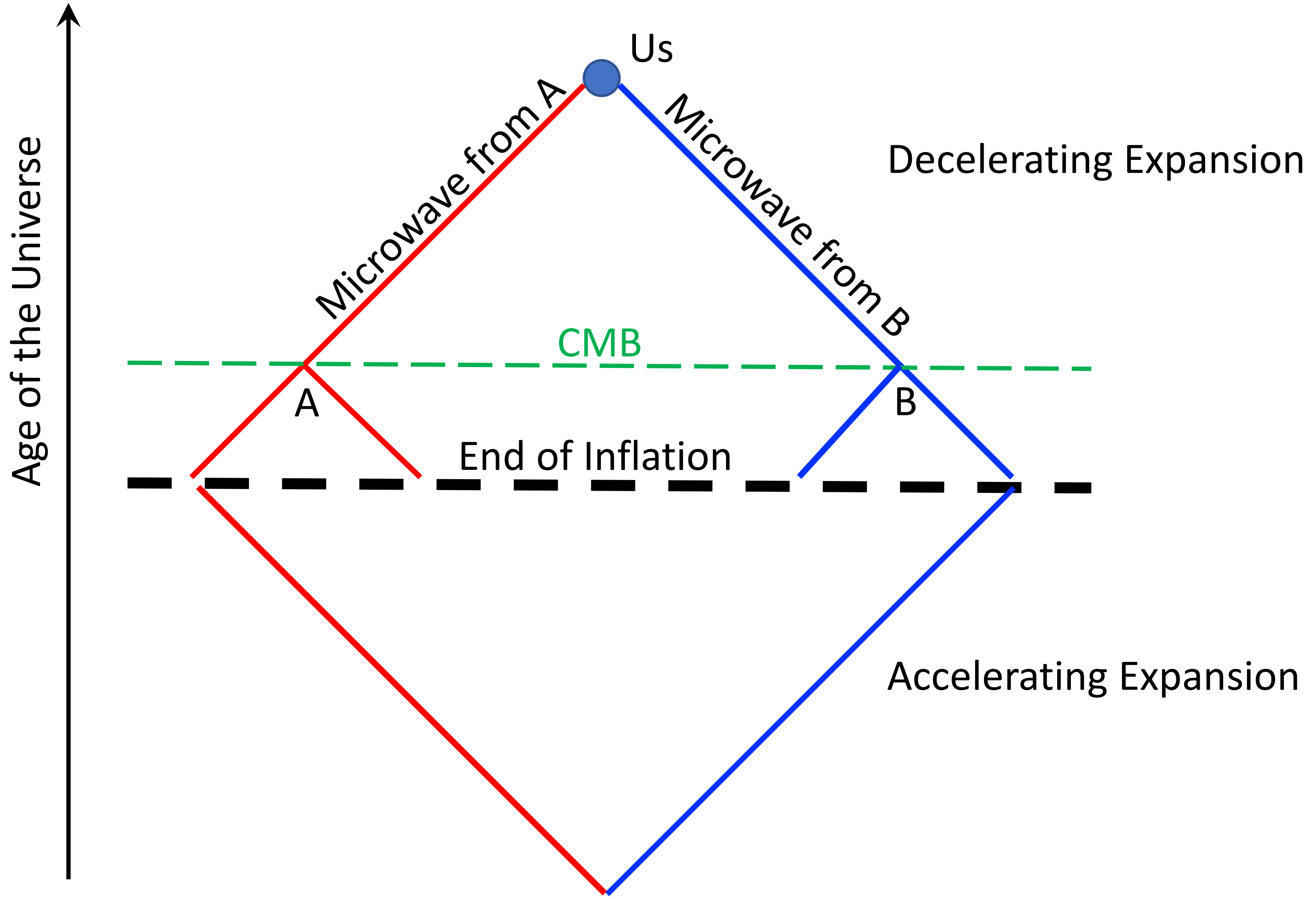
$$\Longrightarrow \quad \tau \in (-\infty, 0)$$

- The event horizon $|\tau|$, becomes smaller and smaller.
- The conformal time τ has a finite ending.

The particle during inflation



How inflation solves the horizon problem?



Physics of Inflation

- Scalar field dynamics

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right)$$

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\end{aligned}$$

$$\rho_\phi \approx -p_\phi \;\; \text{if} \;\; \dot{\phi}^2 \ll V(\phi)$$

We need to keep a large potential for a long time.

Physics of Inflation

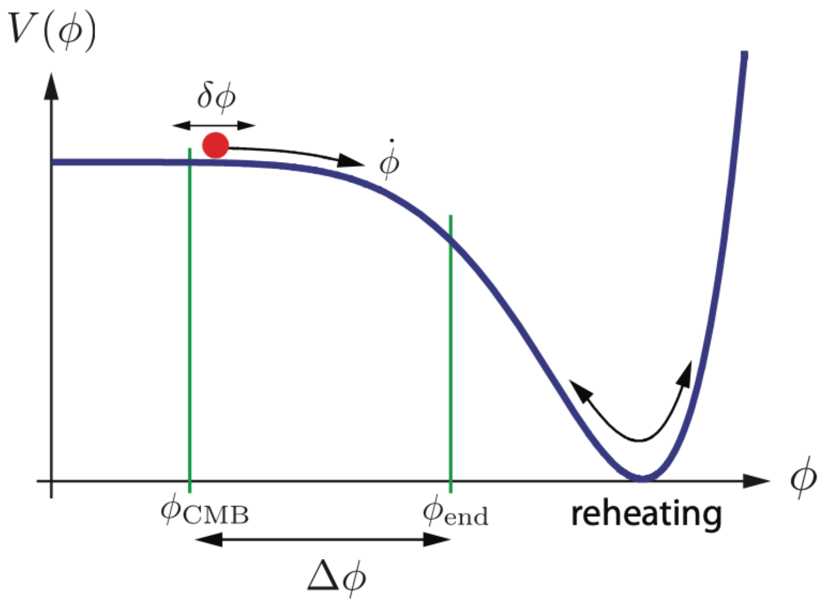
- Evolution of ϕ

$$\frac{\delta S_\phi}{\delta \phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu \phi) + V_{,\phi} = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

Hubble friction

Drives the scalar field down.



$$H^2 \gg V_{\phi\phi}$$

Slow-roll Inflation

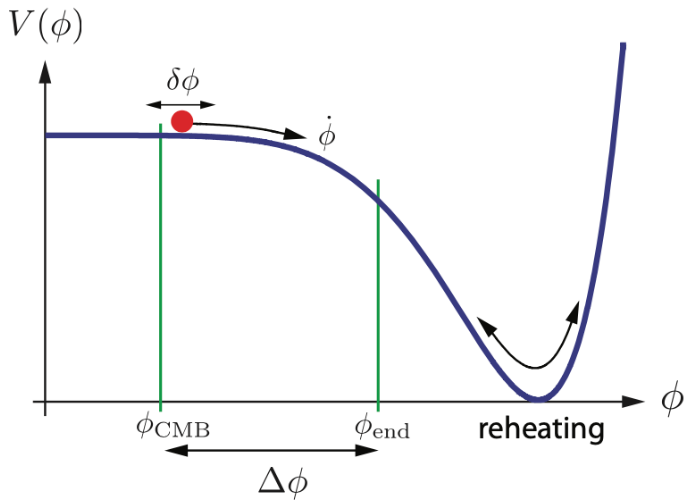
- The evolution of ϕ must be significantly slower than the evolution of the Universe.

$$\varepsilon \equiv \frac{3}{2}(w_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ll 1 \qquad \varepsilon = -\frac{\dot{H}}{H^2}$$

- Accelerated expansion will only be sustained for a sufficiently long period if the second derivative of ϕ is small enough.

$$|\ddot{\phi}| \ll |3H\dot{\phi}|\,,\,|V_{,\phi}| \qquad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1$$

Slow-roll Inflation



- $H^2 \approx \frac{1}{3}V(\phi) \approx \text{const.}$
- $a(t) \sim e^{Ht}$ The space-time is approximately de Sitter.
- Inflation ends when $\varepsilon(\phi_{\rm end}) \equiv 1$
- $N(\phi) \equiv \ln \frac{a_{\rm end}}{a}$

$$= \int_t^{t_{\rm end}} H \mathrm{d}t = \int_\phi^{\phi_{\rm end}} \frac{H}{\dot{\phi}} \mathrm{d}\phi \approx \int_{\phi_{\rm end}}^{\phi} \frac{V}{V_{,\phi}} \mathrm{d}\phi$$

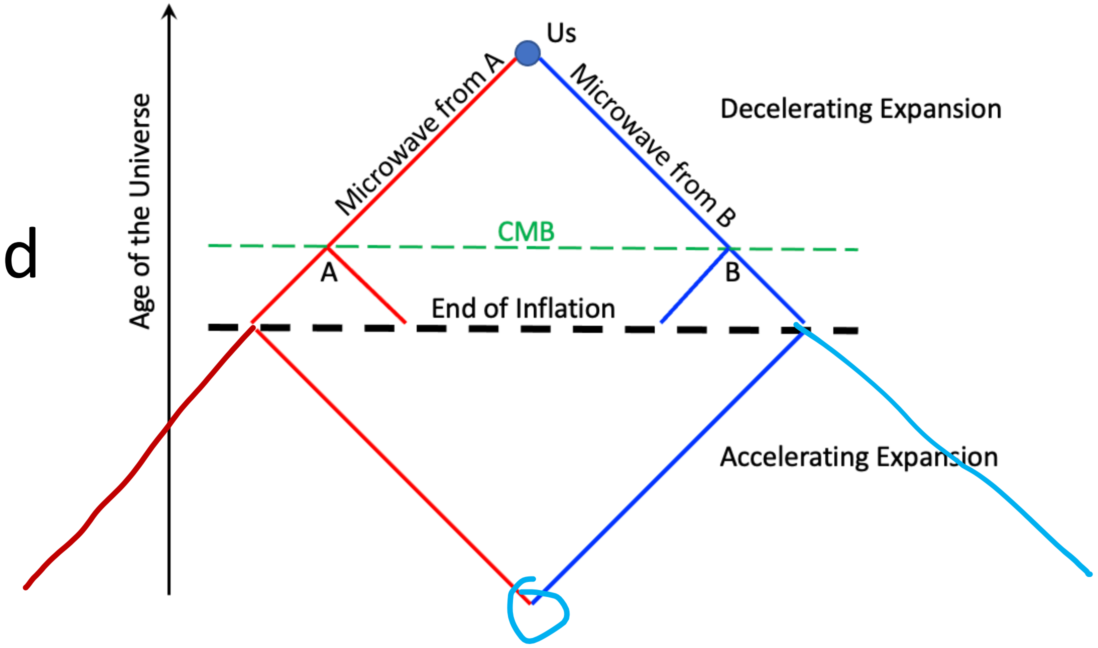
- $N(\phi) = \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon}} \approx \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_{\rm v}}}$

Slow-roll Inflation

- To solve the horizon problem we need

$$N_{\rm tot} \equiv \ln \frac{a_{\rm end}}{a_{\rm start}} \gtrsim 60$$ (HW)

$$\int_{\phi_{\rm end}}^{\phi_{\rm cmb}} \frac{d\phi}{\sqrt{2\varepsilon}} = N_{\rm cmb} \approx 40 - 60$$



Example: $m^2\phi^2$ Inflation

- $$V(\phi) = \frac{1}{2}m^2\phi^2$$

$$\epsilon \approx 2\left(\frac{M_{\rm pl}}{\phi}\right)^2 \qquad \eta \approx 0$$

$$\int_{\phi_{\rm end}}^{\phi_{\rm cmb}} \frac{d\phi}{\sqrt{2\varepsilon}} = N_{\rm cmb} \approx 40 - 60 \Rightarrow N(\phi) = \frac{\phi^2}{4M_{\rm pl}^2} - \frac{1}{2}$$

$$\Rightarrow \phi_{\rm cmb} = 2\sqrt{N_{\rm cmb}}\,M_{\rm pl} \sim 15 M_{\rm pl}$$

Transplanckian problem

Single field inflation models

- Small field model

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^2\right]^2$$

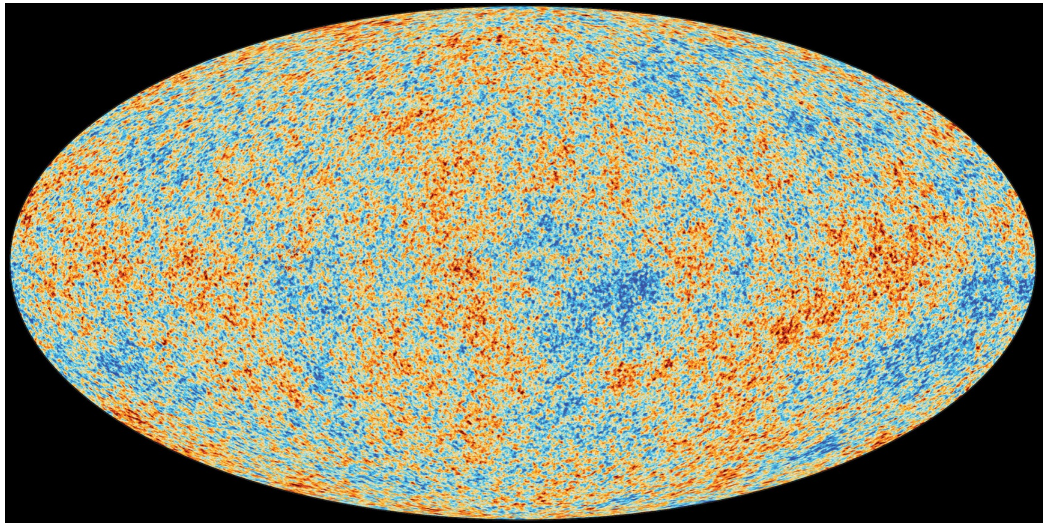
$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu}\right)^4 \left(\ln\left(\frac{\phi}{\mu}\right) - \frac{1}{4}\right) + \frac{1}{4}\right]$$

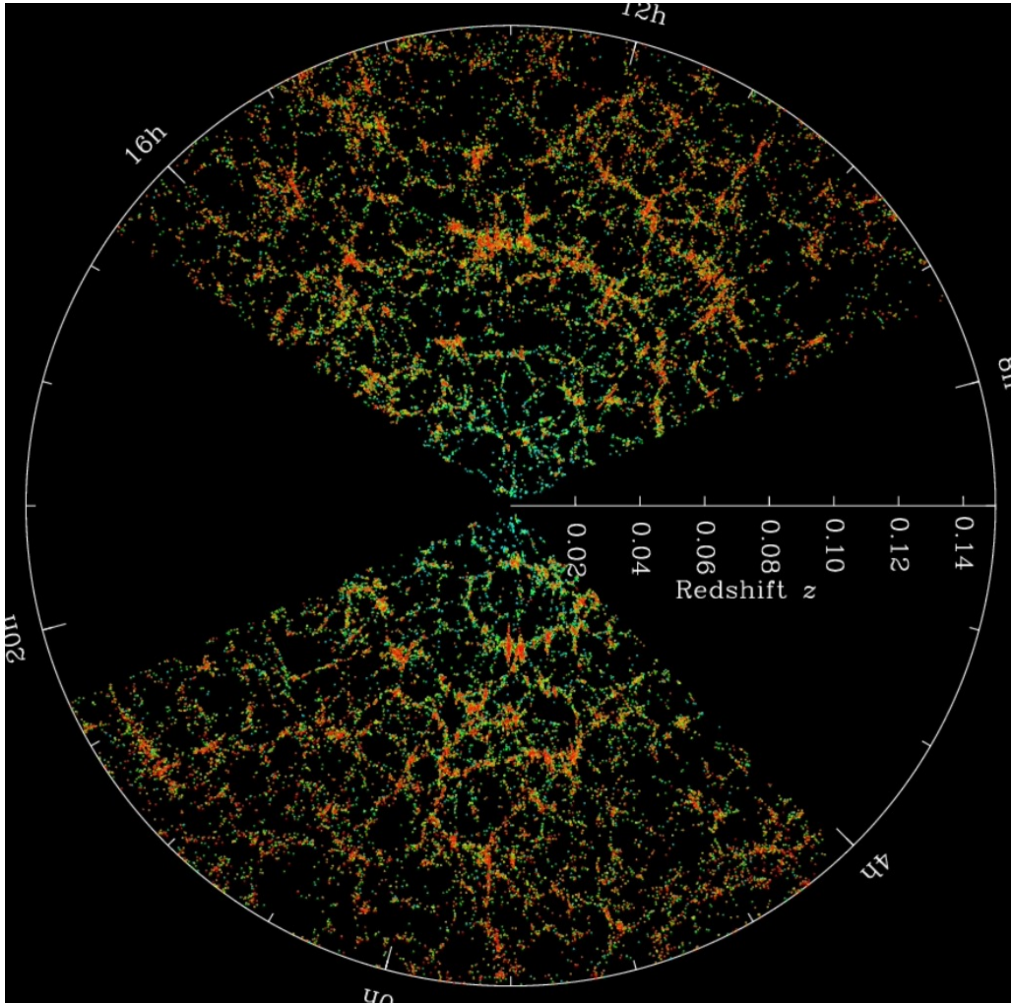
- Large field model

$$V(\phi) = \lambda_p \phi^p$$

$$V(\phi) = V_0 \left[\cos\left(\frac{\phi}{f}\right) + 1\right]$$

Quantum fluctuations during Inflation





Quantum fluctuations during Inflation

- Both the metric and the matter fluctuations.
- Gauge choice is important.
- We need to find gauge invariant quantities.

The Inhomogeneous Universe

- Perburbations

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \delta\phi(t,\mathbf{x})\,, \qquad g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x})$$

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(1+2\Phi)\mathrm{d}t^2 + 2aB_i \mathrm{d}x^i \mathrm{d}t + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]\mathrm{d}x^i \mathrm{d}x^j\,. \end{aligned}$$

- SVT decomposition

$$B_i \equiv \partial_i B - S_i\,, \qquad \text{where} \quad \partial^i S_i = 0$$

$$E_{ij} \equiv 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}\,, \qquad \text{where} \quad \partial^i F_i = 0\,, \quad h^i_i = \partial^i h_{ij} = 0$$

Gauge transformation (coordinate transformation)

- Gauge transformation of the metric:

$$\begin{aligned} t &\to t+\alpha \\ x^i &\to x^i+\delta^{ij}\beta_{,j} \end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned} \Phi &\to \Phi-\dot{\alpha} \\ B &\to B+a^{-1}\alpha-a\dot{\beta} \\ E &\to E-\beta \\ \Psi &\to \Psi+H\alpha\,. \end{aligned}$$

- Tensor part is gauge invariant

Gauge transformation (coordinate transformation)

- Gauge transformation of the energy momentum tensor:

$$
\begin{aligned}
T^0_0 &= -(\bar\rho + \delta\rho) \\
T^0_i &= (\bar\rho + \bar p)\, a v_i \\
T^i_0 &= -(\bar\rho + \bar p)(v^i - B^i)/a \\
T^i_j &= \delta^i_j(\bar p + \delta p) + \Sigma^i_j\,.
\end{aligned}
$$

$$(\delta q)_{,i} \equiv (\bar\rho + \bar p) v_i$$

$$
\begin{aligned}
\delta\rho &\to \delta\rho - \dot{\bar\rho}\,\alpha \\
\delta p &\to \delta p - \dot{\bar p}\,\alpha \\
\delta q &\to \delta q + (\bar\rho + \bar p)\,\alpha
\end{aligned}
$$

- Tensor part is gauge invariant.

Gauge Invariant Quantities

- Tensors h_{ij} and Σ_{ij} are gauge invariant.

- Gauge invariant scalar quantities

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho$$

- ζ remains constant outside the horizon for adiabatic matter perturbations.

$$\boxed{\delta p_{en} \equiv \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\delta\rho} = 0$$

$$\begin{aligned} \delta\rho &\to \delta\rho - \dot{\bar{\rho}}\,\alpha \\ \delta p &\to \delta p - \dot{\bar{p}}\,\alpha \\ \delta q &\to \delta q + (\bar{\rho} + \bar{p})\,\alpha \end{aligned}$$

$$\begin{aligned} \Phi &\to \Phi - \dot{\alpha} \\ B &\to B + a^{-1}\alpha - a\dot{\beta} \\ E &\to E - \beta \\ \Psi &\to \Psi + H\alpha\,. \end{aligned}$$

Gauge Invariant Quantities

- Curvature perturbation on uniform-density hypersurfaces.

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho \qquad\qquad R^{(3)} = 4\nabla^2\Psi/a^2$$

- During slow-roll Inflation: $\delta\rho \approx V'\delta\phi, \dot{\rho} = V'\dot{\phi}$

$$-\zeta \approx \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi$$

- Another gauge invariant quantity: comoving curvature perturbation

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q$$

Gauge Invariant Quantities

- Comoving Curvature Perturbation

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q$$

δq: scalar part of the three momentum density.

$$T_i^0 = \partial_i \delta q.$$

- During slwo-roll inflation: $T_i^0 = -\dot{\bar{\phi}} \partial_i \delta \phi$

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}} \delta \phi$$

- During slow-roll, $\mathcal{R} = -\zeta$ (This relation holds after inflation for superhorizon scales)

Statistics of Cosmological Perturbations

- We are going to calculate the power spectrum of R_k

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \, \delta(\mathbf{k} + \mathbf{k}') \, P_{\mathcal{R}}(k)$$

Spactial homogeneity

$$\Delta_{\mathrm{s}}^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$$

$$n_{\mathrm{s}} - 1 \equiv \frac{d \ln \Delta_{\mathrm{s}}^2}{d \ln k} \qquad \alpha_{\mathrm{s}} \equiv \frac{d n_{\mathrm{s}}}{d \ln k}$$

$$\Delta_{\mathrm{s}}^2(k) = A_{\mathrm{s}}(k_\star) \left(\frac{k}{k_\star} \right)^{n_{\mathrm{s}}(k_\star) - 1 + \frac{1}{2} \alpha_{\mathrm{s}}(k_\star) \ln(k/k_\star)}$$

- If R is Gaussian, the statistics is fully determined by the two-point function.

Statistics of Cosmological Perturbations

- Tensor modes: $h \equiv h^{+}, h^{\times}$

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \, \delta(\mathbf{k} + \mathbf{k}') \, P_h(k) \, , \qquad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k)$$

$$\Delta_{\rm t}^2 \equiv 2\Delta_h^2 \qquad n_{\rm t} \equiv \frac{d \ln \Delta_{\rm t}^2}{d \ln k}$$

$$\Delta_{\rm t}^2(k) = A_{\rm t}(k_\star) \left(\frac{k}{k_\star} \right)^{n_{\rm t}(k_\star)}$$

The Quantum Origin of Structure

- Quantum Mechanics of Harmonic Oscillator

- Quantum Fluctuations in de Sitter Space

Quantum Mechanics of Harmonic Oscillator

- Action $$S = \int \mathrm{d}t \,\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2\right) \equiv \int \mathrm{d}t\, L$$
- EOM: $$\frac{\delta S}{\delta x} = 0 \quad \Rightarrow \quad \boxed{\ddot{x} + \omega^2(t)\, x = 0}$$
- Canonical quantization:

$$p \equiv \frac{dL}{d\dot{x}} = \dot{x} \qquad\qquad [\hat{x}, \hat{p}] = i\hbar$$

Quantum Mechanics of Harmonic Oscillator

- In Heisenberg Picture: $\hat{x} = v(t)\,\hat{a} + v^*(t)\,\hat{a}^\dagger$

Wave functions

- The wave function v satisfies:

$$\ddot{v} + \omega^2(t)v = 0$$

- The commutation relation gives:

$$\langle v, v\rangle[\hat{a}, \hat{a}^\dagger] = 1 \qquad \langle v, w\rangle \equiv \frac{i}{\hbar}\left(v^*\partial_t w - (\partial_t v^*)w\right)$$

- Imposing $[\hat{a}, \hat{a}^\dagger] = 1$ $\quad \langle v, v\rangle \equiv 1$

Quantum Mechanics of Harmonic Oscillator

- Define the vacuum $|0\rangle$, annihilated by a. $\hat{a}|0\rangle = 0$

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \qquad \hat{N} = \hat{a}^\dagger\hat{a}$$

- $\omega(t)$ is time-dependent! This is different from the harmonic oscillator in the textbook. How to define the vacuum??

- We assume that $\omega(t) \to \omega$ when $t \to -\infty$, and define the vacuum there.

Quantum Mechanics of Harmonic Oscillator

$$\begin{aligned}\hat{H} &= \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 \\ &= \frac{1}{2}\left[(\dot{v}^2 + \omega^2 v^2)\hat{a}\hat{a} + (\dot{v}^2 + \omega^2 v^2)^*\,\hat{a}^\dagger\hat{a}^\dagger + (|\dot{v}|^2 + \omega^2|v|^2)(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\right]\end{aligned}$$

$$\hat{H}|0\rangle = \frac{1}{2}\underbrace{(\dot{v}^2 + \omega^2 v^2)^*\,\hat{a}^\dagger\hat{a}^\dagger|0\rangle}_{\text{Should vanish}} + \frac{1}{2}(|\dot{v}|^2 + \omega^2|v|^2)|0\rangle$$

$$\Longrightarrow \quad \dot{v} = \pm i\omega v$$

We choose the standard positive frequency mode $v(t) = \sqrt{\frac{\hbar}{2\omega}}\, e^{-i\omega t}$

Quantum Mechanics of Harmonic Oscillator

- Zero point fluctuations in ground state

$$\begin{aligned}
\langle|\hat{x}|^2\rangle &\equiv \langle 0|\hat{x}^\dagger\hat{x}|0\rangle \\
&= \langle 0|(v^*\hat{a}^\dagger + v\hat{a})(v\hat{a} + v^*\hat{a}^\dagger)|0\rangle \\
&= |v(\omega,t)|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\
&= |v(\omega,t)|^2\langle 0|[\hat{a},\hat{a}^\dagger]|0\rangle \\
&= |v(\omega,t)|^2\,.
\end{aligned}$$

$$\boxed{\langle|\hat{x}|^2\rangle = |v(\omega,t)|^2} = \frac{\hbar}{2\omega}$$

Quantum Fluctuations in de Sitter

- Scalar perturbation:

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\ \left[R - (\nabla\phi)^2 - 2V(\phi)\right]$$

- Using the gauge:

$$\delta\phi = 0\,, \qquad g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]\,, \qquad \partial_i h_{ij} = h^i_i = 0\,.$$

- The free field action of R

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}^4x\, a^3\,\frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]$$

Quantum Fluctuations in de Sitter

- Define the Mukhanov variable

$$v \equiv z\mathcal{R}\,, \qquad \text{where} \qquad z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$$

- Using the conformal time

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \, \left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right] \qquad (\ldots)' \equiv \partial_\tau(\ldots)$$

Quantum Fluctuations in de Sitter

- Fourier modes: $v(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, v_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}$
- EOM: $v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$
- Canonical quantization:

$$v \quad \rightarrow \quad \hat{v} = \int \frac{\mathrm{d}\mathbf{k}^3}{(2\pi)^3} \left[v_k(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\tau) \hat{a}_{\mathbf{k}}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

$$v_{\mathbf{k}} \quad \rightarrow \quad \hat{v}_{\mathbf{k}} = v_k(\tau) \hat{a}_{\mathbf{k}} + v_{-k}^*(\tau) \hat{a}_{-\mathbf{k}}^{\dagger}$$

Quantum Fluctuations in de Sitter

- Impose canonical quantization condition for creation and annihilation operators:

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \qquad \langle v_k, v_k \rangle \equiv \frac{i}{\hbar}(v_k^* v_k' - v_k^{*\prime} v_k) = 1$$

- We need to specify the vacuum. We need a boundary condition.

Quantum Fluctuations in de Sitter

- The vacuum condition $\hat{a}_{\mathbf{k}}|0\rangle = 0\,.$
- We need to specify the v_k to define the vacuum.
- Reexamine the EOM:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \qquad v \equiv z\mathcal{R}\,, \qquad \text{where} \qquad z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$$

- ε changes very slowly. $\frac{z''}{z} \sim \frac{2}{\tau^2}$

Quantum Fluctuations in de Sitter

- At $\tau \to -\infty$, EOM→ $v_k'' + k^2 v_k = 0$
- We impose the boundary condition: $\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}$
- In de Sitter $v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0$
- With the boundary condition, $v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$

Quantum Fluctuations in de Sitter

- Define a field $\hat{\psi}_{\mathbf{k}} \equiv a^{-1}\hat{v}_{\mathbf{k}}$

$$\begin{aligned}
\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{k}'}(\tau)\rangle &= (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{|v_k(\tau)|^2}{a^2} \\
&= (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}(1+k^2\tau^2)
\end{aligned}$$

- Superhorizon modes, $|k\tau| \ll 1$

$$\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{k}'}(\tau)\rangle \to (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}$$

Quantum Fluctuations in de Sitter

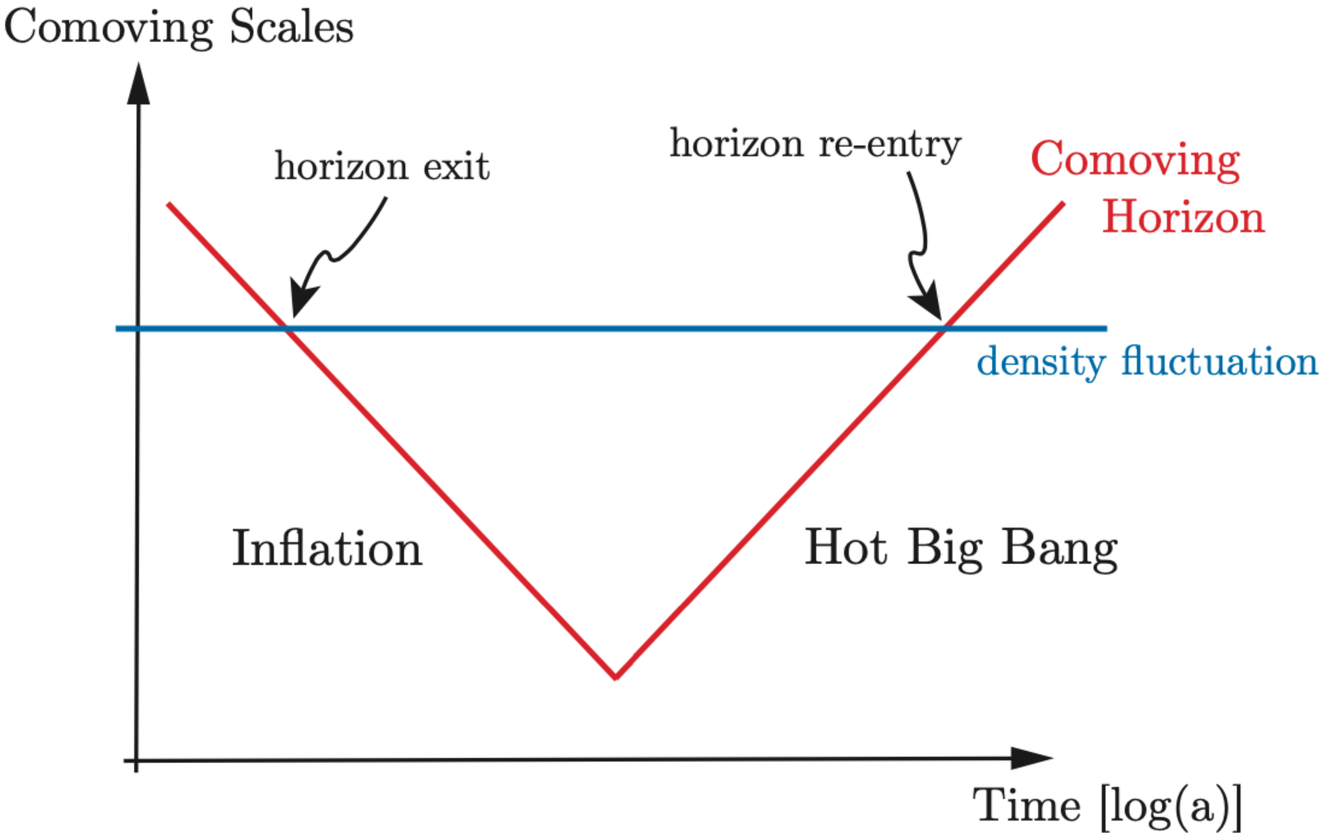
$$\langle \mathcal{R}_{\mathbf{k}}(t)\mathcal{R}_{\mathbf{k}'}(t)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H_\star^2}{2k^3}\frac{H_\star^2}{\dot{\phi}_\star^2} \qquad \Delta_{\mathcal{R}}^2(k) = \frac{H_\star^2}{(2\pi)^2}\frac{H_\star^2}{\dot{\phi}_\star^2}$$

From quantum to classical

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$$

$$v \equiv z\mathcal{R}\,,$$

$$z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$$



Contact with Observations

$$\mathcal{Q}_{\mathbf{k}}(\tau) = T_{\mathcal{Q}}(k, \tau, \tau_\star)\, \mathcal{R}_{\mathbf{k}}(\tau_\star)$$