



华南師範大學
SOUTH CHINA NORMAL UNIVERSITY



Lattice Parton
Collaboration

Advances in the LQCD calculation of light cone distribution

Jun Hua (华俊)

South China Normal University

Dec.11 @ HFCPV2022

CONTENTS

Motivation for
DA research

1

DA Calculation
on LQCD

2

Progress on DA
by LaMET

3

Outlook

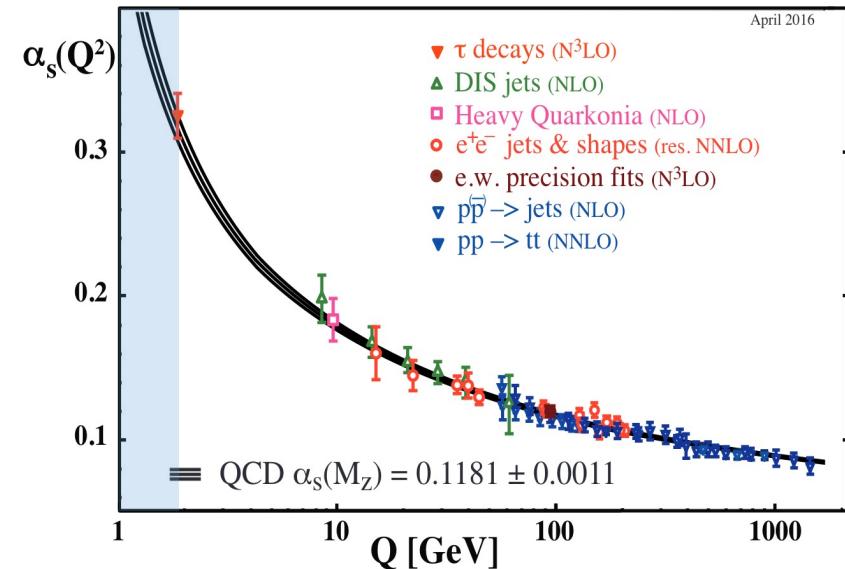
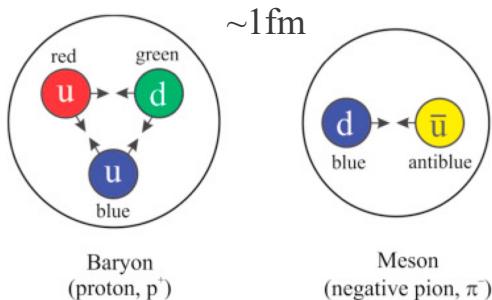
4

Summary

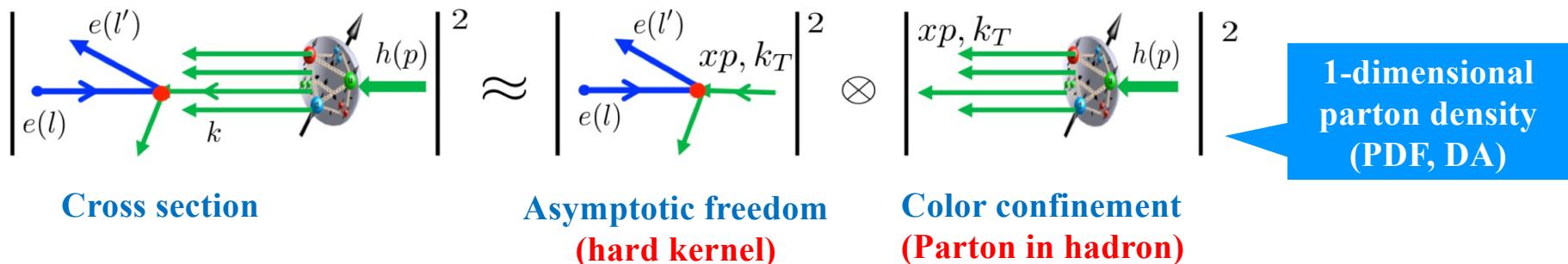
4

Motivation

➤ Color confinement and asymptotic freedom



➤ QCD factorization

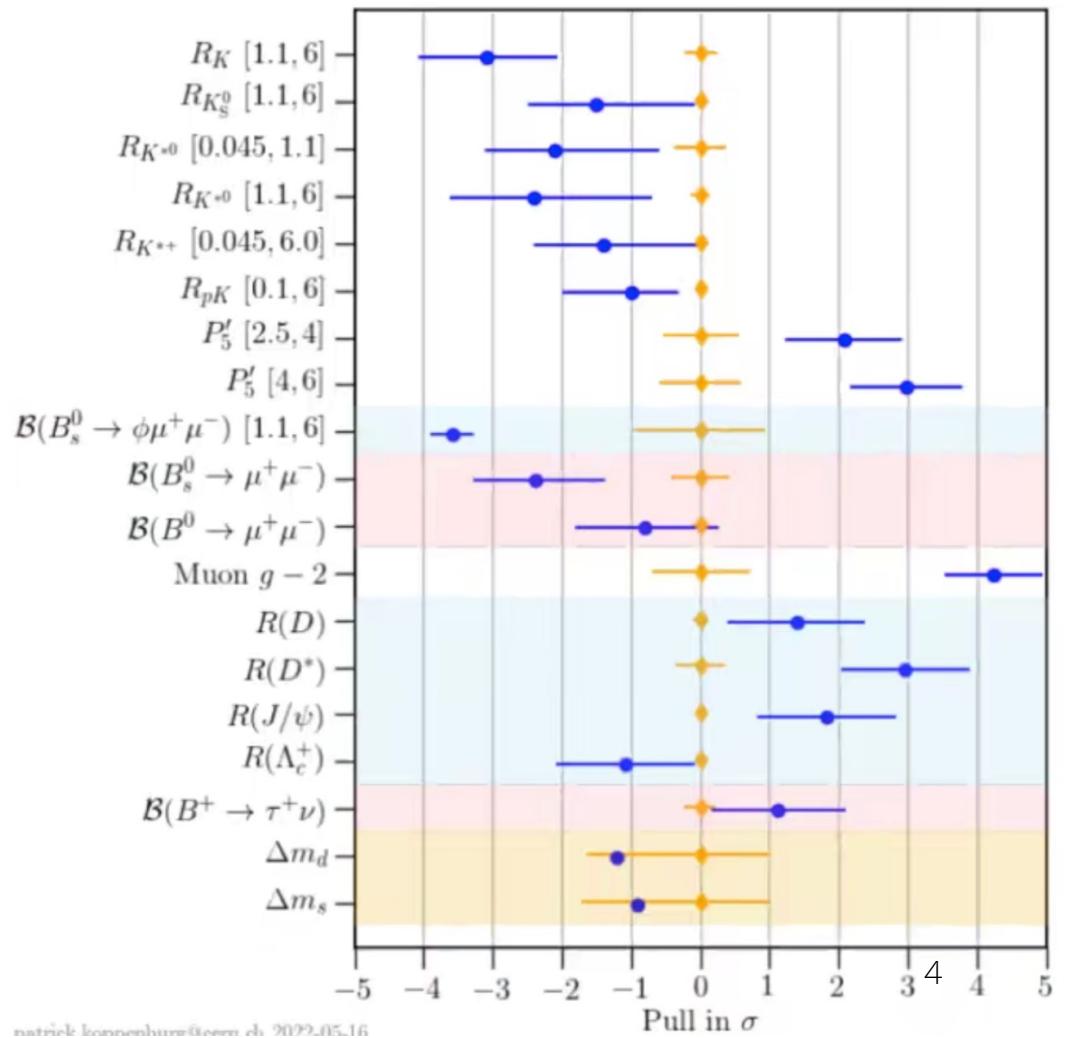
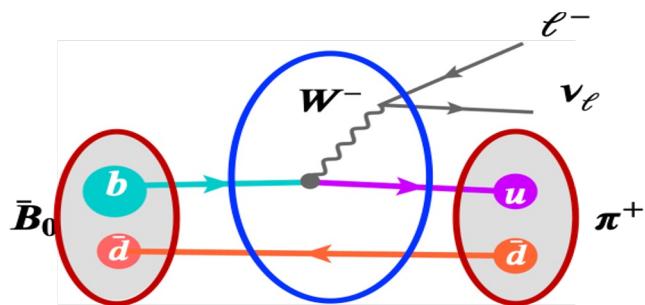


- Distribution amplitude (DA) is non-perturbative fundamental input to collinear factorization for high-energy exclusive QCD processes.

Motivation

➤ Important for many phenomenological applications, e.g.

- $B \rightarrow \pi l \nu_l, B \rightarrow \pi\pi, \dots$
- $\gamma^* \rightarrow \gamma\pi, \gamma\gamma \rightarrow \pi\pi$
- $eN \rightarrow eN\pi$
- $B \rightarrow K^* l^+ l^-$
- $B \rightarrow \phi l^+ l^-$
- ...



Calculation of DAs

➤ Pion DA:

$$\int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0 | \bar{\psi}_1(0) n \cdot \gamma \gamma_5 U(0, \xi^-) \psi_2(\xi^-) | \pi(p) \rangle = i f_\pi \Phi_\pi(x)$$

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\dots} a_{2n-2}^\pi C_{2n-2}^{(3/2)}(2x-1)$$

Gegenbauer expansion

➤ Sum rules limited to first few moments

[P. Ball et.al. PRD71,014015 \(2005\)](#), [P. Ball et.al. JHEP. 03069 \(2007\)](#)

➤ Solutions for DAs from Dyson-Schwinger equations depend on kernels

[F. Gao, L. Chang et.al. PRD 90,014001 \(2014\)](#), [C.D. Roberts,et.al., PPNP.120 , 103883 \(2021\)](#)

➤ Global fits rely on theo and exp precisions

[N. G. Stefanis PRD102.034022\(2020\)](#), [C.Jian, S.Chen and J.Hua arXiv:2209.13312](#)

Calculation of DAs

➤ Pion DA:

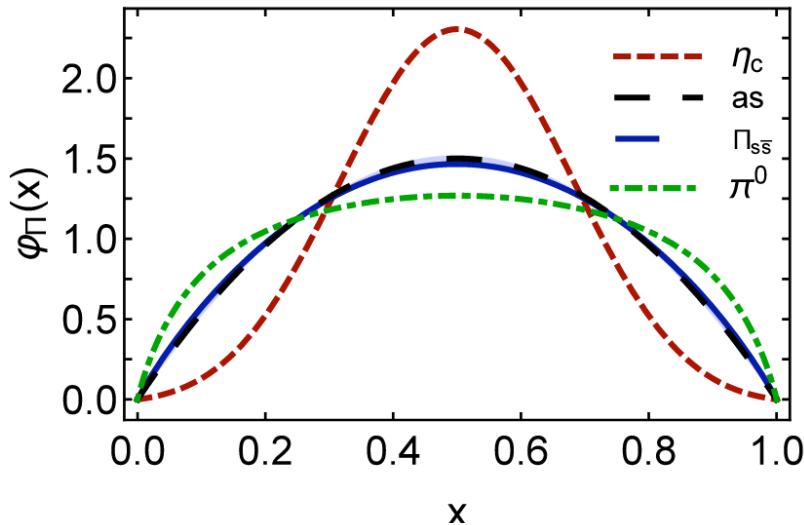
$$\int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0 | \bar{\psi}_1(0) n \cdot \gamma \gamma_5 U(0, \xi^-) \psi_2(\xi^-) | \pi(p) \rangle = i f_\pi \Phi_\pi(x)$$

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\dots} a_{2n-2}^\pi C_{2n-2}^{(3/2)}(2x-1)$$

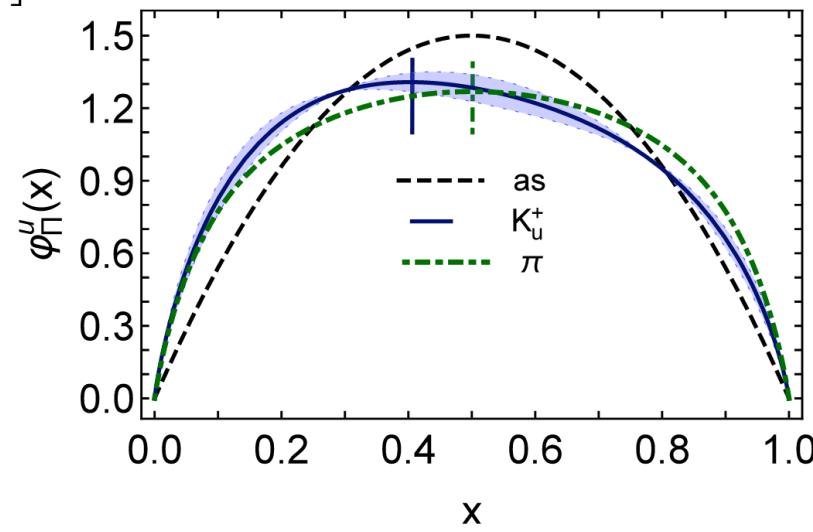
Gegenbauer expansion

➤ Dyson-Schwinger equations :

$$\varphi_\pi(x; \zeta_H) = 18.2x(1-x) \left[1 - 2.33\sqrt{x(1-x)} + 1.79x(1-x) \right]$$



C.D. Roberts,et.al., PPNP.120 , 103883 (2021)



Calculation of DAs

➤ Pion DA:

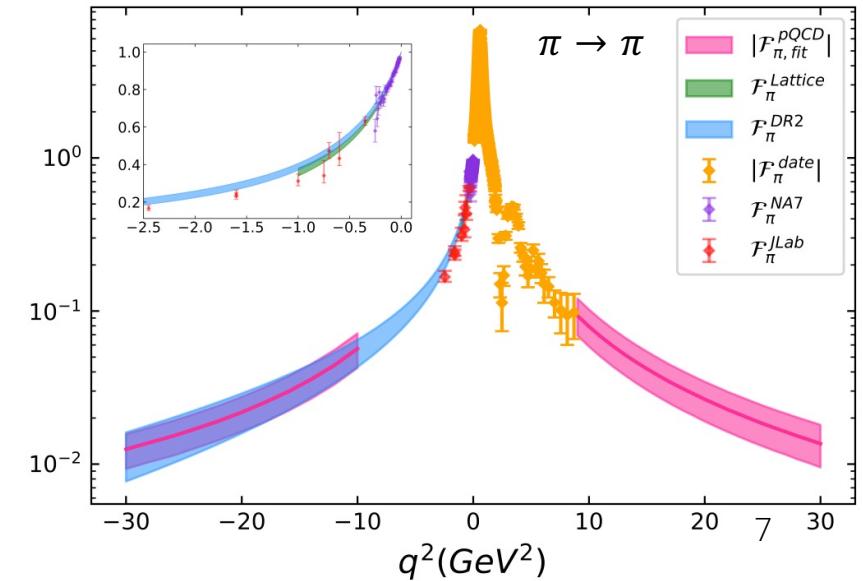
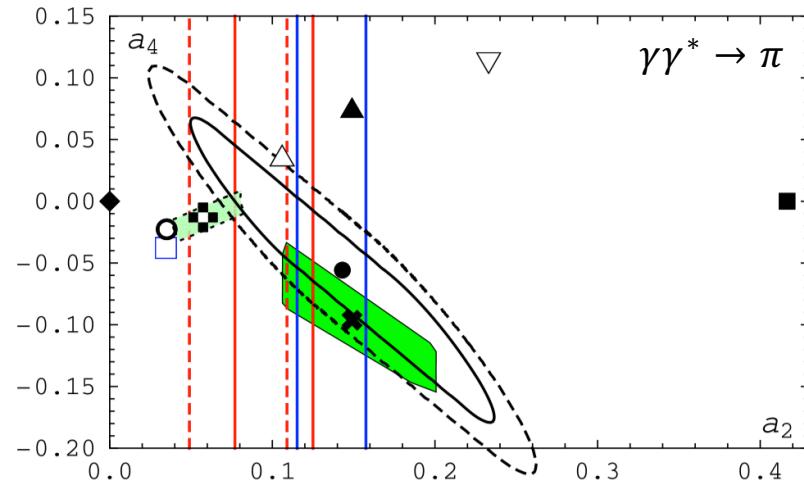
$$\int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0 | \bar{\psi}_1(0) n \cdot \gamma \gamma_5 U(0, \xi^-) \psi_2(\xi^-) | \pi(p) \rangle = i f_\pi \Phi_\pi(x)$$

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\dots} a_{2n-2}^\pi C_{2n-2}^{(3/2)}(2x-1)$$

Gegenbauer expansion

➤ Global fits rely on theo and exp precisions: [C.Jian, S.Chen and J.Hua arXiv:2209.13312](#)

N. G. Stefanis PRD102.034022(2020)



Calculation of DAs

Convergence ?

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\dots} a_n^\pi C_{2n-2}^{(3/2)}(2x-1)$$

Gegenbauer expansion

$$\langle \xi^n \rangle \equiv \int_0^1 dx (2x-1)^n \phi_\pi(x)$$

$$a_0^\pi = \langle \xi^0 \rangle,$$

$$a_2^\pi = \frac{7}{12} (5\langle \xi^2 \rangle - \langle \xi^0 \rangle),$$

$$a_4^\pi = \frac{11}{24} (21\langle \xi^4 \rangle - 14\langle \xi^2 \rangle + \langle \xi^0 \rangle),$$

$$a_6^\pi = \frac{5}{64} (429\langle \xi^6 \rangle - 495\langle \xi^4 \rangle + 135\langle \xi^2 \rangle - 5\langle \xi^0 \rangle),$$

$$a_8^\pi = \frac{19}{384} (2431\langle \xi^8 \rangle - 4004\langle \xi^6 \rangle + 2002\langle \xi^4 \rangle - 308\langle \xi^2 \rangle + 7\langle \xi^0 \rangle),$$

$$a_{10}^\pi = \frac{23}{1536} (29393\langle \xi^{10} \rangle - 62985\langle \xi^8 \rangle + 46410\langle \xi^6 \rangle - 13650\langle \xi^4 \rangle + 1365\langle \xi^2 \rangle - 21\langle \xi^0 \rangle)$$

Huge coefficients ! Theoretical or roundoff errors
can be greatly amplified highly nontrivial task

Good convergence

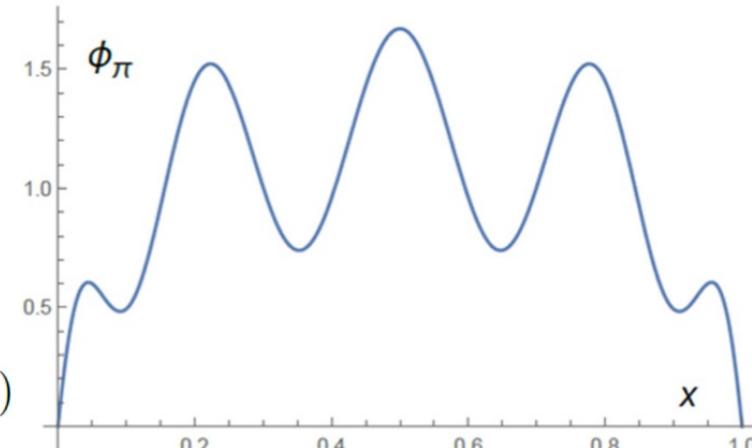
Bad convergence

T.Zhong PRD104. 016021(2021)

$$(\langle \xi^0 \rangle, \langle \xi^2 \rangle, \langle \xi^4 \rangle, \langle \xi^6 \rangle, \langle \xi^8 \rangle, \langle \xi^{10} \rangle)|_{\mu=2 \text{ GeV}} = (1, 0.254, 0.125, 0.077, 0.054, 0.041)$$

$$(a_0^\pi, a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi)|_{\mu=2 \text{ GeV}}$$

$$= (1, 0.157, 0.032, 0.035, 0.098, -0.046)$$



Hsiang-nan Li PRD106. 034015(2022)

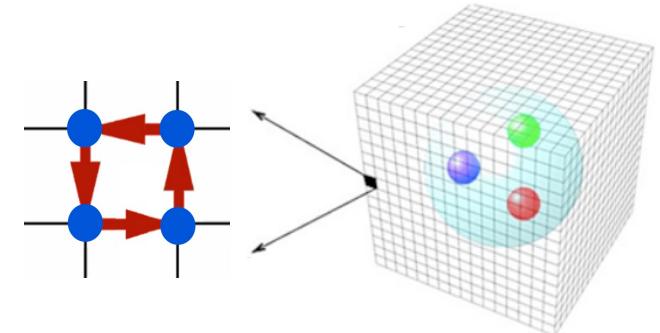
Lattice QCD

➤ Numerical simulation in discretized 4D Euclidean space-time;

➤ Lattice QCD: action

$$S_E^{\text{latt}} = - \sum_{\square} \frac{6}{g^2} \text{Re} \text{tr}_N \left(U_{\square, \mu\nu} \right) - \sum_q \bar{q} \left(D_{\mu}^{\text{latt}} \gamma_{\mu} + am_q \right) q$$

Wilson gauge action
Lattice fermion action



➤ Correlation functions:

$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{\int [DU] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det(D_{\mu}^{\text{latt}} \gamma_{\mu} + am_q) \tilde{\mathcal{O}}(U)}{\int [DU] e^{-S_{\text{glue}}^{\text{latt}}} \prod_q \det(D_{\mu}^{\text{latt}} \gamma_{\mu} + am_q)}$$

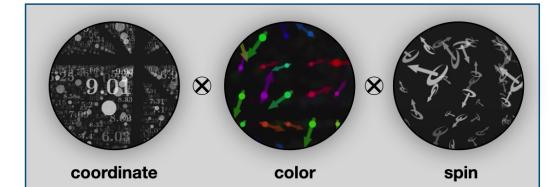
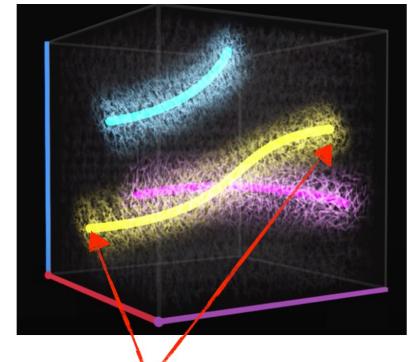
➤ Monte Carlo simulation:

- The integration is performed for all link variables: $n_s^3 \times n_t \times N_{\text{color}} \times N_{\text{spin}}$

- Importance sampling: $e^{-S_{\text{glue}}^{\text{latt}}}(U) \prod_q \det(D_{\mu}^{\text{latt}}(U) \gamma_{\mu} + am_q)$

- Therefore

$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$



Lattice Calculation of DAs

➤ Pion DA:

$$\int \frac{d\xi^-}{2\pi} e^{ixp^+\xi^-} \langle 0 | \bar{\psi}_1(0) n \cdot \gamma \gamma_5 U(0, \xi^-) \psi_2(\xi^-) | \pi(p) \rangle = i f_\pi \Phi_\pi(x)$$

$$\phi_\pi(x) = 6x(1-x) \sum_{n=1,2,\dots} a_{2n-2}^\pi C_{2n-2}^{(3/2)}(2x-1)$$

Gegenbauer expansion

➤ Lattice by OPE limited to first few moments

V.M.Braun et.al. PRD **92.014504 (2015)**, V.M.Braun et.al. JHEP **04082 (2017)**,
(RQCD) G.S.Bali et.al. JHEP **08065 (2019)**

➤ Quasi-correlation(LaMET) allows access to entire x range, but not reliable
near endpoints of x

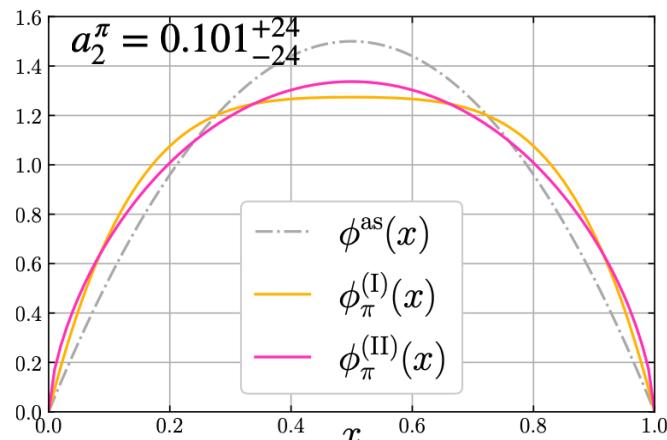
J.H.Zhang PRD**95. 094514(2017)**, R.Zhang H.W.Lin et.al. PRD**102. 094519(2020)**,
(LPC)J.Hua et.al. PRL**127. 062002(2021)**, (LPC)J.Hua et.al. PRL**129. 132001(2022)**

Lattice Calculation of DAs

Lattice by OPE

$$\bar{d}(z_2 n) \not{\partial} \gamma_5 [z_2 n, z_1 n] u(z_1 n) = \sum_{k,l=0}^{\infty} \frac{z_2^k z_1^l}{k! l!} n^\rho n^{\mu_1} \dots n^{\mu_{k+l}} \mathcal{M}_{\rho \mu_1 \dots \mu_{k+l}}^{(k,l)},$$

$$\mathcal{M}_{\rho \mu_1 \dots \mu_{k+l}}^{(k,l)} = \bar{d}(0) \overleftarrow{D}_{(\mu_1} \dots \overleftarrow{D}_{\mu_k} \overrightarrow{D}_{\mu_{k+1}} \dots \overrightarrow{D}_{\mu_{k+l})} \gamma_\rho \gamma_5 u(0).$$



(RQCD) G.S.Bali et.al. JHEP 08065 (2019)

Very precise low order moment!

However

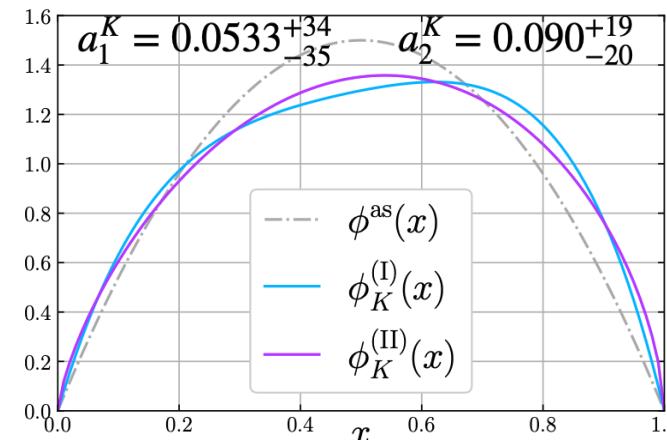
$$a_2^\pi = \frac{7}{12} (5\langle\xi^2\rangle - \langle\xi^0\rangle),$$

$$a_4^\pi = \frac{11}{24} (21\langle\xi^4\rangle - 14\langle\xi^2\rangle + \langle\xi^0\rangle), \quad a_4 = 0.002 \pm 0.071$$

$$\langle\xi^2\rangle = 0.235 \pm 0.008$$

$$\langle\xi^4\rangle = 0.109 \pm 0.005$$

$$a_4 = 0.002 \pm 0.071$$



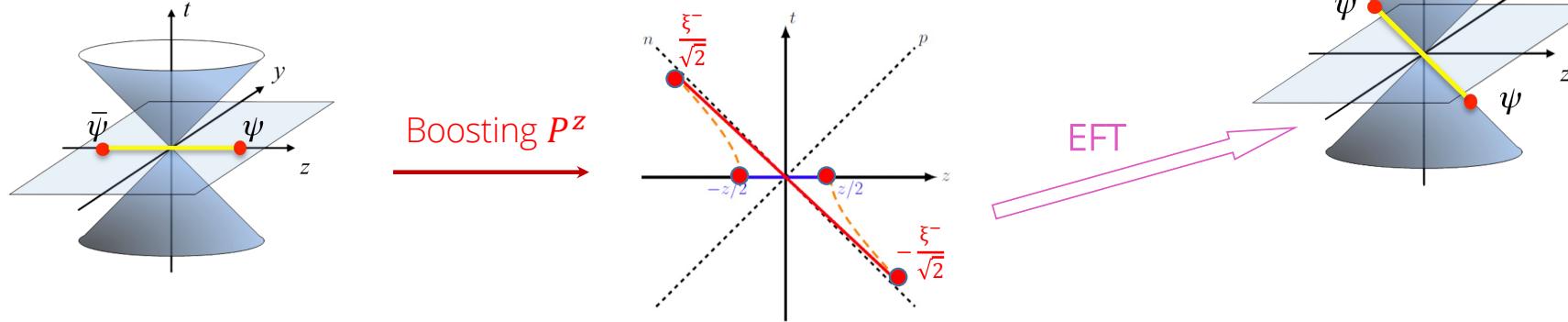
How to push high order moment ?



- 1. Operator mixing**
- 2. Computing power**

Progress on DAs by LaMET

- Define a lattice calculable, equal-time correlation: **quasi-DA**



- Effective field theory:

- Instead of taking $P^z \rightarrow \infty$ calculation, one can perform an expansion for **large but finite P^z** :

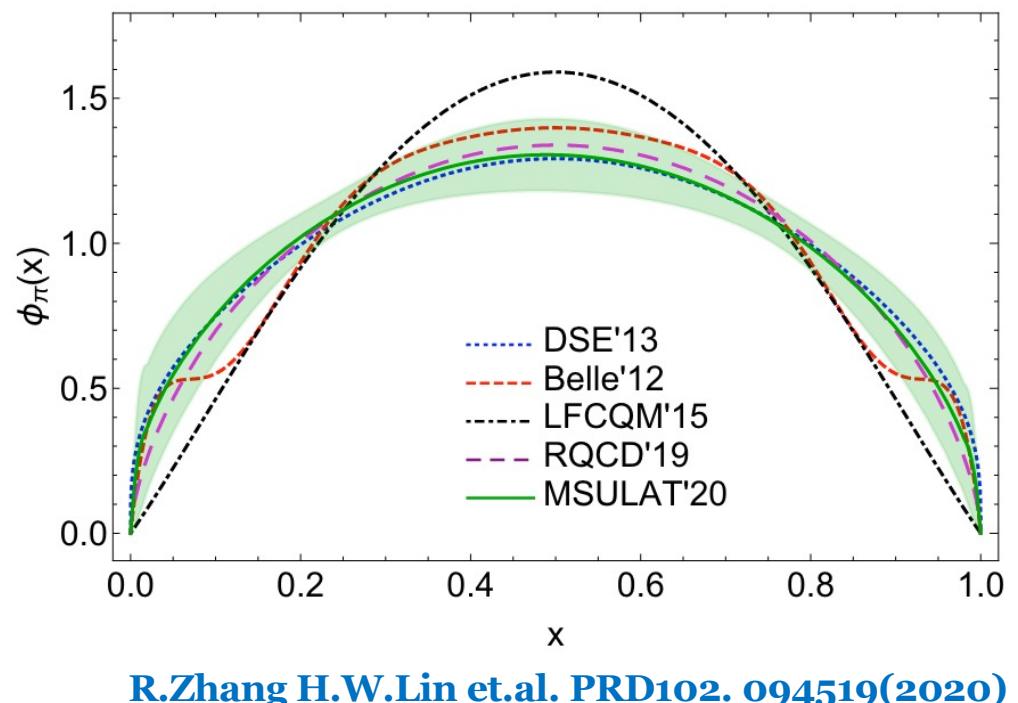
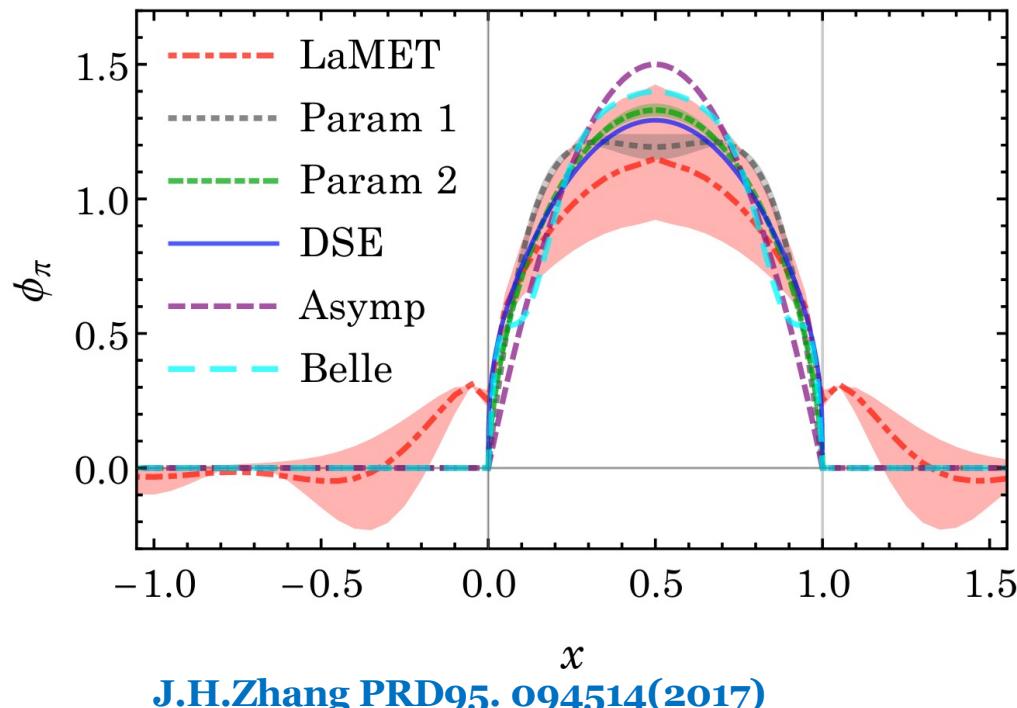
$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} \underset{\text{Quasi-DA}}{C(x, y, P^z, \mu)} \underset{\text{LCDA}}{q(y, \mu)} + \mathcal{O}\left(\frac{\Lambda^2, M^2}{(P^z)^2}\right)$$

Matching kernel

Power suppressed by $m^2/(P^z)^2, \Lambda^2/(P^z)^2$

Progress on DAs by LaMET

Stage I



Not a correct renormalization for non-local operator (linear divergence).

Progress on DAs by LaMET

Stage II

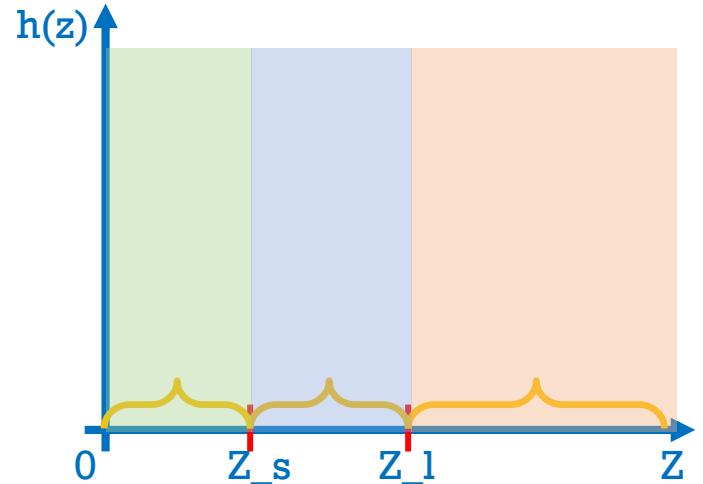
How to build ?

➤ Hybrid scheme

Separate the short and long distance quasi-LF correlations and renormalize them differently

$$\langle P | \bar{\psi}(z) \Gamma W(z, 0) \psi(0) | P \rangle / \langle X | \bar{\psi}(z) \Gamma W(z, 0) \psi(0) | X \rangle$$

- RI/MOM: [Alexandrou et al, NPB 17'](#), [Stewart, Zhao, PRD 18'](#)
 $|X\rangle$ is chosen as a single off-shell quark state
- Ratio: [Radyushkin, PRD 17'](#)
 $|X\rangle$ is chosen as a zero momentum hadron state
- VEV: [Braun et al, PRD 19'](#)
 $|X\rangle$ is chosen as the vacuum



Progress on DAs by LaMET

Stage II

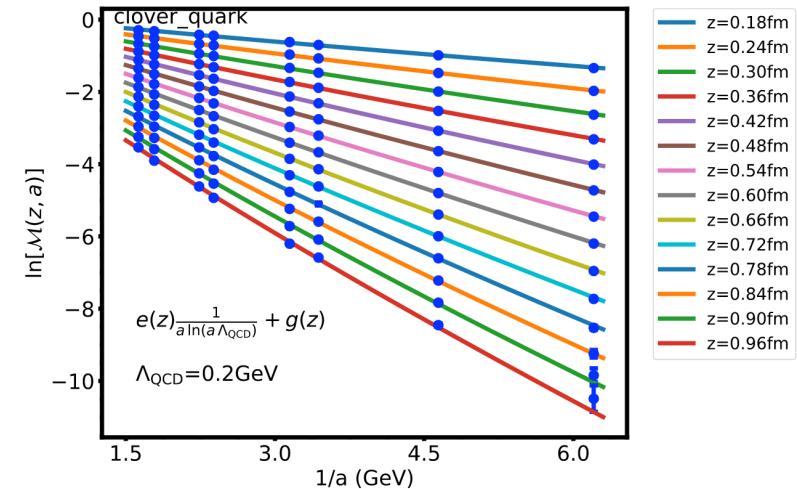
How to build ?

➤ Self renormalization

Fitting the bare matrix elements at multiple lattice spacings to :

$$\ln \mathcal{M}(z, a) = \frac{kz}{a \ln[a \Lambda_{\text{QCD}}]} + m_0 z + g'(z) + f(z)a + \frac{3C_F}{b_0} \ln \left[\frac{\ln[1/(a \Lambda_{\text{QCD}})]}{\ln[\mu/\Lambda_{\text{QCD}}]} \right] + \ln \left[1 + \frac{a}{\ln(a \Lambda_{\text{QCD}})} \right]$$

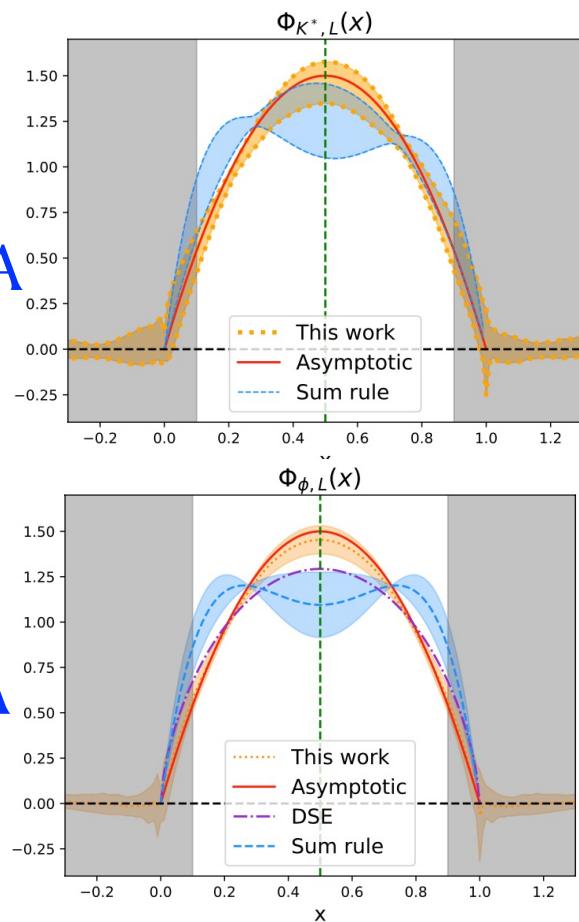
- The pieces other than $g'(z)$ are renormalization factors
- Renormalon ambiguity m_0 can be determined by matching the renormalized matrix element to the continuum $\overline{\text{MS}}$ result at short distance
- Such renormalized matrix element can then, in principle, be matched to the light-cone distribution using the $\overline{\text{MS}}$ matching



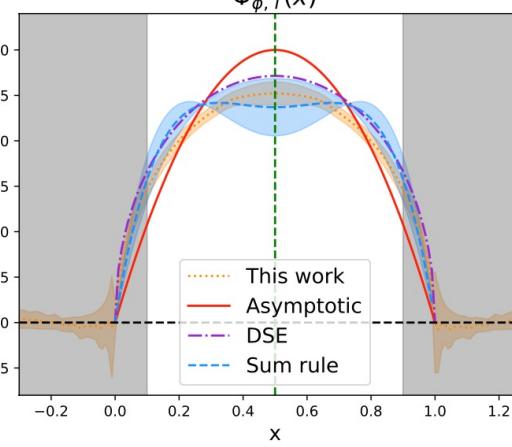
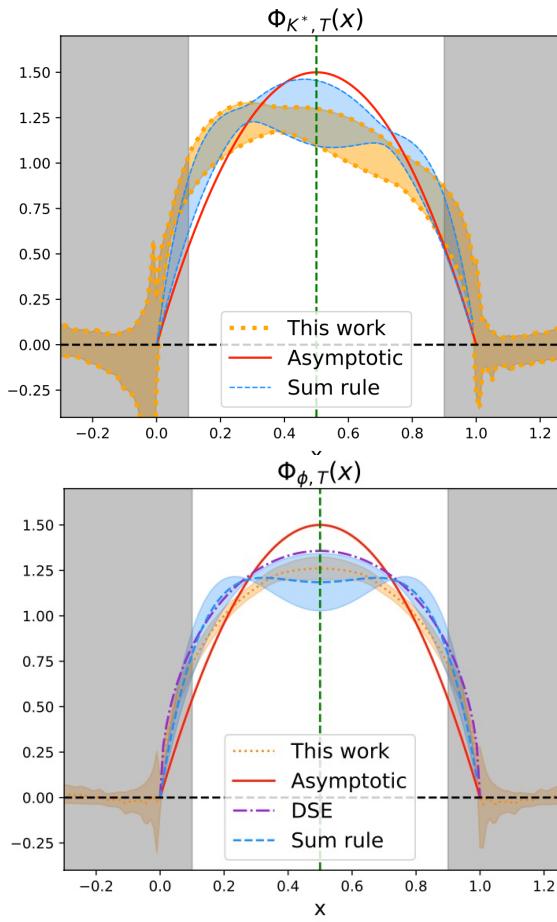
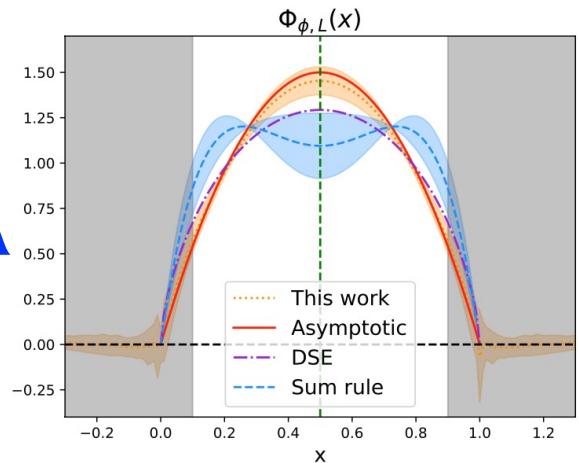
Progress on DAs by LaMET

Stage II

K^* LCDA



ϕ LCDA



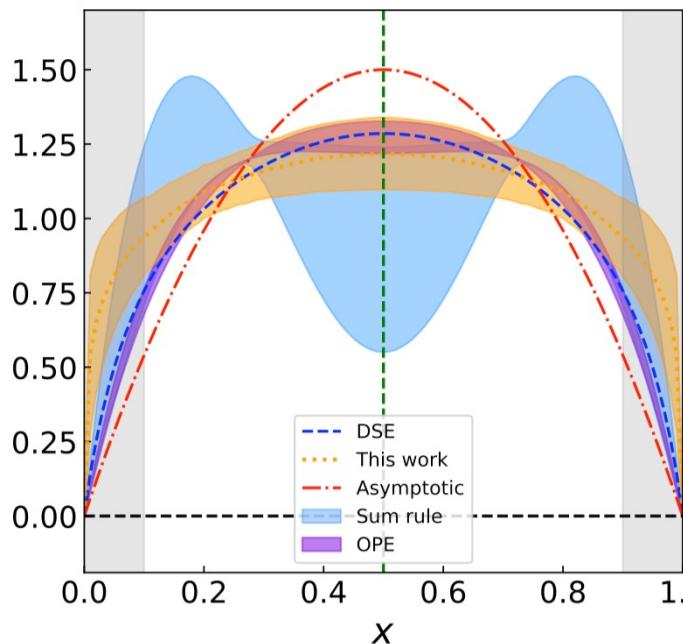
(LPC) J.Hua et.al. PRL127. 062002(2021)

- 3 lattice spacings:
(0.12, 0.09, 0.06) fm,
largest volume ($96^3 \times 192$)
- 3 momentum:
(1.29, 1.72, 2.15) GeV
- mass:
 $K^*: 0.89 \text{ GeV}, \phi: 1.02 \text{ GeV}$
- Hybrid scheme (based on RI/MOM)

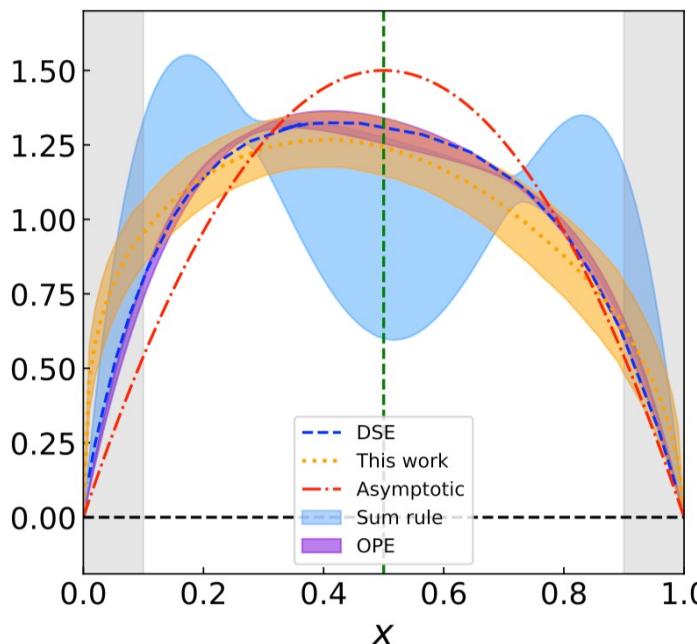
Progress on DAs by LaMET

Stage II

π LCDA:



K LCDA:



(LPC) J.Hua et.al. PRL129. 132001(2022)

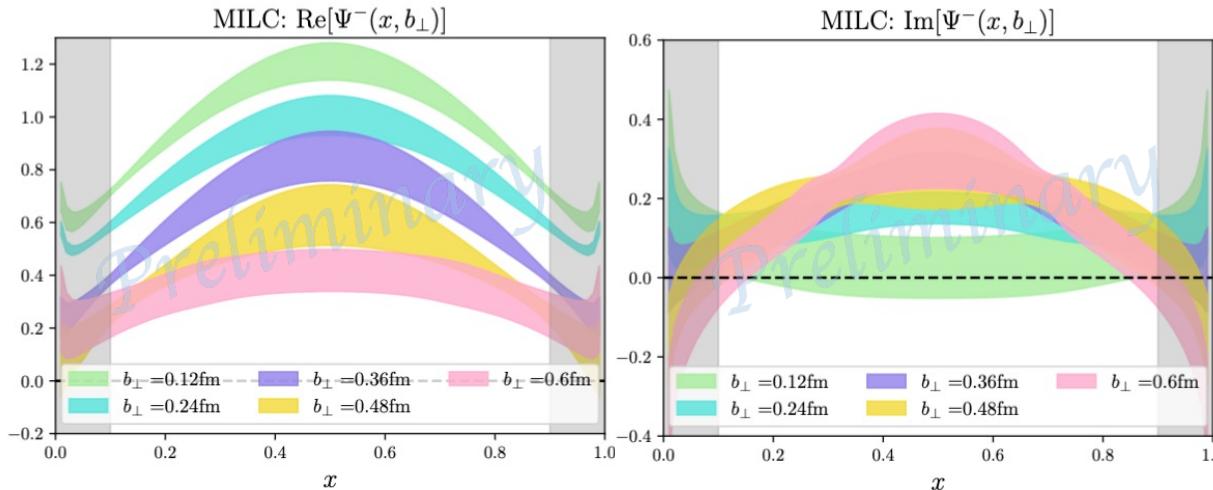
$$q(x, P^z, \mu) = \int dy C^{-1}(x, y, P^z, \mu) \tilde{q}(y, \mu) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{(xP^z)^2}, \frac{\Lambda_{QCD}^2}{((1-x)P^z)^2}\right)$$

- 3 lattice spacings:
(0.12, 0.09, 0.06) fm,
- largest volume ($96^3 \times 192$)
- 3 momentum:
(1.29, 1.72, 2.15) GeV
- mass:
 π : 0.13 GeV, K : 0.49 GeV
- Hybrid scheme (Self renormalization)

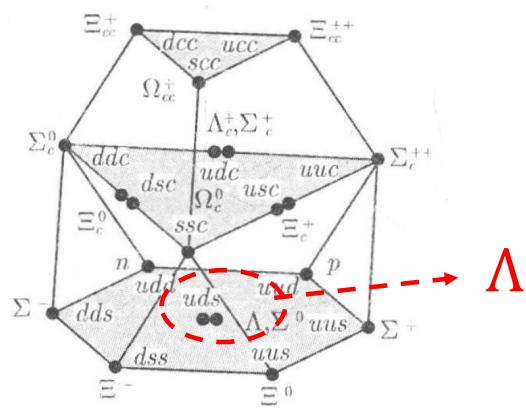
Large momentum expansion
breaks done in end point region

Outlook

➤ Transverse Momentum Dependent Wave Functions, TMDWFs (1D → 3D)



➤ LCDA for Baryons:



Summary

- Sum rules: limited to first few moments
- Dyson-Schwinger equations: model dependent
- Lattice by OPE: limited to first few moments
- Quasi-correlation(LaMET): not reliable near endpoints of x
- Global fits: rely on theory and experimental precisions
- Quantum Computing: first attempt

Summary

- We have several different methods for calculating LCDA, but each has limitations should be solved:
 - Lattice by OPE: Push to higher order moment
 - LaMET: The correct renormalization scheme has been proposed, how to extend the reliable region in next step.
- We are on different paths in search of a common goal.
- For other future, we are promoting :
 - Three-dimensional TMDWFs
 - LCDAs for baryons

Thanks for your attentions!