Two-loop EW corrections to $e^+e^- \to ZH$ with one closed fermionic loop

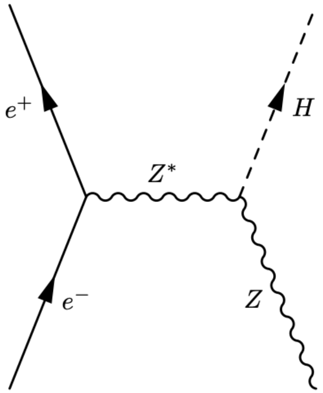
Qian Song
In collaboration with Ayres Freitas and Keping Xie
University of Pittsburgh
CEPC Workshop, 23-25 May, 2022

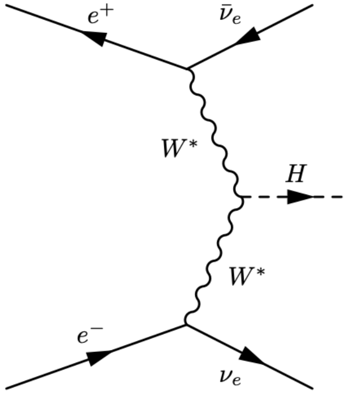
Content

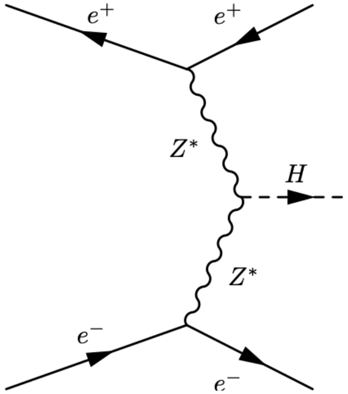
- Introduction
- Evaluation method
    - UV finite diagrams
    - UV divergent diagrams
- Summary

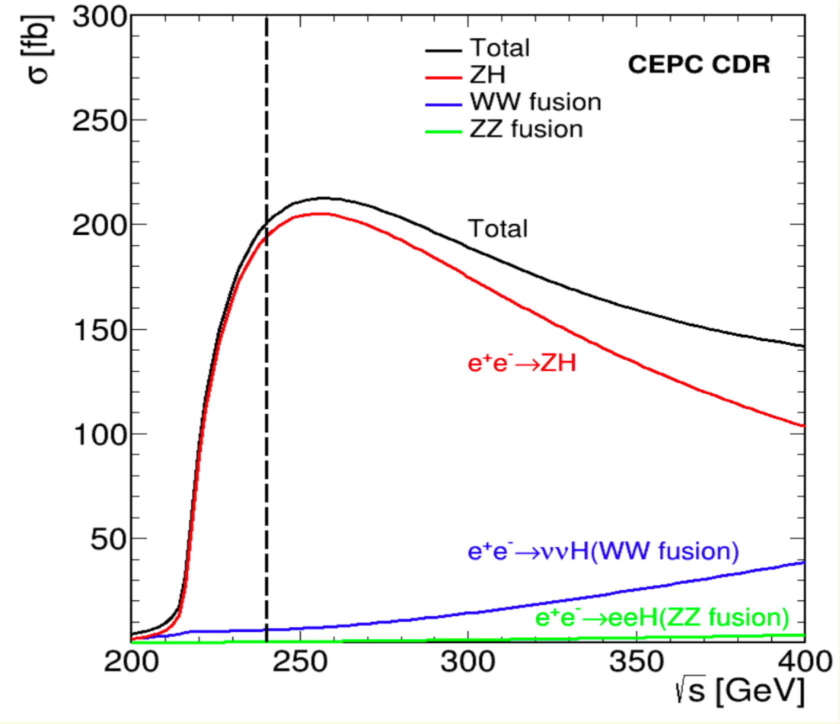
Experimental precision

- FCC-ee, CEPC, ILC: e+e- collider ($\sqrt{s}$=240-250GeV)
- Measure H properties with very high precision due to large statistics, high luminosity, clean environment
- Expected precision on $\sigma(e^+e^- \to ZH)$
  - ILC:1.2% H.Baer et al `13
  - FCC-ee:0.4% A.Abada et al `19
  - CEPC:0.5%. Y.Fang et al `18









Theoretical corrections

- LO : only consider s channel
- NLO: unpolarized beam: 5-10%

  J.Fleischer, F.Jegerlehner `83

  B.A.Kniehl `92

  A.Denner, J.Kublbeck, R.Mertig, M.Bohm `92

  polarized beam: 10-20%

  S.Bondarenko, Y.Dydyshka, L.Kalinovskaya, L.Rumyantsev, R.Sadykov and V.Yermolchyk `18

- NNLO(EW+QCD):

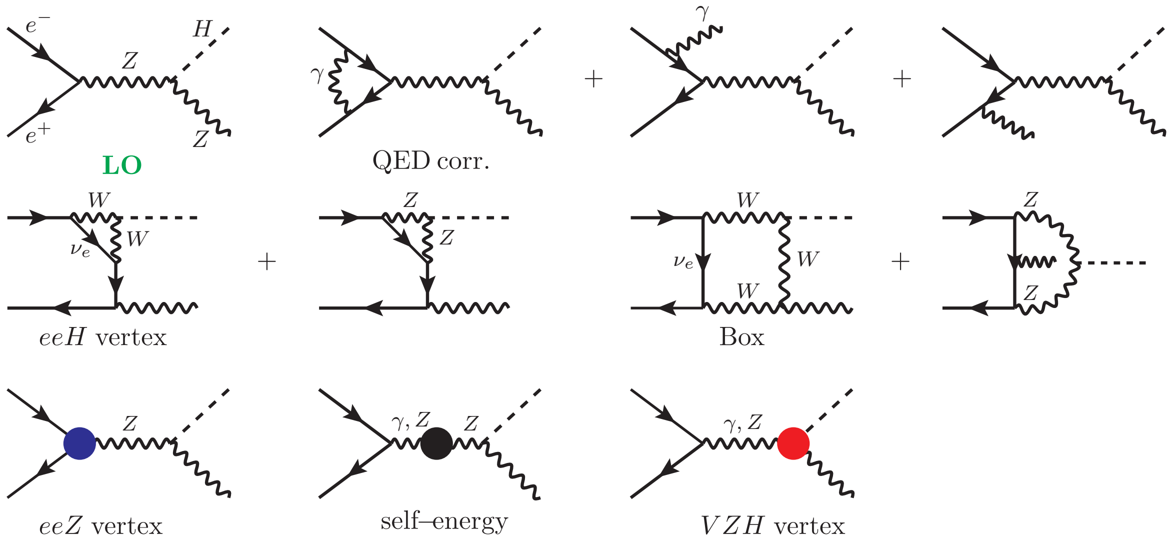
  0.4-1.3% $(\alpha(0), \alpha(M_Z), G_\mu)$

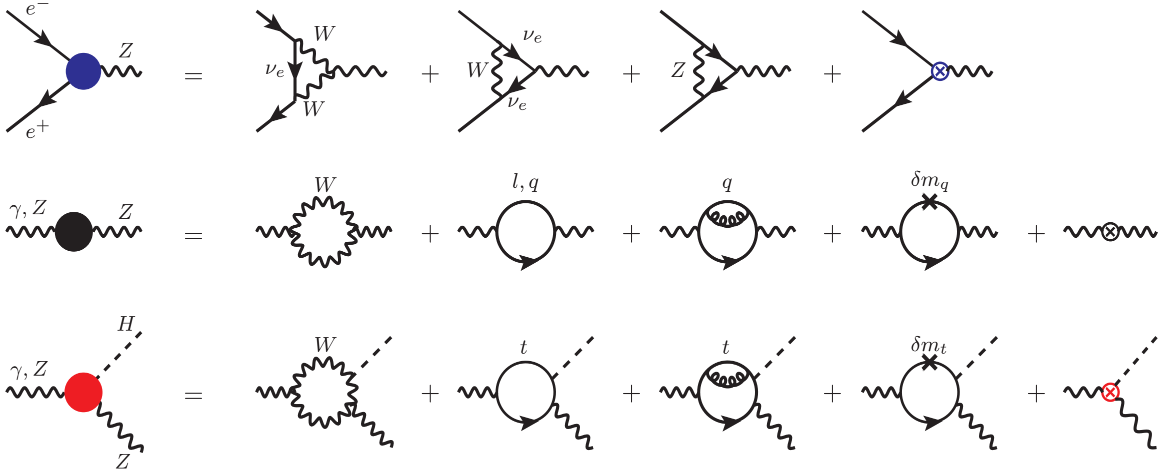
  Q. F. Sun, F. Feng, Y. Jia and W. L. Sang `16

  1.3% $(\overline{MS}, \alpha(M_Z))$

  Y. Gong, Z. Li, X. Xu, L. L. Yang and X. Zhao `17

- NNLO(EW+EW): with fermionic loop(this talk)



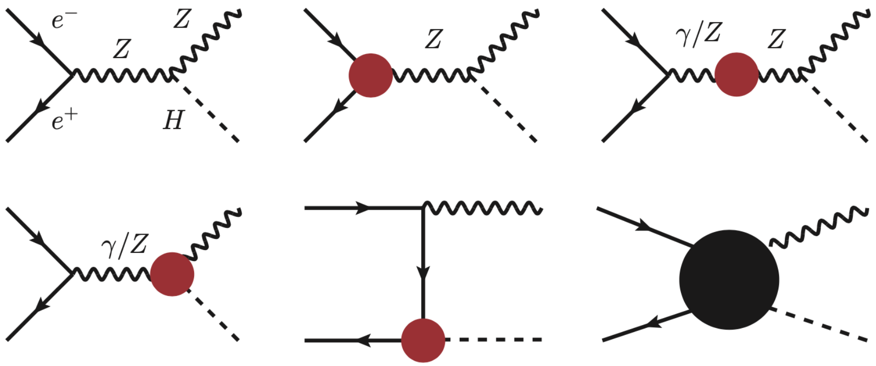


Two-loop EW correction

- NNLO(EW+EW): have an impact of $O(1\%)$ A.Freitas et al `19
- Diagrams with closed fermion loop dominant due to large top-quark Yukawa coupling and large multiplicity of fermions in SM I.Dubovyk, A.Freitas, J.Gluza,T,Riemann and J.Usovitsch '19
  - double box diagrams with top-loop: UV finite A.Freitas and Q.Song `21
    - most challenging
    - with 7 denominators
    - 4 independent mass scales(mz,mh,mw,mt), 2 independent energy scales(s,t)

  - box with triangle/self-energy subloop
  - two-loop vertices(eeZ, VZH, eeH)
  - self-energy diagrams

} simpler than double-box, but UV divergent



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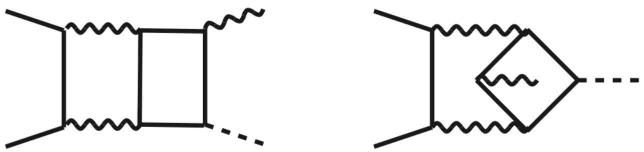

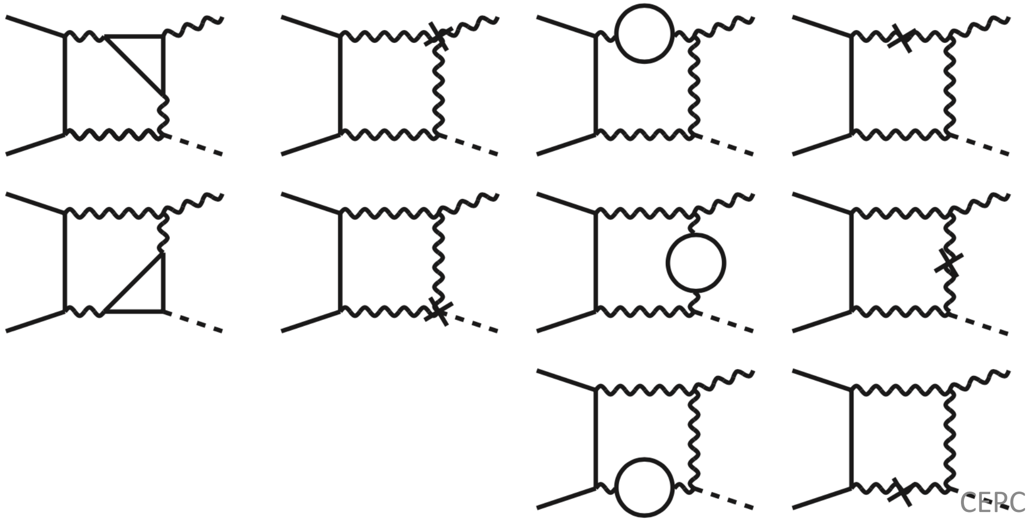

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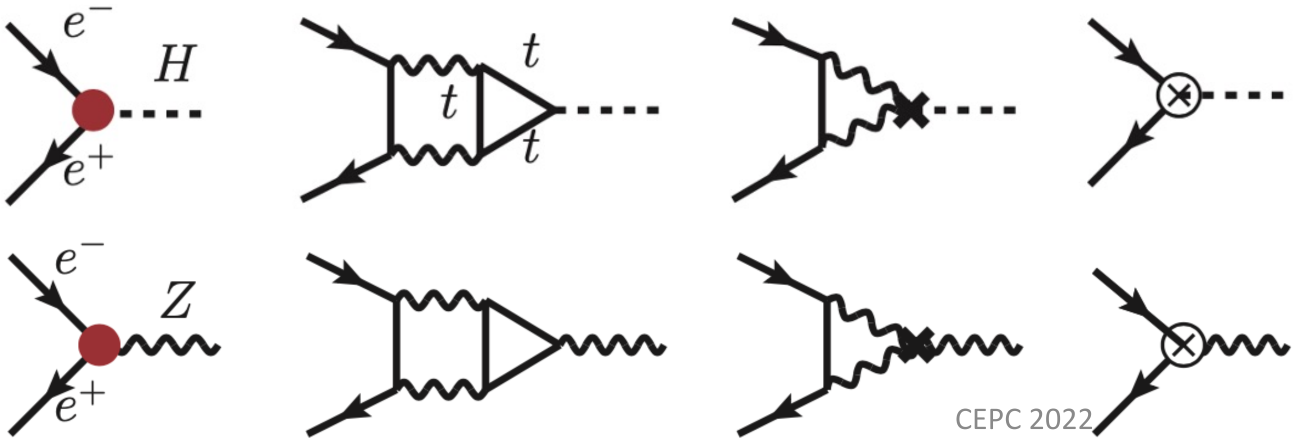
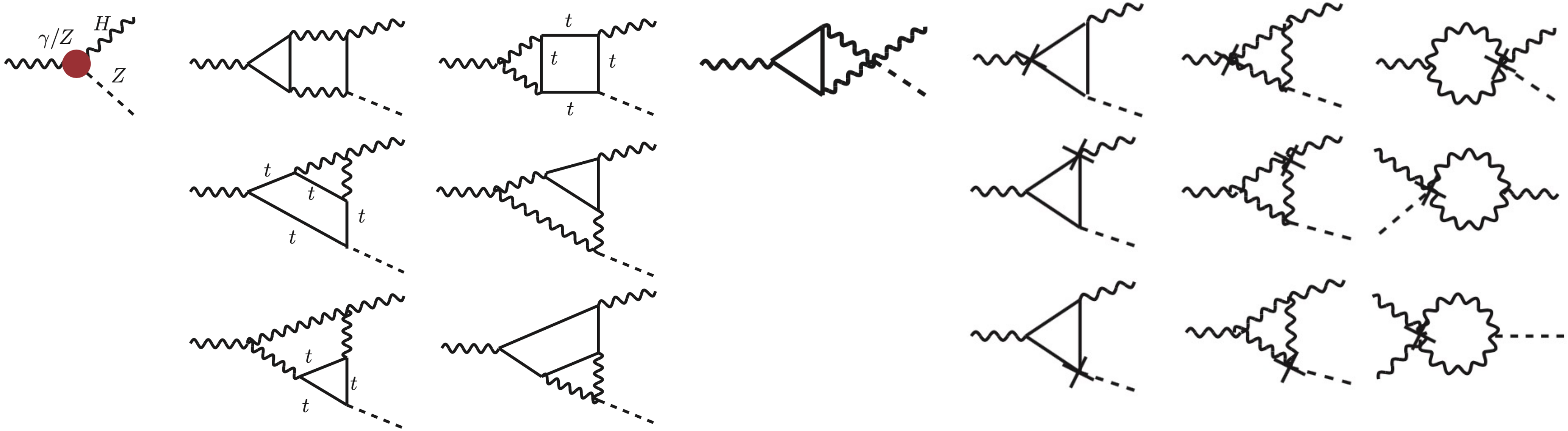
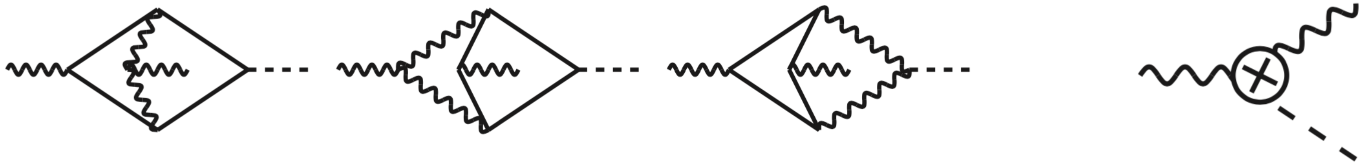


Two-loop EW correction

- two-loop planar VZH vertex:



- two-loop nonplanar VZH vertex:



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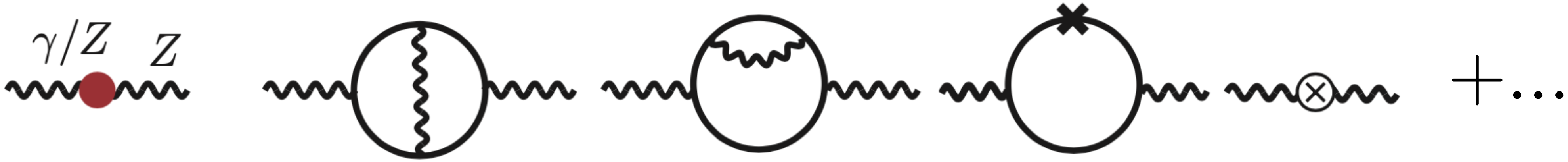

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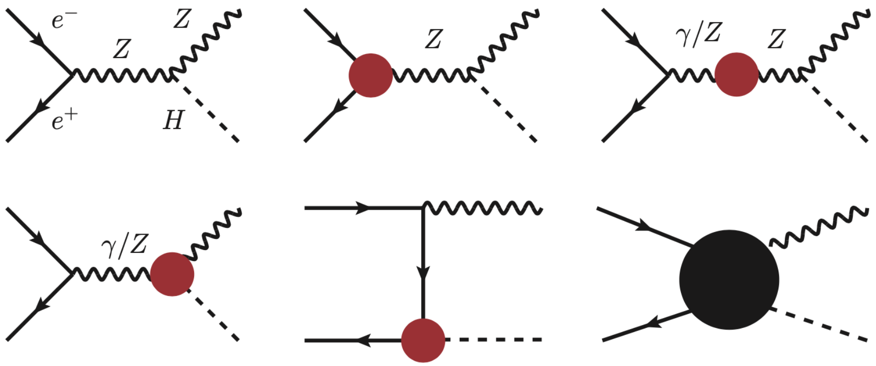
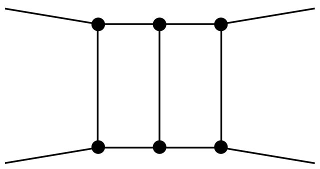


Evaluation method

- Analytical calculation: can be done for 1-loop, simple diagrams for 2-loop; generally difficult in 2-loop: require more knowledge about special functions(harmonic polylogarithmic functions, iterated elliptic integrals)



  - massless double-box diagram V.A.Smirnov `99
  - double-box diagram with 4 massive and 3 massless lines V.A.Smirnov `00

- Numerical calculation:

  - double-box diagram with arbitrary mass configuration: Feynman parametrization takes few days because integrand converges slowly F. Yuasa, E. de Doncker, N. Hamaguchi, T. Ishikawa, K. Kato, Y. Kurihara, J. Fujimoto and Y. Shimizu `12

$$I_{planar} = -\int_0^1 d\rho \int_0^1 d\xi \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^1 du_3 \int_0^{1-u3} du_4 \, \frac{\mathcal{C}}{(\mathcal{D} - i\epsilon\mathcal{C})^3} \rho^3 \xi^2 (1-\xi)^2$$

  - massless double-box diagram: Mellin-Barnes representation C.Anastasiou, J.B. Tausk and M.E.Tejeda-teomans '00

Evaluation method

- Our method: Feynman parametrization and dispersion relation
- Two momentum loops are completely decoupled, squared amplitude is expressed as

$$|MM^*| = \int dx \int d\sigma \, \Delta B_{\mu\nu}(\sigma, m_1^2, m_2^2) \times (c_1 A_0 + c_2 B_0 + c_3 C_0 + c_4 D_0 + c_{ij} D_{ij} + ...)$$

Integration variable from Feynman Parametrization and dispersion relation

Imaginary part of B functions, analytically known

Tensor/scalar PaVe functions from FeynCalc/LoopTools

- Double-box diagram: 3-fold numerical integral
- Box with triangle/self-energy subloop: 2-fold/1-fold numerical integral
- Two-loop vertex diagram: 2-fold numerical integral
- Two-loop self-energy diagram: This method can be used as crosscheck
  - Tensor reduction: FIRE/pySecDec or private code
  - Evaluate master integral: TVID/pySecDedc

A.V.Smirnov and F.S.Chukharev `19

B.Stefan, F.Ayres and W.Daniel `19

- Numerically stable(4-digit precision) and takes few minutes

Precision is confined by LoopTools(double/quadrupole precision), we use double precision

Calculation process

- Generate Feynman diagrams with FeynArts T.Hahn `00
- Calculate unpolarized squared amplitude with FeynCalc V. Shtabovenko, R. Mertig and F. Orellana `01
  - Trace not involving γ_5 is evaluated in D dimension(NDR) because of UV div
  - Trace involving γ_5 is treated in 4 dimension because that part is UV finite
- CT is calculated using on-shell scheme S. Bauberger, F. A. Berends, M. B˙ohm and M. Buza `94
- Use dispersion relation and Feynman parameterization to simplify squared amplitude
  - Using private code, squared amplitude is expressed as

$$|MM^*| = \int dx \int d\sigma \Delta B_{\mu\nu}(\sigma, m_1^2, m_2^2) \times (c_1 A_0 + c_2 B_0 + c_3 C_0 + c_4 D_0 + c_{ij} D_{ij} + \ldots)$$

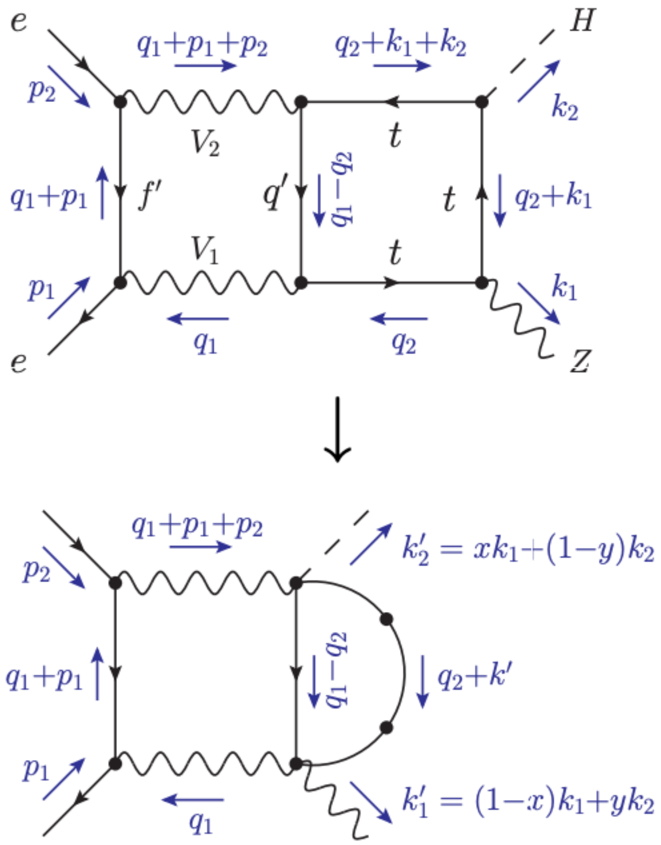
  - Squared amplitude is separated into UV finite and UV div part with dispersion relation
  - UV evaluation is different for triangle subloop, self-energy subloop, two-loop vertex
- Squared amplitude is evaluated numerically in C++ with LoopTools package, Gauss-Kronrod quadrature in Boost package T.Hahn `98 https://www.boost.org/

Planar double-box diagram

Use Feynman parametrization to simplify the denominators

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2} \underbrace{\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}}_{\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}}$$

$$= \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d^D_{q_1} \frac{B_0((q_1+k')^2, m_{q'}^2, m'^2)}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



Loop momentum q1 appears in B0 functions,
so cannot integrate over q1.
→ use dispersion relation to put q1 outside B0 function

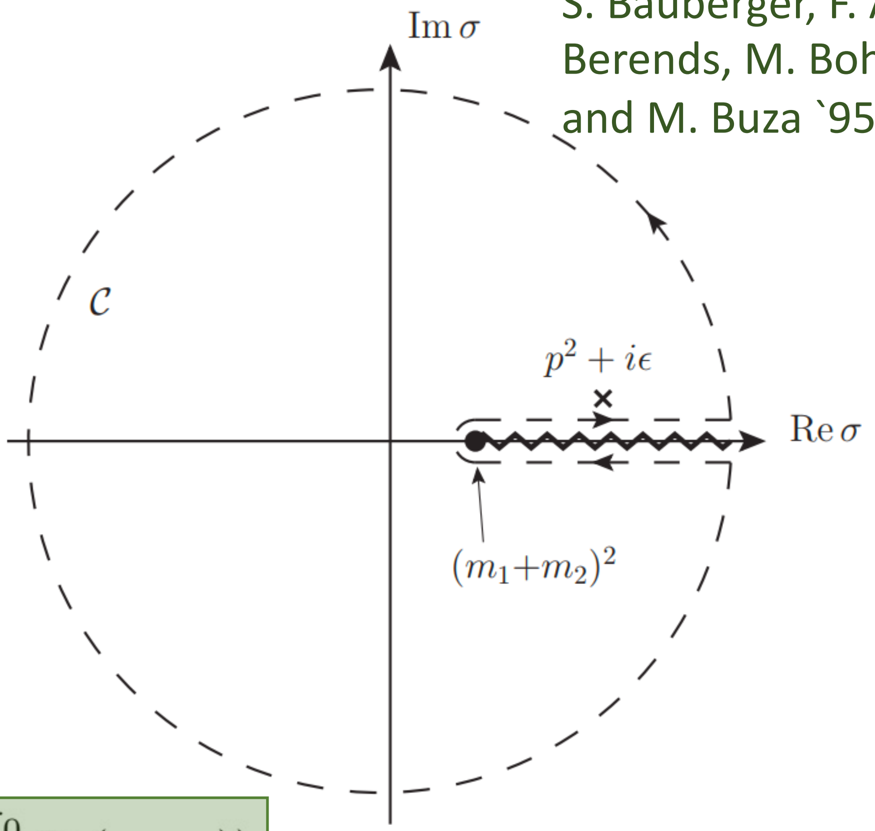
Planar double-box diagram

dispersion relation:

$$B_0(p^2, m_1^2, m_2^2) = \frac{1}{2\pi i} \oint_C d_\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\varepsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{\Delta B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$

$$= \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{1}{\pi} \frac{Im B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$



S. Bauberger, F. A. Berends, M. Bohm and M. Buza `95

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m_{q'}^2) D_0(..., \sigma)$$

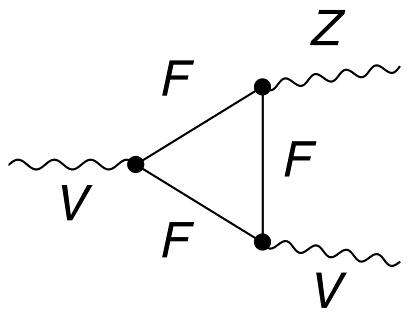
$$= \int_0^1 dx \int_0^{1-x} dy \int_{(m'+m_{q'})^2}^{\infty} d\sigma \partial^2_{m'^2} \Delta B_0(s, m'^2, m_{q'}^2)(D_0(..., \sigma) - \frac{\sigma_0}{\sigma} D_0(..., \sigma_0))$$

$$+ \int_0^1 dx \int_0^{1-x} dy \sigma_0 D_0(..., \sigma_0) \partial^2_{m'^2} B_0(0, m'^2, m_{q'}^2)$$

cancel the unphysical divergence at lower bound

Box diagram with triangle fermionic subloop

- Triangle loop is UV divergent.



$$\int d^4q \frac{q^\mu q^\nu}{(q^2 - m_0^2)((q+p_1)^2 - m_1^2)((q+p_2)^2 - m_2^2)} \overset{q\to\infty}{\approx} \int dq \frac{1}{q} = log(\infty)$$

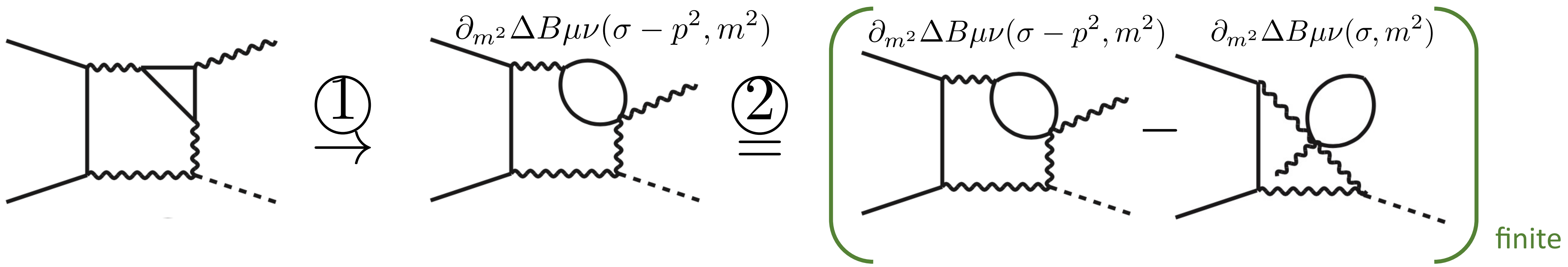
- With Feynman parametrization and dispersion relation, the UV divergence at large q becomes UV divergence at large σ

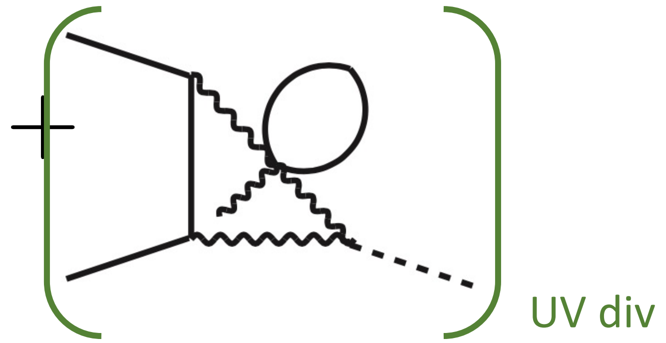
$$I = \int d^Dq \frac{q^\mu q^\nu}{(q^2 - m_0^2)((q+p_1)^2 - m_1^2)((q+p_2)^2 - m_2^2)} = \int d^Dq \int_0^1 dx \partial_{m^2} \frac{q^\mu q^\nu}{(q^2 - m_0^2)((q+p)^2 - m^2)}$$

$$= \int_0^1 dx \partial_{m^2} B^{\mu\nu}(p^2, m_0^2, m^2)$$

$$= \int_0^1 dx \int_{\sigma_0}^{\infty} d\sigma \frac{\partial_{m^2} \Delta B^{\mu\nu}(\sigma, m_0^2, m^2)}{\sigma - p^2} \quad \overset{\sigma\to\infty}{\approx} \int d\sigma \frac{1}{\sigma} = log(\infty)$$

Box diagram with triangle fermionic subloop





1. Feynman parametrization and dispersion relation
   divergence of fermionic loop → $\Delta B_{\mu\nu}(\sigma - p^2)$ at large σ

2. $\lim_{\sigma\to\infty} \partial_{m^2}\Delta B\mu\nu(\sigma - p^2, m^2) - \partial_{m^2}\Delta B\mu\nu(\sigma, m^2) \to \text{finite}$

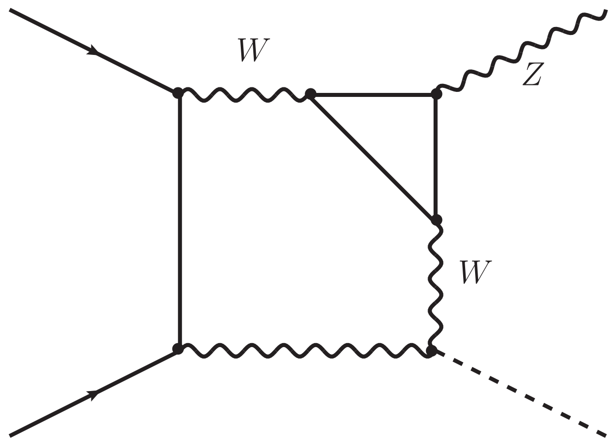
3. $\partial_{m^2}\Delta B\mu\nu(\sigma, m^2)$ can be integrated analytically:

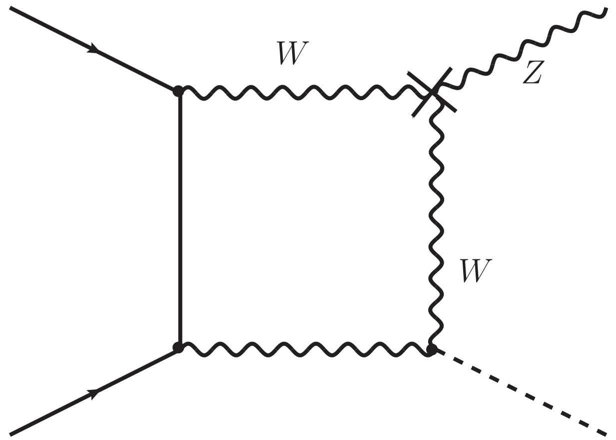
   $$\int d\sigma \partial_{m^2}\Delta B\mu\nu(\sigma, m^2) = \partial_{m^2} B_{\mu\nu}(0, m^2)$$

   bosonic subloop is expressed using scalar PaVe functions

4. Divergent part cancels with CT diagram

Box diagram with triangle fermionic subloop





div: (-2.808566*10^-6 + 1.176922*10^-6*I)/ϵ
finite: 2.871395*10^-5 - 1.562527*10^-5*I

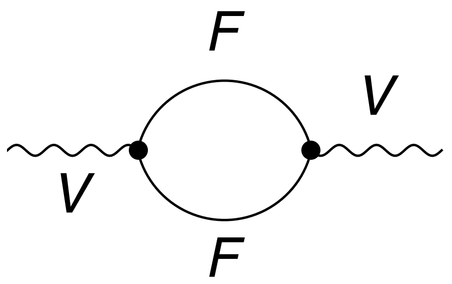
div: (2.808566*10^-6 - 1.176922*10^-6*I)/ϵ
finite: (-2.317469*10^-5 + 1.049388*10^-5*I) -
(1.288853*10^-4 - 5.102569*10^-5*)*deltaAlpha

Sum of loop and CT is UV finite

Box diagram with self-energy fermionic subloop

- Self-energy loop is UV divergent.

$$\int d^4q \frac{q^\mu q^\nu}{(q^2 - m_1^2)((q+p)^2 - m_2^2)} \overset{q\to\infty}{\approx} \int dq\, q = \infty$$

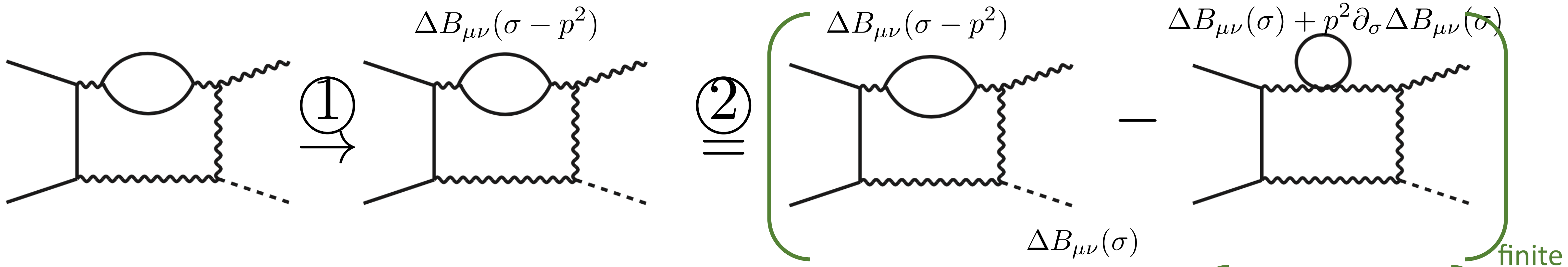


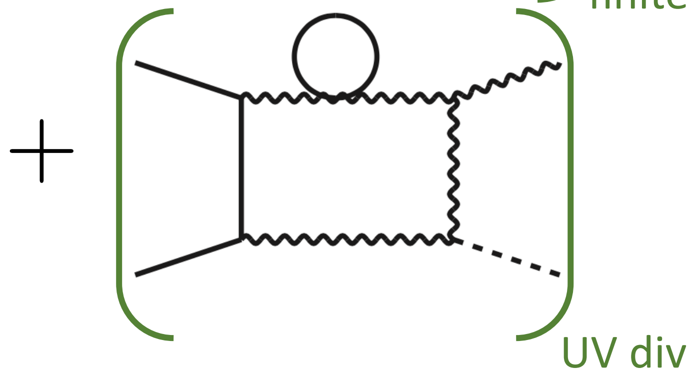
- With dispersion relation, the UV performance at large q becomes UV divergence at large σ

$$\int d^4q \frac{q^\mu q^\nu}{(q^2 - m_1^2)((q+p)^2 - m_2^2)}$$

$$= \int_{\sigma_0}^{\infty} d\sigma \frac{\Delta B^{\mu\nu}(\sigma, m_1^2, m_2^2)}{\sigma - p^2} \overset{\sigma\to\infty}{\approx} \int d\sigma = \infty$$

Box diagram with self-energy fermionic subloop





1. Feynman parametrization and dispersion relation
   divergence of fermionic loop $\rightarrow \Delta B_{\mu\nu}(\sigma - p^2)$ at large σ

2. $\lim_{\sigma\to\infty} \Delta B_{\mu\nu}(\sigma - p^2) - \Delta B_{\mu\nu}(\sigma) - p^2\partial_\sigma \Delta B_{\mu\nu}(\sigma) \rightarrow \text{finite}$

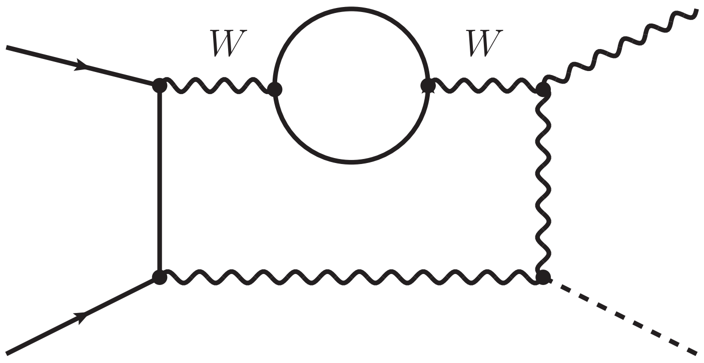
3. $\Delta B_{\mu\nu}(\sigma) + p^2\partial_\sigma \Delta B_{\mu\nu}(\sigma)$ can be integrated analytically:

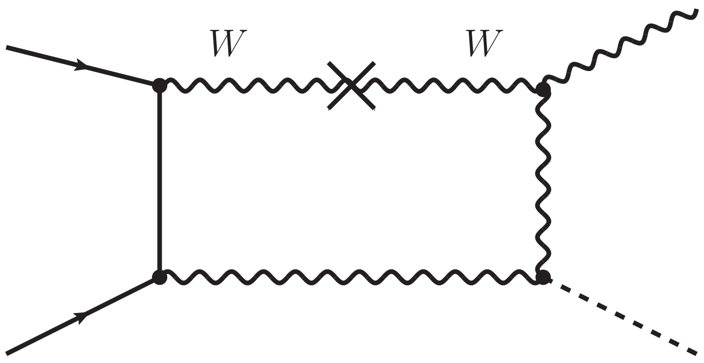
$$\int d\sigma \Delta B_{\mu\nu}(\sigma) + p^2\partial_\sigma \Delta B_{\mu\nu}(\sigma) = B_{\mu\nu}(0) + p^2 \frac{\partial B_{\mu\nu}(ps)}{\partial ps}|_{ps\to 0}$$

   bosonic subloop is expressed using scalar PaVe functions

4. Divergent part cancels with CT diagram

Box diagram with self-energy fermionic subloop



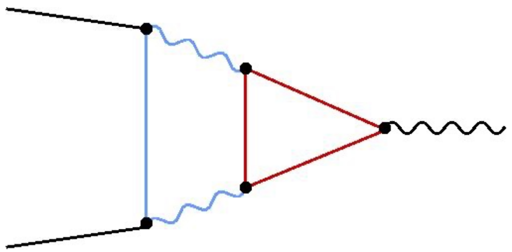


div: (6.449289*10^-7 + 3.513202*10^-15*I)/ϵ
finite: -6.847403*10^-6 – 3.854979*10^-14*I

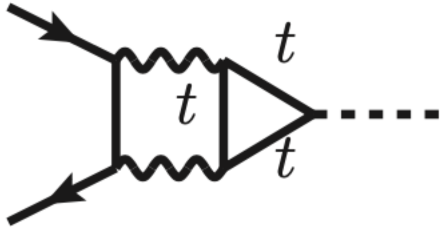
div: (-6.449289*10^-7 -7.518501*10^-15*I)/ϵ
finite: 6.963528*10^-6 + 8.438749*10^-14*I

Sum of loop and CT is UV finite

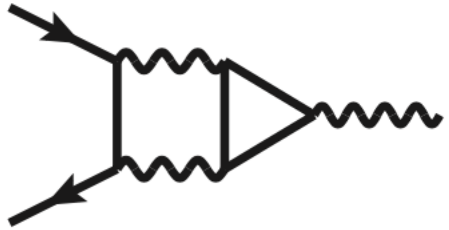
Two-loop vertex diagram



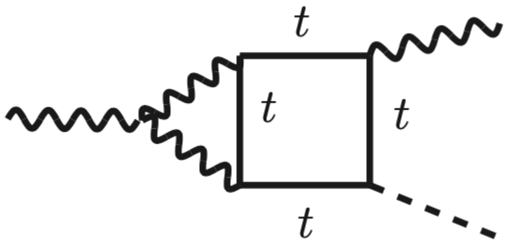
- Two loop vertex has two different UV divergence
  - local divergence($\frac{1}{\epsilon}$): one subloop(triangle fermionic loop), dependent on the other loop(blue)
  - global divergence($\frac{1}{\epsilon^2}$): highest order divergence, independent on loop external momentum



- Two-loop vertex has 3 UV behaviors
  - only local divergence(eeH vertex):
    - use dispersion relation to separate UV divergence(same as box with triangle subloop)
  - 1 local + global divergence(eeZ vertex):
    - dispersion relation fails to separate overlapping divergence
    - subtract a diagram that can cancel global divergence, and add it back
    - use dispersion to separate local divergence

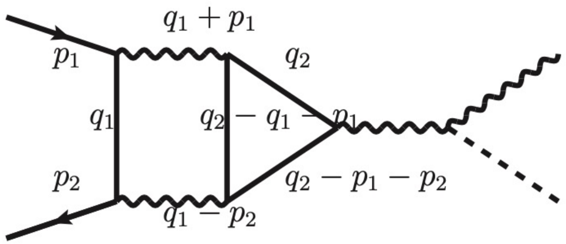


  - 2 local + global divergence(VZH vertex):
    - dispersion relation fails to separate overlapping divergence
    - subtract a diagram that can cancel global divergence ,and add it back
    - use dispersion relations twice to cancel 2 local divergences separately

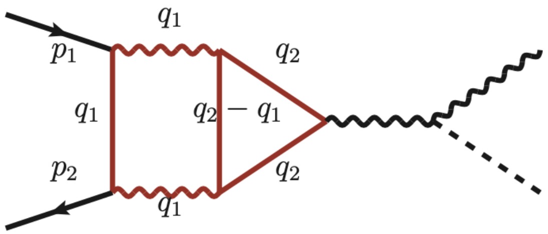


Global divergence

- The diagram we subtract satisfies two points
  - cancel global divergence: global div is decided by the vertex, not external momentum
  - can be calculated analytically: mass scale is as less as possible
- The proper choice is two-loop vertex with same structure and zero external momentum $p_1 = p_2 = 0$



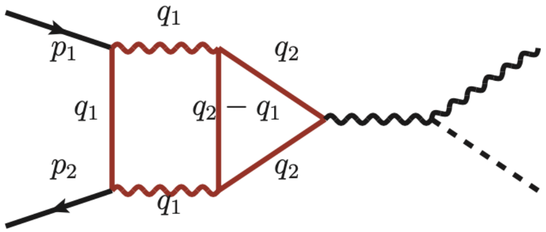
$-$



$$\Rightarrow \frac{C_1}{\epsilon} + C_2$$

Local divergence C1 is separated using dispersion relation

$+$

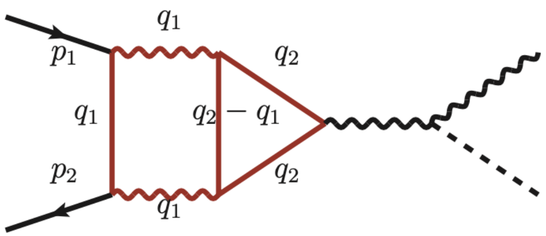


$$\Rightarrow \frac{D_1}{\epsilon^2} + \frac{D_2}{\epsilon} + D_3$$

Similar to vacuum bubble and can be calculated analytically

Red loop: external momentum are set to be 0

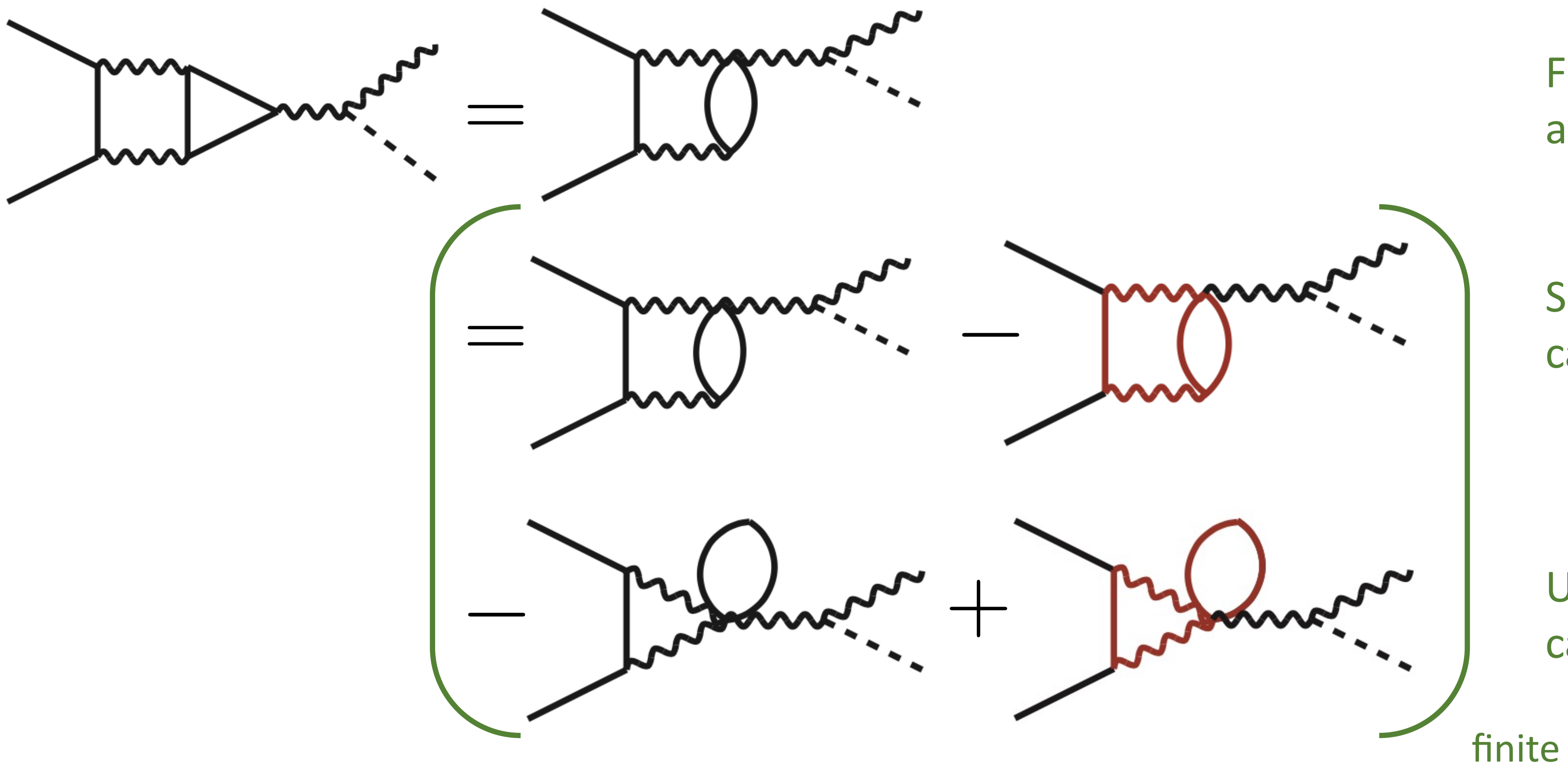
Global divergence



$$\Rightarrow \frac{D_1}{\epsilon^2} + \frac{D_2}{\epsilon} + D_3$$

- Squared amplitude is calculated analytically
- Tensor decomposition: reduce tensor integral to scalar integral
- Reduce scalar integral to master one by using FIRE — A.V.Smirnov and F.S.Chukharev `19
- Master integral can be evaluated numerically and analytically with TVID — B.Stefan, F.Ayres and W.Daniel `19

Two-loop vertex: global+1 local divergence

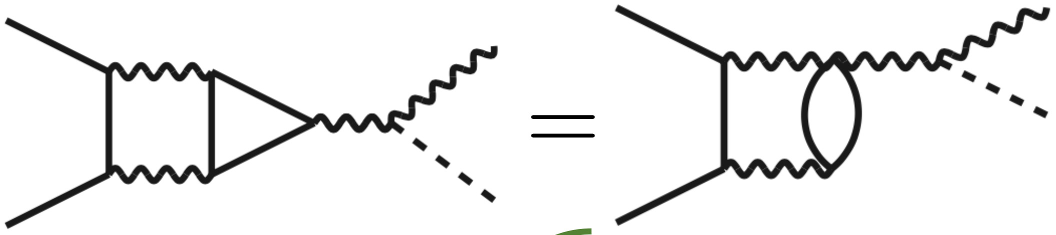


Feynman parametrization and dispersion relation

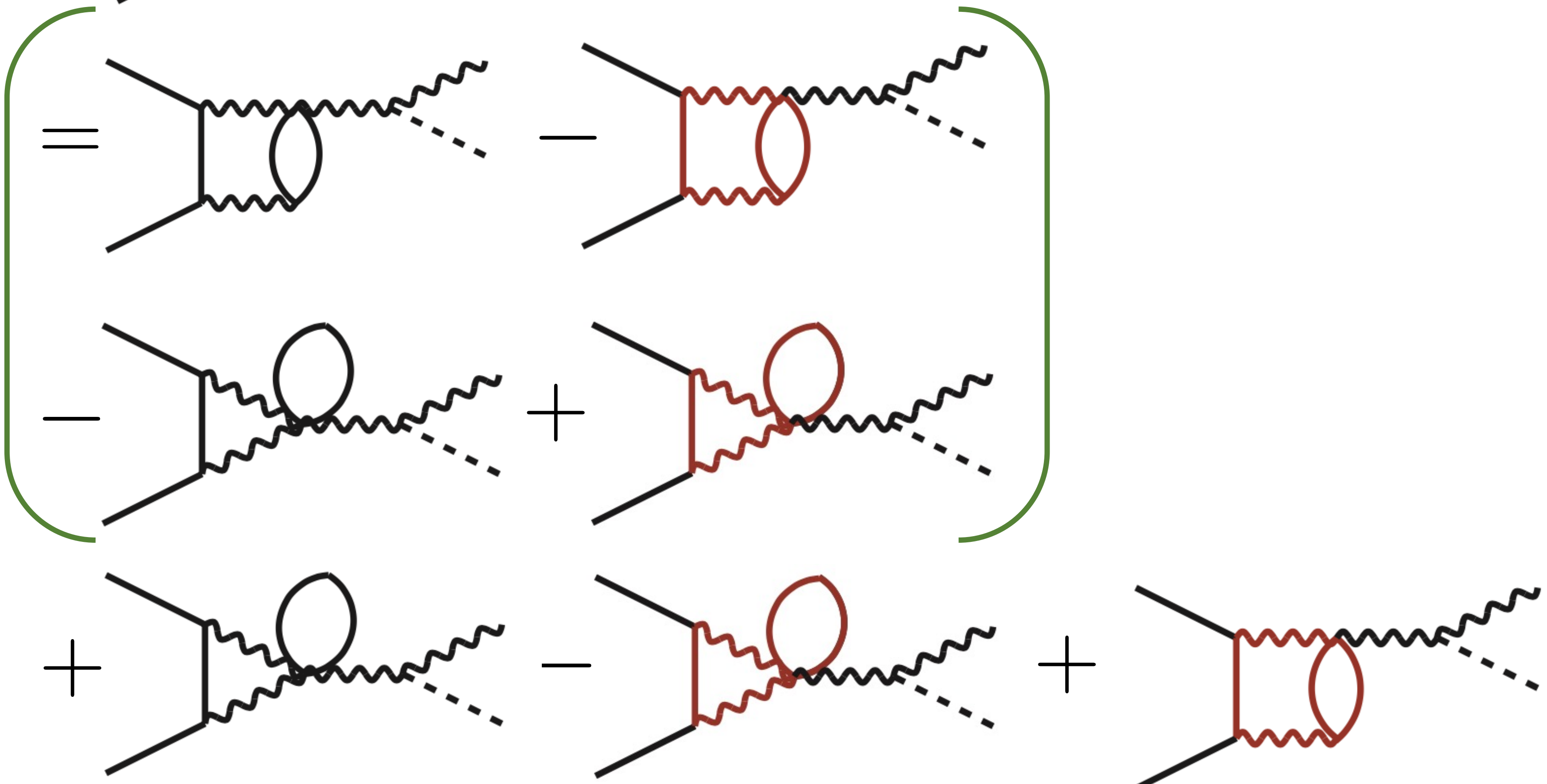
Subtract a diagram to cancel global divergence

Use dispersion relation to cancel local divergence

Two-loop vertex: global+1 local divergence

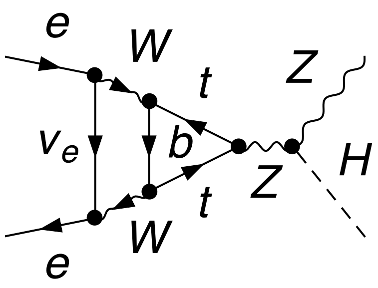


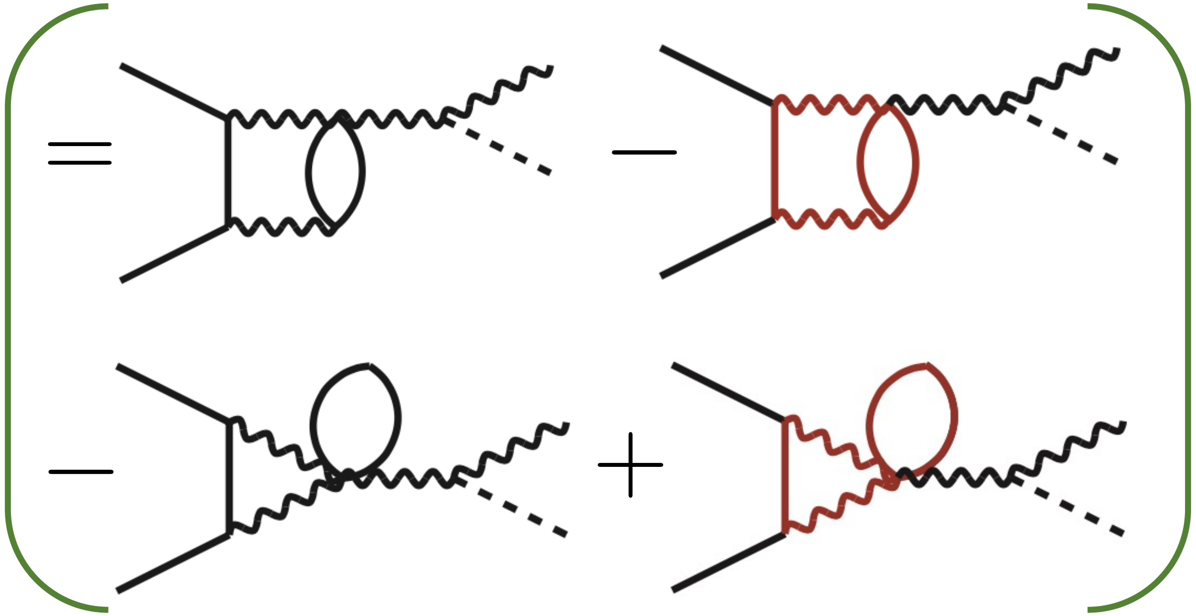
UV finite: are integrated together in C++



Evaluated analytically in Mathematica

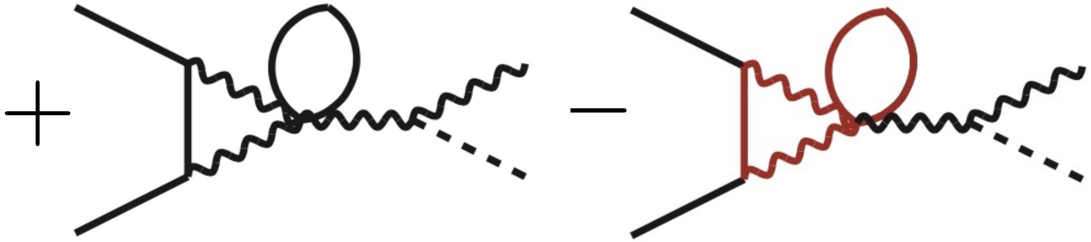
Two-loop vertex: global+1 local divergence





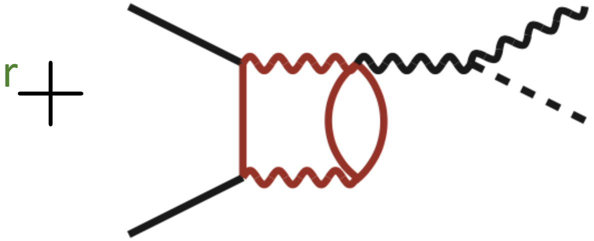
$$\frac{0}{\epsilon^2} + \frac{0}{\epsilon} + 1.12765 \times 10^{-7}$$

These two diagrams also have the same global divergence, so no $1/\epsilon^2$



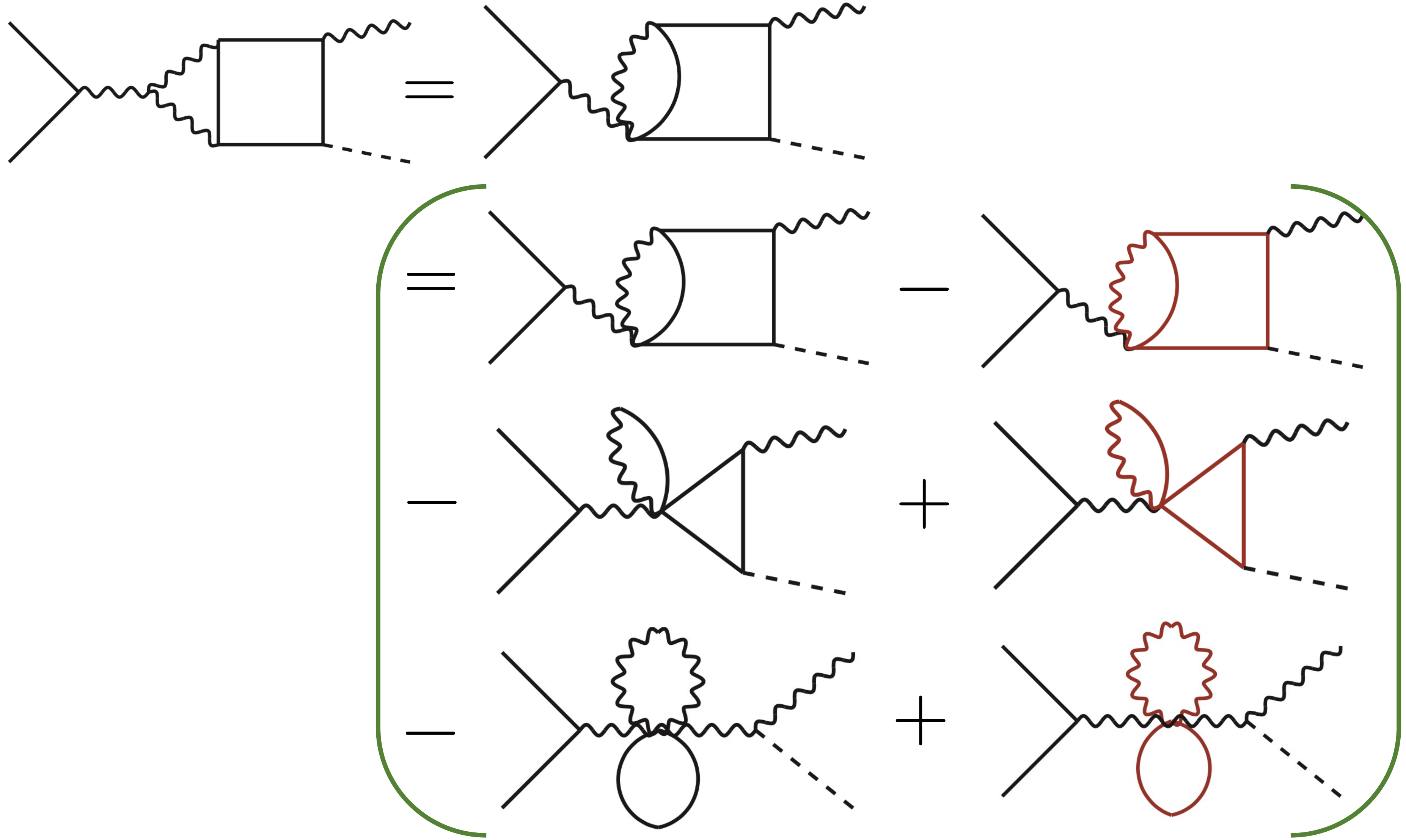
$$\frac{0}{\epsilon^2} + \frac{1.13118 \times 10^{-8}}{\epsilon} - 4.19241 \times 10^{-7}$$

Divergence should cancel after combining all CT diagrams



$$\frac{1.58587 \times 10^{-8}}{\epsilon^2} + \frac{-2.78273 \times 10^{-7}}{\epsilon} + 2.236153 \times 10^{-6}$$

Two-loop vertex: global+2 local divergence



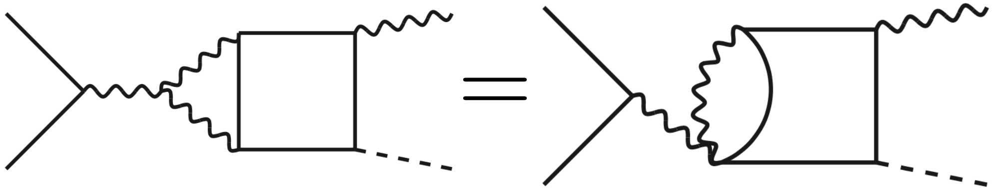
Feynman parametrization and dispersion relation

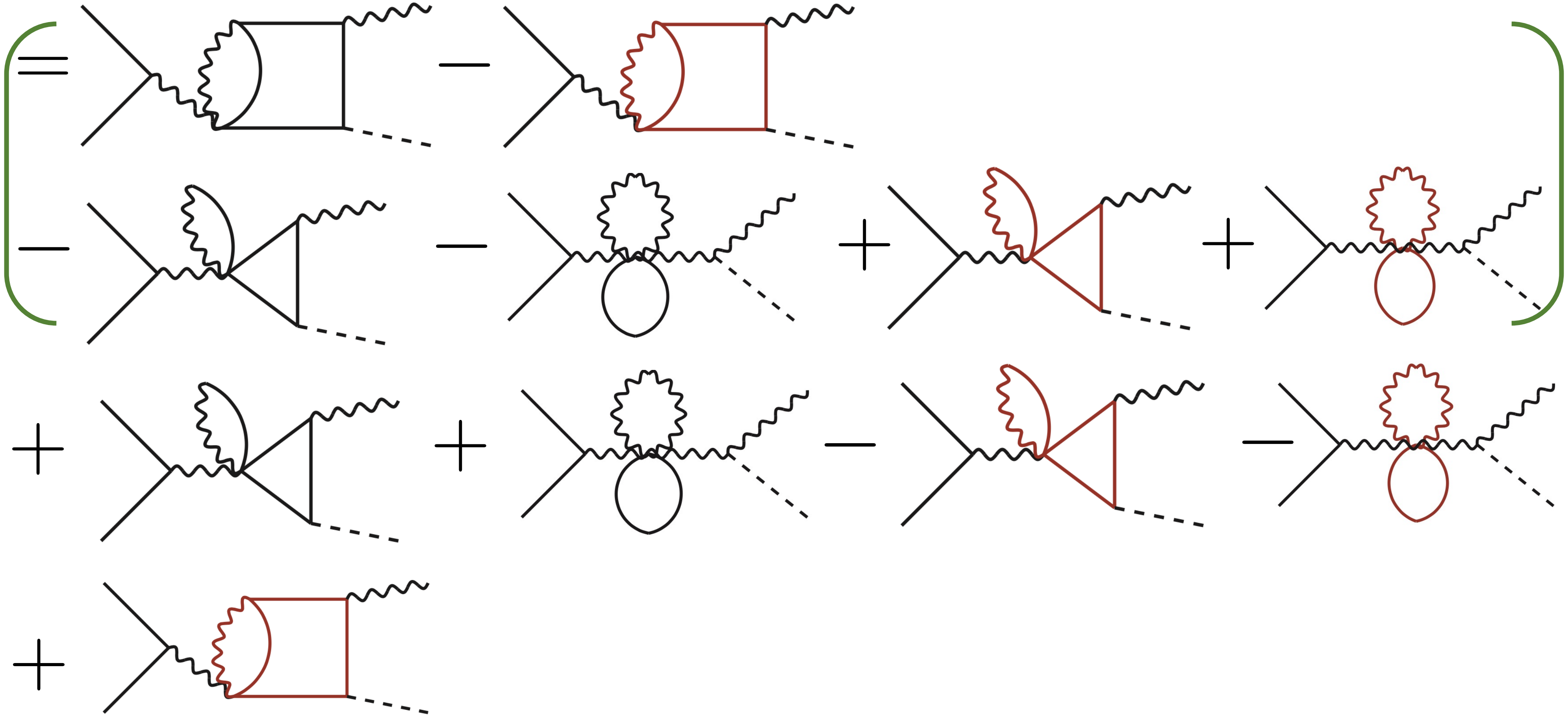
Subtract a diagram to cancel global divergence

Use dispersion relation twice to cancel two local divergence separately

finite

Two-loop vertex: global+2 local divergence

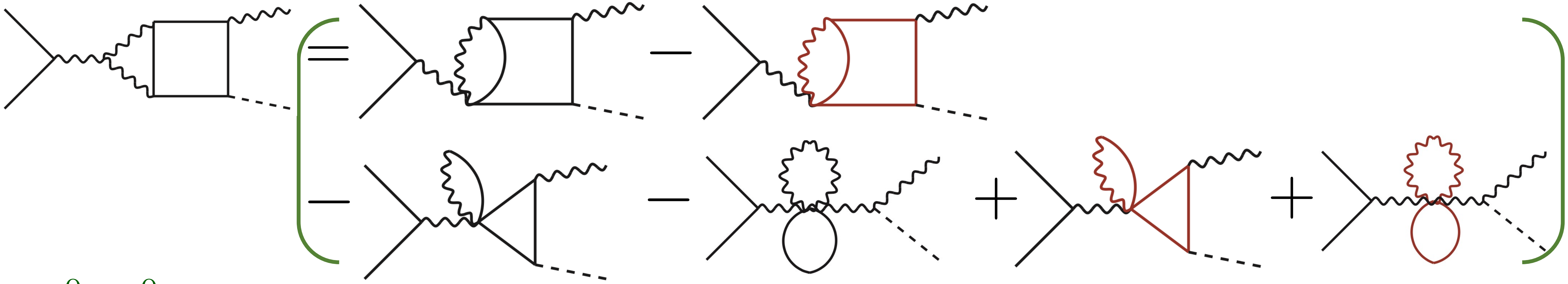




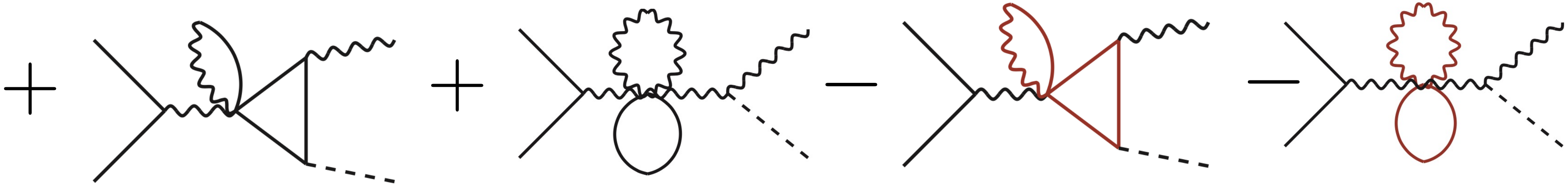
UV finite: are integrated together in C++

Evaluated analytically in Mathematica

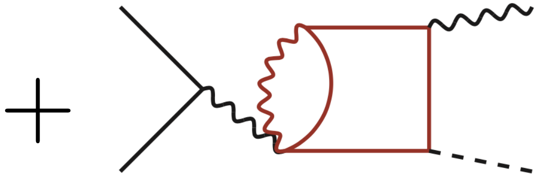
Two-loop vertex: global+2 local divergence



$$\frac{0}{\epsilon^2}+\frac{0}{\epsilon}+7.61565\times 10^{-7}$$



$$\frac{0}{\epsilon^2}-\frac{1.98406\times 10^{-8}}{\epsilon}-7.2425\times 10^{-6}$$



$$\frac{1.76463\times 10^{-7}}{\epsilon^2}-\frac{3.97028\times 10^{-6}}{\epsilon}+0.0000453674$$

Divergence should cancel after combining all CT diagrams

Summary

- Two-loop Electroweak correction must be included because of expected high precision at future e^+e^- colliders.
- Diagrams with closed fermionic subloop are dominated
- Double-box diagrams are the most challenging one
- Box with triangle/SE subloop, two-loop vertex, and self-energy diagrams: numerically simpler, but UV divergent
- Using Feynman parametrization and dispersion relation, squared amplitude can be simplified to 3/2/1-fold integral.
- Numerical integration takes few minutes with 4-digit precision.
- Dispersion relation is used to separate local UV divergence, canceled by CT.
- For diagrams with global and local divergences, additional diagram is needed to cancel global divergence and additional terms are needed to cancel local divergence.
- Additional diagram/terms are added back and calculated both analytically and numerically.

Thank you!

1. Numerical integration technique

- Feynman Parametrization
  - Introduce 3 set Feynman parameters to square amplitude for all propagators with loop momentum q1,q2,q1+q2 respectively
  - With decomposition and derivative with respect to mass, momentum, square amplitude can be written as a linear combination of minimal set of scalar integrals
  - UV divergent part of the scalar integrals are analytically known, and finite part can be integrated numerically

1. Numerical integration technique

- Sector decomposition
  - Introduce N Feynman parameters to square amplitude with N propagators
  - Introduce N primary sector such that every sector I_k is N-1 fold integral
  - Decomopose I_k into subsectors such that all divergence is decomposed and factorized to variable
  - Taylor expansion is performed to separate divergent and finite part
  - Left integral can be integrated numerically
- Subtraction terms
- Differential equations
- …

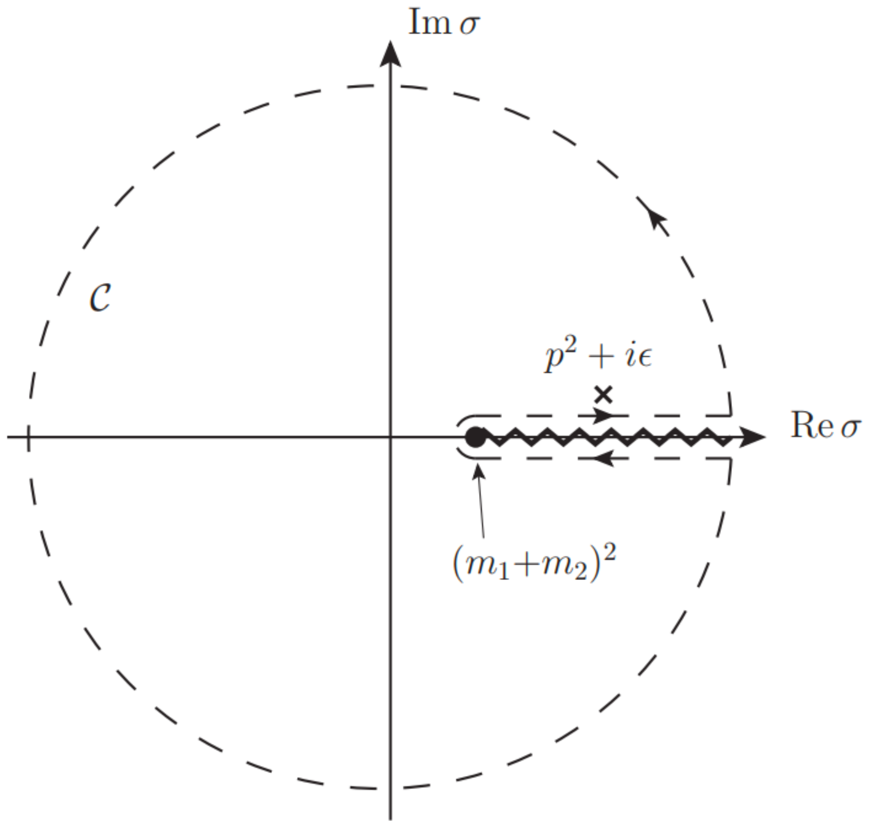
2. Dispersion relation

$m_1'^2 > 0, m_2'^2 > 0$

dispersion relation

$$B_0(p^2, m_1^2, m_2^2) = \frac{1}{\pi}\int_{\sigma_0}^{\infty} d\sigma \frac{ImB_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$

$$\begin{aligned}
B_0(p^2, m_1^2, m_2^2) &= \frac{1}{2\pi i}\oint_C d\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{2\pi i}\left(\int_{\sigma_0}^{\infty} + \int_{c1} + \int_{c2} + \int_{\infty}^{\sigma_0}\right) d\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{2\pi i}\int_{\sigma_0}^{\infty} d\sigma \frac{B_0(\sigma + i\delta, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} + \int_{\infty}^{\sigma_0} d\sigma \frac{B_0(\sigma - i\delta, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{2\pi i}\int_{\sigma_0}^{\infty} d\sigma \frac{B_0(z, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} + \int_{\infty}^{\sigma_0} d\sigma \frac{B_0(z^*, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{2\pi i}\int_{\sigma_0}^{\infty} d\sigma \frac{B_0(z, m_1^2, m_2^2) - B_0^*(z, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{2\pi i}\int_{\sigma_0}^{\infty} d\sigma \frac{2iImB_0(z, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} \\
&= \frac{1}{\pi}\int_{\sigma_0}^{\infty} d\sigma \frac{ImB_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}
\end{aligned}$$



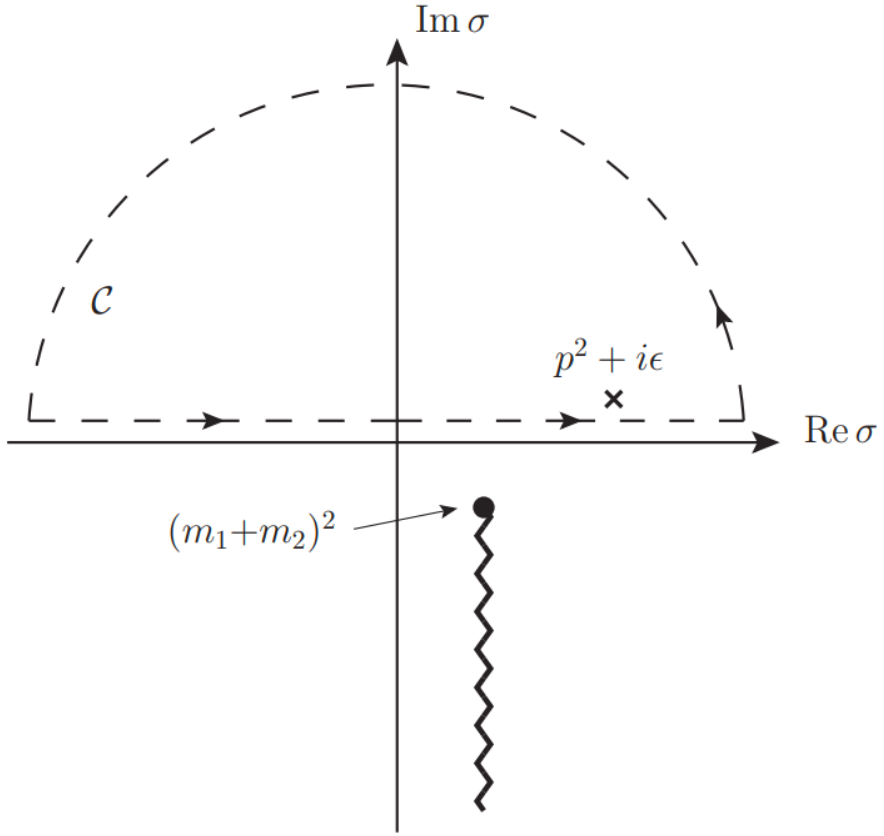
S. Bauberger, F. A. Berends, M. B̈ohm and M. Buza, Nucl. Phys. B434, 383 (1995)[hep-ph/9409388]

2. Dispersion relation

- $m_1'^2 < 0, m_2'^2 > 0$

$$B_0(p^2, {m'}_1^2, {m'}_2^2) = \frac{1}{2\pi i}\oint_C d\sigma \frac{B_0(\sigma, {m'}_1^2, {m'}_2^2)}{\sigma - p^2 - i\varepsilon}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\sigma \frac{B_0(\sigma, {m'}_1^2, {m'}_2^2)}{\sigma - p^2 - i\epsilon}$$



3. UV div evaluation: box with triangle subloop

- Subtract one term to make the integrand finite at large σ and add it back
- The term we add back can be separated into divergent part and finite part

$$= \int_0^1 dx \int_{\sigma_0}^{\infty} d\sigma \partial_{m^2} \Delta B^{\mu\nu}(\sigma, m_0^2, m^2)(\frac{1}{\sigma - p^2} - \frac{1}{\sigma} + \frac{1}{\sigma})$$

$$= \int_0^1 dx \int_{\sigma_0}^{\infty} d\sigma \partial_{m^2} \Delta B^{\mu\nu}(\sigma, m_0^2, m^2)(\frac{1}{\sigma - p^2} - \frac{1}{\sigma}) \overset{\sigma\to\infty}{\approx} \int d\sigma \frac{1}{\sigma^2} \to \text{UV finite}$$

$$+ \int_0^1 dx \partial_{m^2} \Delta B^{\mu\nu}(0, m_0^2, m^2) \to \underbrace{\partial_{m^2} \Delta B^{\mu\nu}|_{div}}_{\text{Cancel with CT}} + \partial_{m^2} \Delta B^{\mu\nu}|_{finite}$$

4. UV div evaluation: box with triangle subloop

- Subtract one term can't make the integrand finite
- Subtract two terms to make the integrand finite at large σ and add it back
- Two terms we add back can be separated into divergent part and finite part

$$\int_{\sigma_0}^{\infty} d\sigma \Delta B^{\mu\nu}(\sigma, m_1^2, m_2^2)\left(\frac{1}{\sigma - p^2} - \frac{1}{\sigma} - \frac{p^2}{\sigma^2} + \frac{1}{\sigma} + \frac{p^2}{\sigma^2}\right)$$

$$\int_{\sigma_0}^{\infty} d\sigma \Delta B^{\mu\nu}(\sigma, m_1^2, m_2^2)\left(\frac{1}{\sigma - p^2} - \frac{1}{\sigma} - \frac{p^2}{\sigma^2}\right) \overset{\sigma\to\infty}{\approx} \int d\sigma \frac{1}{\sigma^2} \to \text{UV finite}$$

$$+ B^{\mu\nu}(0, m_1^2, m_2^2) + \partial_{m^2} B^{\mu\nu}(m^2, m_1^2, m_2^2)|_{m^2=0} \to \underbrace{B^{\mu\nu}|_{div} + \partial_{m^2} B^{\mu\nu}|_{div}}_{\text{Cancel with CT}} + B^{\mu\nu}|_{finite} + \partial_{m^2} B^{\mu\nu}|_{finite}$$

5. double-box diagram

Integrating over q1 gets the D0 function.
Use Leibiniz's rule to put the derivative inside the integral: ΔB_0 is divergent at the lower bound, it can be fixed by subtracting one term to make the integrand become 0 at the lower bound.

$$I_{plan} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \int d^D_{q_1} \Delta B_0(s, m'^2, m^2_{q'})$$
$$\frac{1}{(q_1^2 - m^2_{V_1})((q_1+p_1)^2 - m^2_{f'})((q_1+p_1+p_2)^2 - m^2_{V_2})(s-(q_1+k')^2)}$$
$$= -\int_0^1 dx \int_0^{1-x} \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'}) D_0(p_1^2, p_2^2, k'^2_2, k'^2_1, s, t, m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma)$$

Leibiniz's rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx}$$

5. double-box diagram

$$\partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0) + \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$\frac{\Delta B_0(s, m_1^2, m_2^2)}{s - s_0}\ |_{s=s_0} = \frac{1}{0}$$

$$= \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

→ 0 at the lower bound, so derivative van be put inside the integral

$$+ \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})\frac{\sigma_0}{\sigma}D_0(...,\sigma_0)$$

→ integrate over σ gives $B_0(0, m'^2, m^2_{q'})$ (dispersion relation)

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m^2_{q'}) D_0(p_1^2, p_2^2, {k'}_2^2, {k'}_1^2, s, t, m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma)$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_{(m'+m_{q'})^2}^{\infty} d\sigma \partial^2_{m'^2}\Delta B_0(s, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$+ \int_0^1 dx \int_0^{1-x} dy \sigma_0 D_0(...,\sigma_0)\partial^2_{m'^2} B_0(0, m'^2, m^2_{q'})$$

6. Missing term estimation

- A simple method is based on the assumption that the perturbation series follows roughly a geometric progression, such as

$$\mathcal{O}(\alpha^2\alpha_s) = \frac{\mathcal{O}(\alpha^2)}{\mathcal{O}(\alpha)}\mathcal{O}(\alpha\alpha_s^2)$$

- One is called the "Traditional Blue Band Method". It is based on the fact that the results by using different method, different renormalization shceme, differ from each other.
- A different approach is that for each type of unknown corrections the relevant enhancement factors are kept and remaining loop integral is set to be 1.
- Use Bayesian models(QCD)

https://indico.cern.ch/event/1107840/contributions/4822317/