

Feynman Integrals of Grassmannians

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I. Introduction

1. Some progresses

- Considering Feynman integrals as the generalized hypergeometric functions, one finds that the D -module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) D -module.
- 1-loop 2-point function B_0 : Gauss functions $_2F_1$.
1-loop 3-point function C_0 : Appell functions F_1 .
1-loop 4-point function D_0 : Lauricella functions F_s .
- C_0 is convenient for analytic continuation because continuation of F_1 has been analyzed thoroughly. However, how to perform continuation of F_s outside its convergent domain is still a challenge.

I. Introduction

2. Previous research

- Hypergeometric functions of some Feynman integrals are obtained from Mellin-Barnes representations.

Feng, Chang, Chen, Gu, Zhang, NPB 927(2018)516, arXiv:1706.08201;

Feng, Chang, Chen, Zhang, NPB 940(2019)130, arXiv:1809.00295;

Gu, Zhang, CPC 43(2019)083102, arXiv:1811.10429;

Gu, Zhang, Feng, IJMPA 35(2020)2050089.

- Using GKZ-hypergeometric system of a general compact manifold, we obtain the fundamental solution systems in neighborhoods of the regular singularities 0, ∞ .

Feng, Chang, Chen, Zhang, NPB 953(2020)114952, arXiv:1912.01726;

Feng, Zhang, Dong, Zhou, arXiv:2209.15194.

I. Introduction

3. Generally strategy

- **The fundamental solution systems of neighborhoods of different regular singularities can be regarded as analytic continuations of each other. We can formulate the Feynman integrals as hypergeometric functions over Grassmannians manifold.**
- **Steps:** (1) we embed parametric representation of Feynman integral in the subvarieties of proper Grassmannians $G_{k,n}$, and write out the GKZ-system satisfied by the Feynman integral.
(2) **fundamental solution systems** are constructed in neighborhoods of regular singularities of the GKZ-system. The combination coefficients can be determined from the Feynman integral with some special kinematic parameters.
- **Feng, Zhang, Chang, Feynman Integrals of Grassmannians, arXiv: 2206.04224.**

II. 1-loop self-energy

1. $m_1^2 = m_2^2 = 0$

- Adopting Feynman parametric representation, we get the integral of zero virtual masses as

$$\begin{aligned}
 A_{1SE}(p^2, 0, 0) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q+p)^2} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 dt_1 t_1^{D/2-2} (p^2 t_1 - p^2)^{D/2-2} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2},
 \end{aligned}$$

with the homogeneous coordinate $t_2 = -1$, the volume element of projective line $\omega_2(t) = t_2 dt_1 - t_1 dt_2$.

II. 1-loop self-energy

1. $m_1^2 = m_2^2 = 0$

- The integral

$$A_{1SE}(p^2, 0, 0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2},$$

can be embedded in the subvariety of the Grassmannian $G_{2,3}$, with splitting local coordinates as

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}.$$

- first row: integration variable t_1 , second row: t_2 ,
 first column: power function $t_1^{D/2-2}$, second column: t_2^{2-D} ,
 third column: $(z_{1,3}t_1 + z_{2,3}t_2)^{D/2-2} = (t_1 p^2 + t_2 p^2)^{D/2-2}$.

II. 1-loop self-energy

1. $m_1^2 = m_2^2 = 0$

Splitting local coordinates: $A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}$.

$$A_{1SE}(p^2, 0, 0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2},$$

satisfies the following **GKZ-system**

$$\left\{ \vartheta_{1,1} + \vartheta_{1,3} \right\} A_{1SE} = -A_{1SE}, \quad \left\{ \vartheta_{2,2} + \vartheta_{2,3} \right\} A_{1SE} = -A_{1SE},$$

$$\vartheta_{1,1} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE}, \quad \vartheta_{2,2} A_{1SE} = (2 - D) A_{1SE},$$

$$\left\{ \vartheta_{1,3} + \vartheta_{2,3} \right\} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE},$$

where the Euler operator $\vartheta_{i,j} = z_{i,j} \partial / \partial z_{i,j}$.

Exponent matrix: $\begin{pmatrix} \frac{D}{2} - 2 & 0 & 1 - \frac{D}{2} \\ 0 & 2 - D & D - 3 \end{pmatrix}$.

II. 1-loop self-energy

1. $m_1^2 = m_2^2 = 0$

- Exponent matrix:

$$\begin{pmatrix} \frac{D}{2} - 2 & 0 & 1 - \frac{D}{2} \\ 0 & 2 - D & D - 3 \end{pmatrix}.$$

- $G_{2,3}$ splitting local coordinates:

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}.$$

- One obtains solution of the GKZ-system

$$A_{1SE}(p^2, 0, 0) = C_{1SE}^{(0)}(p^2)^{1-D/2} (p^2)^{D-3} = C_{1SE}^{(0)}(p^2)^{D/2-2}.$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$\begin{aligned}
 A_{1SE}(p^2, 0, m_2^2) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2((q+p)^2 - m_2^2)} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(-)^{2-D/2} (4\pi)^{D/2}} \int_0^1 dt_1 dt_3 \frac{t_1^{D/2-2} \delta(t_1 + t_3 - 1)}{(t_3 p^2 - m_2^2)^{2-D/2}} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(-)^{2-D/2} (4\pi)^{D/2}} \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} ,
 \end{aligned}$$

with homogeneous coordinate $t_2 = -1$, volume element of projective space $\omega_3(t) = t_1 dt_2 dt_3 - t_2 dt_1 dt_3 + t_3 dt_1 dt_2$.

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- The integral

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}},$$

can be embedded in the subvariety of the **Grassmannian** $G_{3,5}$, with **splitting local coordinates** as

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}.$$

- first row: t_1 , second row: t_2 , third row: t_3 ,
 first column: $t_1^{D/2-2}$, second column: t_2^{2-D} , third column:
 $t_3^0 = 1$, fourth column represents the function $\delta(t_1 + t_2 + t_3)$,
 fifth column: $(z_{1,5} t_1 + z_{2,5} t_2 + z_{3,5} t_3)^{D/2-2} = (t_2 m_2^2 + t_3 p^2)^{D/2-2}$.

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} ,$$

satisfies the following **GKZ-system**

$$\left\{ \vartheta_{1,1} + \vartheta_{1,4} \right\} A_{1SE} = -A_{1SE} , \quad \left\{ \vartheta_{2,2} + \vartheta_{2,4} + \vartheta_{2,5} \right\} A_{1SE} = -A_{1SE} ,$$

$$\left\{ \vartheta_{3,3} + \vartheta_{3,4} + \vartheta_{3,5} \right\} A_{1SE} = -A_{1SE} , \quad \vartheta_{1,1} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE} , \quad \vartheta_{2,2} A_{1SE} = (2 - D) A_{1SE} ,$$

$$\vartheta_{3,3} A_{1SE} = 0 , \quad \left\{ \vartheta_{1,4} + \vartheta_{2,4} + \vartheta_{3,4} \right\} A_{1SE} = -A_{1SE} , \quad \left\{ \vartheta_{2,5} + \vartheta_{3,5} \right\} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE} .$$

Exponent matrix:

$$\begin{pmatrix} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & 0 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix} ,$$

$$\alpha_{2,4} + \alpha_{2,5} = D - 3 , \quad \alpha_{3,4} + \alpha_{3,5} = -1 , \quad \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2 , \quad \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2 .$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- Dual space of the GKZ-system: 3×5 matrix $(0_{3 \times 3} \mid E_3^{(1)})$ with

$$E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} .$$

- Integer lattice $(0_{3 \times 3} \mid nE_3^{(1)})$ ($n \geq 0$) is compatible with two choices of the exponents. Through

$$\alpha_{2,4} + \alpha_{2,5} = D - 3, \quad \alpha_{3,4} + \alpha_{3,5} = -1, \quad \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2, \quad \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2,$$

the first choice is written as

$$\alpha_{2,4} = 0, \quad \alpha_{2,5} = D - 3, \quad \alpha_{3,4} = \frac{D}{2} - 2, \quad \alpha_{3,5} = 1 - \frac{D}{2} .$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

Splitting local coordinates: $A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}$.

Integer lattice $(0_{3 \times 3} \mid nE_3^{(1)})$: $nE_3^{(1)} = \begin{pmatrix} 0 & 0 \\ n & -n \\ -n & n \end{pmatrix}$.

Exponent matrix: $\begin{pmatrix} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & 0 & D - 3 \\ 0 & 0 & 0 & \frac{D}{2} - 2 & 1 - \frac{D}{2} \end{pmatrix}$.

Hypergeometric function:

$$\begin{aligned} \psi_{\{1,2,3\}}^{(1)} &\sim (m_2^2)^{\alpha_{2,5}} (p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5} + n) \Gamma(-\alpha_{3,4} + n)}{\Gamma(1 + \alpha_{2,4} + n) \Gamma(1 + \alpha_{3,5} + n)} (m_2^2)^{-n} (p^2)^n \\ &\sim (m_2^2)^{D-3} (p^2)^{1-D/2} \sum_{n=0}^{\infty} \frac{\Gamma(3 - D + n)}{n!} \left(\frac{p^2}{m_2^2}\right)^n. \end{aligned}$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- For integer lattice ($0_{3 \times 3} \Big| nE_3^{(1)}$) ($n \geq 0$), second choice is written as

$$\alpha_{3,5} = 0, \quad \alpha_{2,4} = \frac{D}{2} - 1, \quad \alpha_{2,5} = \frac{D}{2} - 2, \quad \alpha_{3,4} = -1.$$

- Adopting integer lattice and the corresponding exponents matrices, we obtain hypergeometric function as

$$\psi_{\{1,2,3\}}^{(2)}(p^2, 0, m_2^2)$$

$$\sim (m_2^2)^{\alpha_{2,5}} (p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5} + n) \Gamma(-\alpha_{3,4} + n)}{\Gamma(1 + \alpha_{2,4} + n) \Gamma(1 + \alpha_{3,5} + n)} (m_2^2)^{-n} (p^2)^n$$

$$\sim (m_2^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{D}{2} + n)}{\Gamma(\frac{D}{2} + n)} \left(\frac{p^2}{m_2^2}\right)^n.$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- For integer lattice $(0_{3 \times 3} \mid -nE_3^{(1)})$ ($n \geq 0$), two possibilities:

$$\alpha_{2,5} = 0, \quad \alpha_{2,4} = D - 3, \quad \alpha_{3,4} = 1 - \frac{D}{2}, \quad \alpha_{3,5} = \frac{D}{2} - 2 ;$$

$$\alpha_{3,4} = 0, \quad \alpha_{2,4} = \frac{D}{2} - 2, \quad \alpha_{2,5} = \frac{D}{2} - 1, \quad \alpha_{3,5} = -1 .$$

- Adopting integer lattice and exponents matrices, we obtain two linear independent hypergeometric functions as

$$\psi_{\{1,2,3\}}^{(3)}(p^2, 0, m_2^2) \sim (p^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(3-D+n)}{n!} \left(\frac{m_2^2}{p^2}\right)^n ;$$

$$\psi_{\{1,2,3\}}^{(4)}(p^2, 0, m_2^2) \sim \frac{(m_2^2)^{D/2-1}}{p^2} \sum_{n=0}^{\infty} \frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)} \left(\frac{m_2^2}{p^2}\right)^n .$$

II. 1-loop self-energy

2. $m_1^2 = 0, m_2^2 \neq 0$

- $\det(A_{\{1,2,3\}}^{(1)}) = 1$, $G_{3,5}$ splitting local coordinates

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}.$$

Convergent regions: $|1/x| \leq 1$, $|x| \leq 1$ ($x = m_2^2/p^2$). Neighborhoods: $x = \infty, 0$.

- $\det(A_{\{1,2,5\}}^{(1)}) = p^2$, $G_{3,5}$ splitting local coordinates

$$\left(A_{\{1,2,5\}}^{(1)}\right)^{-1} \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{m_2^2}{p^2} & 1 - \frac{m_2^2}{p^2} & 0 \\ 0 & 0 & \frac{1}{p^2} & \frac{1}{p^2} & 1 \end{pmatrix}.$$

Convergent regions: $|1/(1 - 1/x)| \leq 1$, $|1 - 1/x| \leq 1$. Neighborhoods: $x = 0, 1$.

- $\det(A_{\{1,3,5\}}^{(1)}) = -m_2^2$, $\det(A_{\{2,3,4\}}^{(1)}) = 1$, $\det(A_{\{2,4,5\}}^{(1)}) = -p^2$,
 $\det(A_{\{3,4,5\}}^{(1)}) = m_2^2$.
- **24 fundamental solutions**

II. 1-loop self-energy

3. $m_1^2 \neq 0, m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$\begin{aligned} A_{1SE}(p^2, m_1^2, m_2^2) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m_1^2)((q+p)^2 - m_2^2)} \\ &\propto \int_0^1 dt_1 dt_2 \delta(t_1 + t_2 - 1) (t_1 t_2 p^2 - t_1 m_1^2 - t_2 m_2^2)^{D/2-2} \\ &\propto \int \omega_3(t) \delta(t_1 + t_2 + t_3) t_3^{2-D} (t_1 t_2 p^2 + t_1 t_3 m_1^2 + t_2 t_3 m_2^2)^{D/2-2}. \end{aligned}$$

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$, with splitting local coordinates as

$$A^{(1S)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & p^2 \\ 0 & 1 & 0 & 1 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & 1 & m_1^2 \end{pmatrix}.$$

- 72 fundamental solutions

III. 1-loop triangle

$$m_1^2 = m_2^2 = m_3^2 = 0$$

- Adopting Feynman parametric representation

$$\begin{aligned} C(p_1^2, p_2^2, p_3^2) &= \left(\Lambda_{\text{RE}}^2\right)^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q-p_1)^2(q+p_2)^2} \\ &\propto \int_0^1 dt_1 dt_2 dt_3 \frac{\delta(t_1 + t_2 + t_3 - 1)}{(t_1 t_2 p_3^2 + t_2 t_3 p_1^2 + t_1 t_3 p_2^2)^{3-D/2}} \\ &\propto \int_0^1 \omega_4(t) t_4^{3-D} \frac{\delta(t_1 + t_2 + t_3 + t_4)}{(t_1 t_2 p_3^2 + t_2 t_3 p_1^2 + t_1 t_3 p_2^2)^{3-D/2}}. \end{aligned}$$

- Grassmannian $G_{4,7}$, with splitting local coordinates as

$$A^{(1T)} = \begin{pmatrix} 1 & 0 & 0 & 0 & p_2^2 & 1 & 1 \\ 0 & 1 & 0 & 0 & p_3^2 & 1 & 1 \\ 0 & 0 & 1 & 0 & p_1^2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

- 192 fundamental solutions

IV. 2-loop vacuum

$$m_1^2 \neq 0, m_2^2 \neq 0, m_3^2 \neq 0$$

- Adopting Feynman parametric representation

$$\begin{aligned} A_{2Vac}(m_1^2, m_2^2, m_3^2) &\propto \int_0^\infty \prod_{i=1}^3 dt_i \frac{(t_1 t_2 t_3)^{1-D/2} \delta(t_1 t_2 t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3)}{(t_1 m_1^2 + t_2 m_2^2 + t_3 m_3^2)^{3-D}} \\ &\propto \int \omega_4(t) \frac{(t_1 t_2 t_3)^{1-D/2} t_4^{D/2-1} \delta(t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4)}{(t_1 m_1^2 + t_2 m_2^2 + t_3 m_3^2)^{3-D}} . \end{aligned}$$

- Grassmannian $G_{4,8}$, with splitting local coordinates as

$$A^{(2V)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & m_1^2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & m_2^2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & m_3^2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} .$$

- 48 fundamental solutions

V. Summary

- **Feynman integrals** can be taken as functions on the subvarieties of Grassmannians through homogenizing the parametric representation. The GKZ-systems can be obtained in splitting local coordinates.
- **Fundamental solution systems** in neighborhoods of the regular singularities, using matrices of integer lattice and exponent matrices.
- **multi-loop diagrams:** the sizes of Grassmannians are too large to construct the fundamental systems through Feynman parametric representations. To efficiently derive the solution, we can embed the integrals into the subvarieties of Grassmannians using α -parametric representation.
- **Plan:** 2-loop self-energy, 3-loop vacuum . . .
Writing a numerical calculation program.



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Thanks!