Improved Asymptotic Formulae

with an application to the Di-Higgs resonance search

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SYSU-PKU Collider Physics Forum for Young Scientists 5th,
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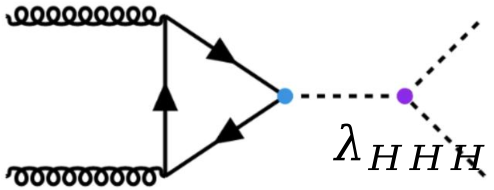
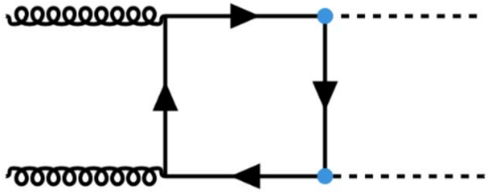
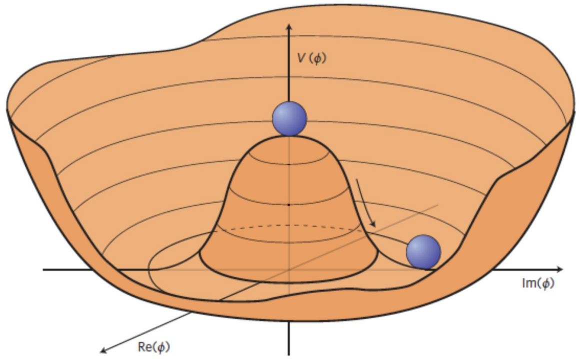
Di-Higgs production

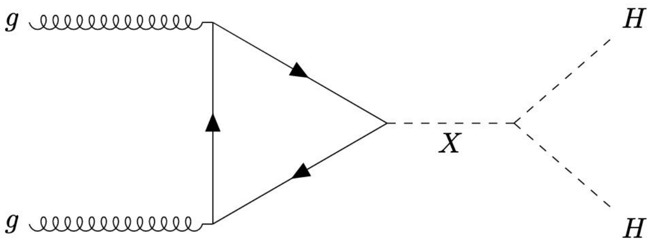
Non-resonant production of double higgs events   triple-higgs coupling strength, …

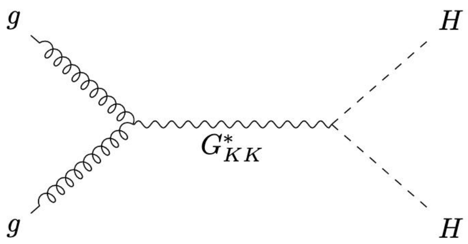






Resonant production of double higgs events   search for high-mass resonance, graviton, …

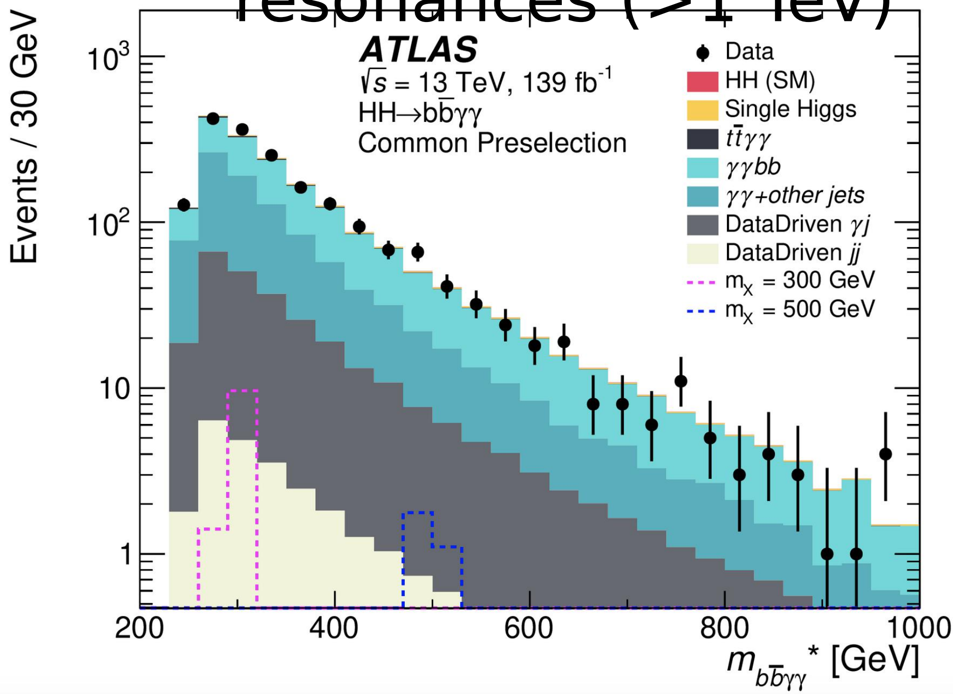


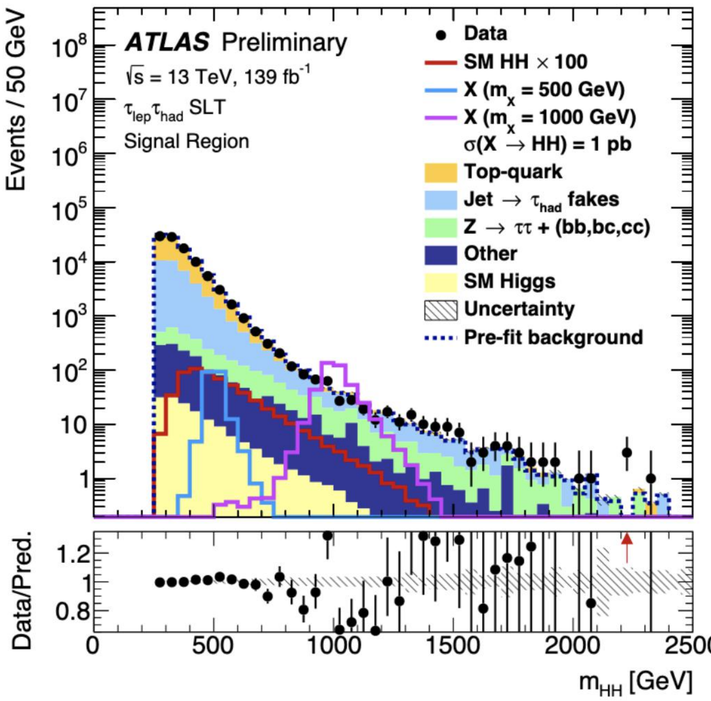


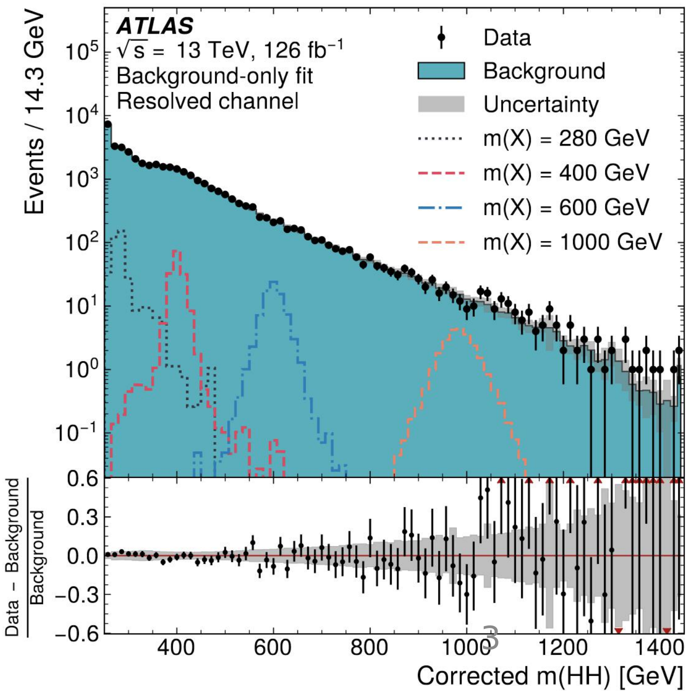
Di-Higgs resonance search in ATLAS

The search has been performed in several final states.

- (H bb)(H $\gamma\gamma$): low BR, clean, sensitive to low-mass resonances (< 1 TeV)
- (H bb)(H $\tau\tau$): sensitive to median-mass resonances
- (H bb)(H bb): high BR, dirty, sensitive to high-mass resonances (>1 TeV)



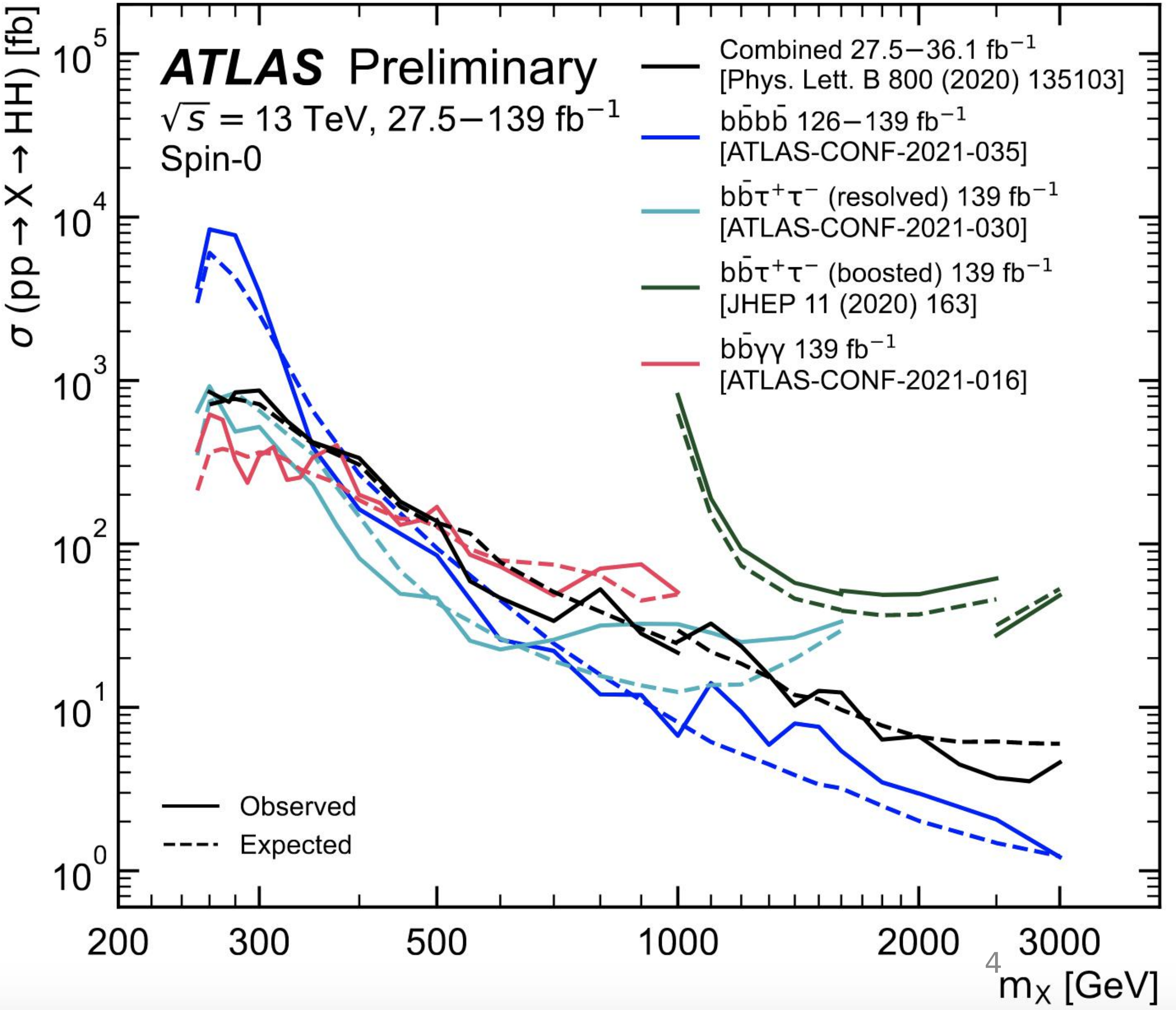




Di-Higgs resonance search in ATLAS

The search has been performed ir

- (H   bb)(H $\gamma\gamma$): sensitive to lo
- (H   bb)(H $\tau\tau$): sensitive to m
- (H   bb)(H   bb): sensitive to hi



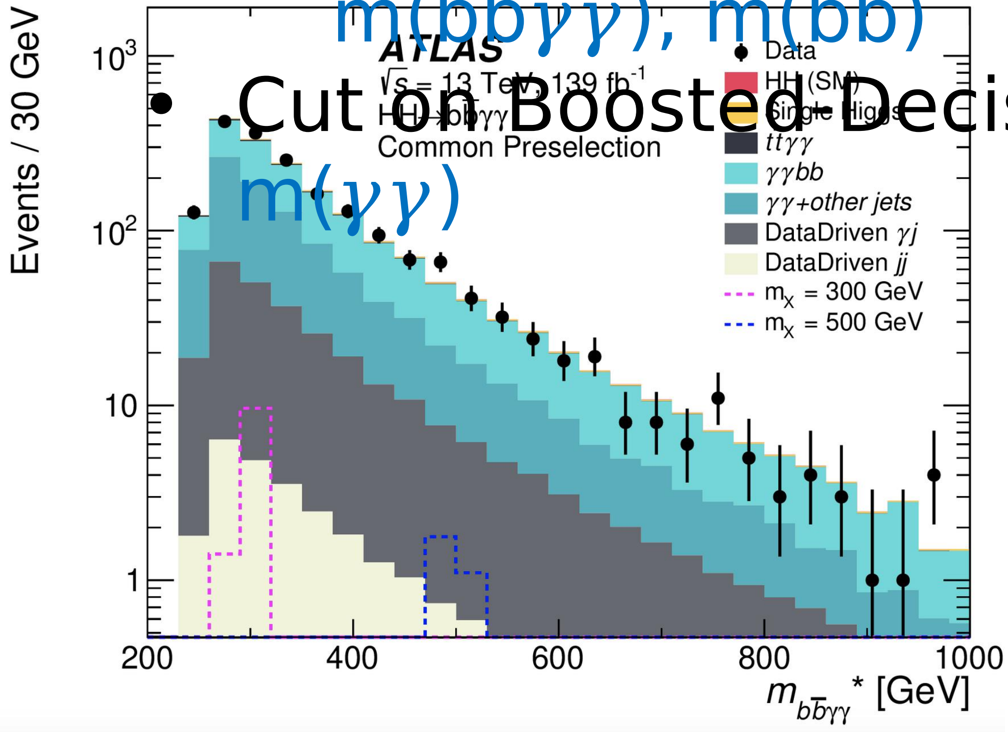
Current status:

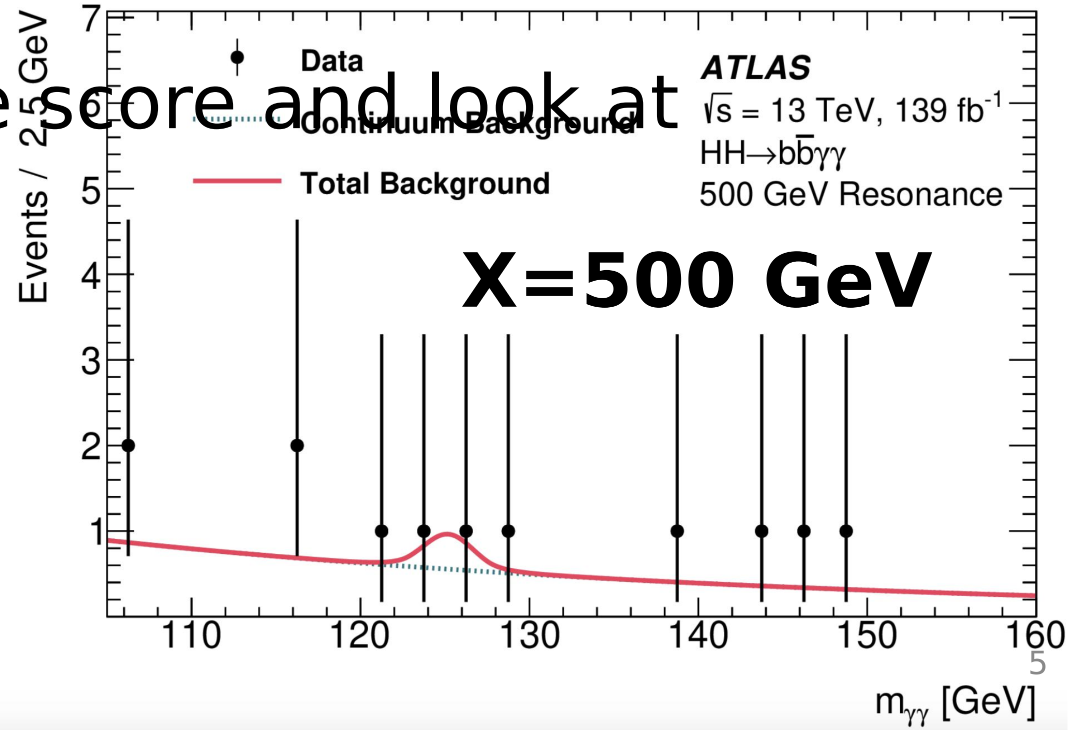
- Using full run2 dataset, no significant evidence for an HH resonance up to 3 TeV
- Upper limits on production cross section are provided.

Di-Higgs resonance search using bb $\gamma\gamma$

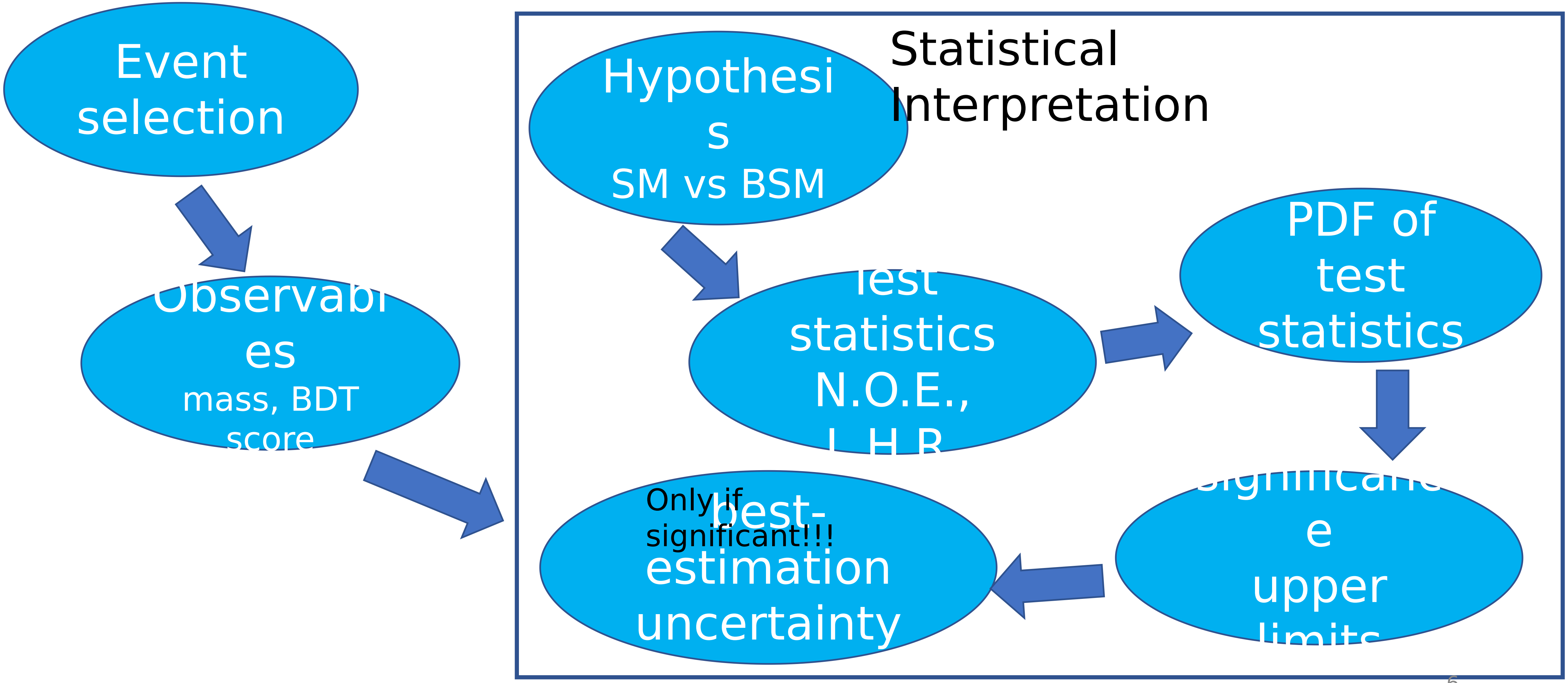
Search for HH resonance using (H bb, H $\gamma\gamma$) in ATLAS

- Multi-Variant-Analysis-based event selection:
  - training on many observables including m(bb$\gamma\gamma$), m(bb)
- Cut on Boosted-Decision-Tree score and look at m($\gamma\gamma$)

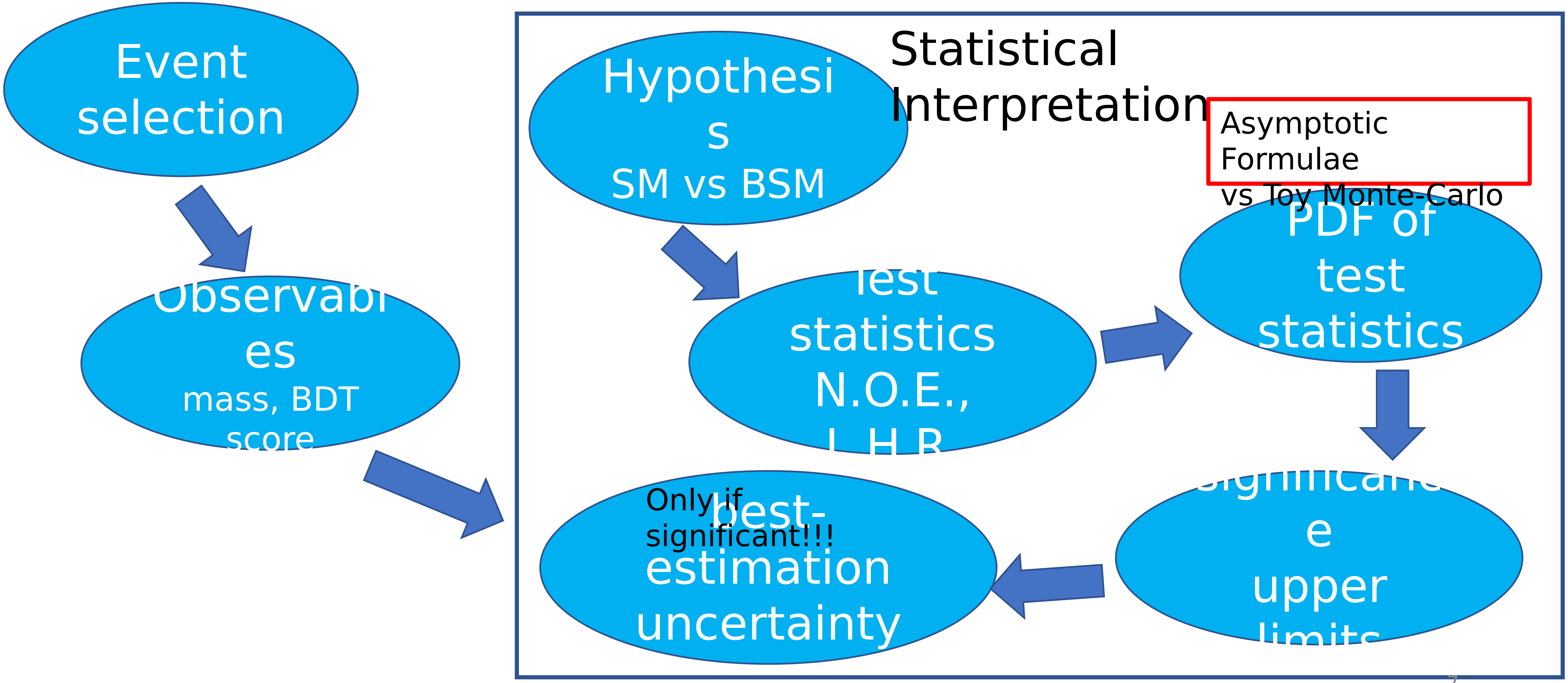




Statistical Interpretation



Statistical Interpretation



Summary of 6 Test statistics

Test statistic	Note
t_0	To establish the discovery of a signal
q_0	To establish the discovery of a positive signal
t_{μ}	To set an interval at a given confidence level
$\tilde{t}_{\mu}$	To set an interval for a positive signal at a given confidence level
q_{μ}	To set an upper limit of a signal at a given confidence level
$\tilde{q}_{\mu}$	To set an upper limit of a positive signal at a given confidence level

Classical Asymptotic Formulae

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Special Article - Tools for Experiment and Theory

arXiv:1007.1727

Asymptotic formulae for likelihood-based tests of new physics

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TESTS OF STATISTICAL HYPOTHESES CONCERNING SEVERAL PARAMETERS WHEN THE NUMBER OF OBSERVATIONS IS LARGE(1)

BY
ABRAHAM WALD

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Likelihood ratio in a measurement

Signal region:
sensitive to Parameter Of Interest ($\boldsymbol{\mu}$)

Control region:
sensitive to Nuisance Parameters (θ)

The likelihood function is the product of Poisson probabilities for all bins:

$$L(\mu, \boldsymbol{\theta}) = \prod_{j=1}^{N} \frac{(\mu s_j + b_j)^{n_j}}{n_j!} e^{-(\mu s_j + b_j)} \prod_{k=1}^{M} \frac{u_k^{m_k}}{m_k!} e^{-u_k} .$$

To test a hypothesized value of μ we consider the profile likelihood ratio

$$\lambda(\mu) = \frac{L(\mu, \hat{\hat{\boldsymbol{\theta}}})}{L(\hat{\mu}, \hat{\boldsymbol{\theta}})} .$$

Most powerful test (Neyman-Pearson

The $\boldsymbol{\mu}$ value we want to test

Optimal estimate of $\boldsymbol{\mu}$

Wald's theorem

The likelihood function is the product of Poisson probabilities for all bins:

$$L(\mu, \boldsymbol{\theta}) = \prod_{j=1}^{N} \frac{(\mu s_j + b_j)^{n_j}}{n_j!} e^{-(\mu s_j + b_j)} \prod_{k=1}^{M} \frac{u_k^{m_k}}{m_k!} e^{-u_k} .$$

To test a hypothesized value of μ we consider the profile likelihood ratio

$$\lambda(\mu) = \frac{L(\mu, \hat{\hat{\boldsymbol{\theta}}})}{L(\hat{\mu}, \hat{\boldsymbol{\theta}})} .$$

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same limit distribution. The limit distribution of $-2 \log \lambda_n$ is the χ^2-distribution with r degrees of freedom if the hypothesis to be tested is true. If the true parameter point θ_n is not an element of ω, the distribution of $-2 \log \lambda_n$ approaches the distribution of a sum of non-central squares

$$U^2 = u_1^2 + \cdots + u_r^2,$$

where the variates $u_1, \cdots, u_r$ are independently and normal
with unit variances and

$$\sum_{p=1}^{r} (Eu_p)^2 = n \sum$$

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Can we go one small step further?

$$t_\mu \equiv -2 \ln \lambda(\mu) = \frac{(\hat{\mu} - \mu)^2}{\sigma^2} + \mathcal{O}(\frac{1}{\sqrt{N}}) \qquad (17)$$

Here $\hat{\mu}$ follows a Gaussian distribution with a mean μ' and standard deviation σ, and N represents the data sample size. The standard deviation σ of $\hat{\mu}$ is obtained from the covariance matrix of the estimators for all the parameters, $V_{ij} = \mathrm{cov}[\hat{\theta}_i, \hat{\theta}_j]$, where here the θ_i represent

Example: $\widetilde{q_\mu}$

$$\tilde{q}_\mu = \begin{cases} 0 & \hat{\mu} > \mu \,, \\ -2\ln \frac{\mathcal{L}(\mu,\hat{\hat{\boldsymbol{\theta}}}(\mu))}{\mathcal{L}(\hat{\mu},\hat{\boldsymbol{\theta}})} & \mu \geq \hat{\mu} \geq 0 \\ -2\ln \frac{\mathcal{L}(\mu,\hat{\hat{\boldsymbol{\theta}}}(\mu))}{\mathcal{L}(0,\hat{\hat{\boldsymbol{\theta}}}(0))} & \hat{\mu} < 0 \,. \end{cases}$$

$\widehat{\mu} > \mu$: not inconsistent with upper limit mu
$\widehat{\mu} < 0$: assume non-negative signal strength and we compare with 0

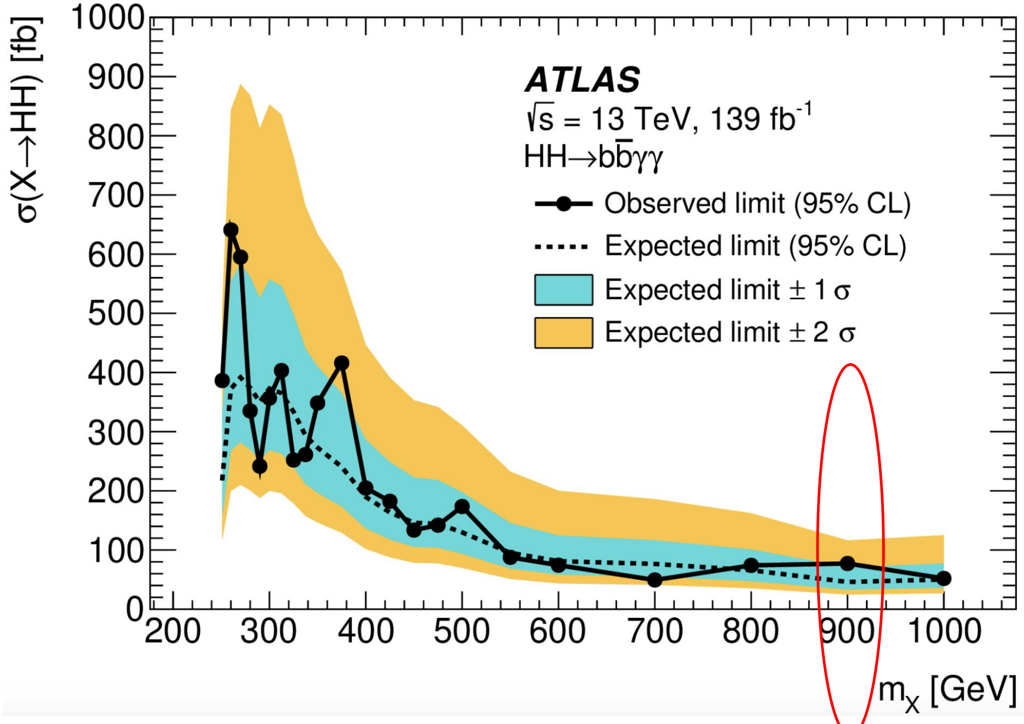
$$\tilde{q}_\mu = \begin{cases} 0 & \hat{\mu} > \mu \,, \\ \frac{(\hat{\mu}-\mu)^2}{\sigma^2} & \mu \geq \hat{\mu} \geq 0 \,, \\ \frac{\mu^2-2\mu\hat{\mu}}{\sigma^2} & \hat{\mu} < 0 \,. \end{cases}$$

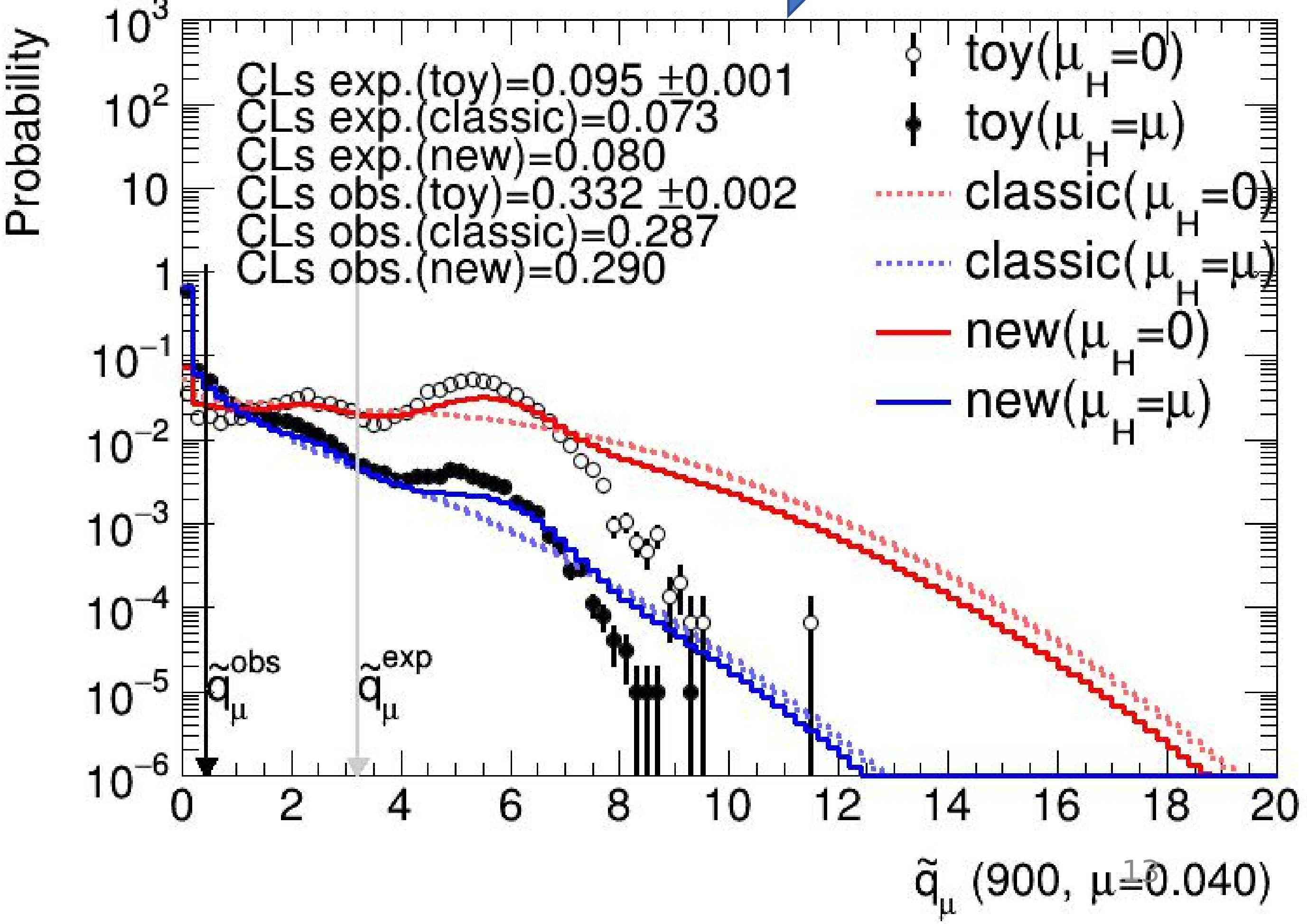
Classic asymptotic formulae

$$f(\tilde{q}_\mu|\mu') = \Phi\left(\frac{\mu'-\mu}{\sigma}\right)\delta(\tilde{q}_\mu) + \begin{cases} \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\tilde{q}_\mu}}\exp\left[-\frac{1}{2}\left(\sqrt{\tilde{q}_\mu}-\frac{\mu-\mu'}{\sigma}\right)^2\right] & 0 < \tilde{q}_\mu \leq \mu^2/\sigma^2 \,, \\ \frac{1}{\sqrt{2\pi}(2\mu/\sigma)}\exp\left[-\frac{1}{2}\frac{(\tilde{q}_\mu-(\mu^2-2\mu\mu')/\sigma^2)^2}{(2\mu/\sigma)^2}\right] & \tilde{q}_\mu > \mu^2/\sigma^2 \,. \end{cases}$$

Example: X(900GeV) HH$\gamma\gamma$ bb

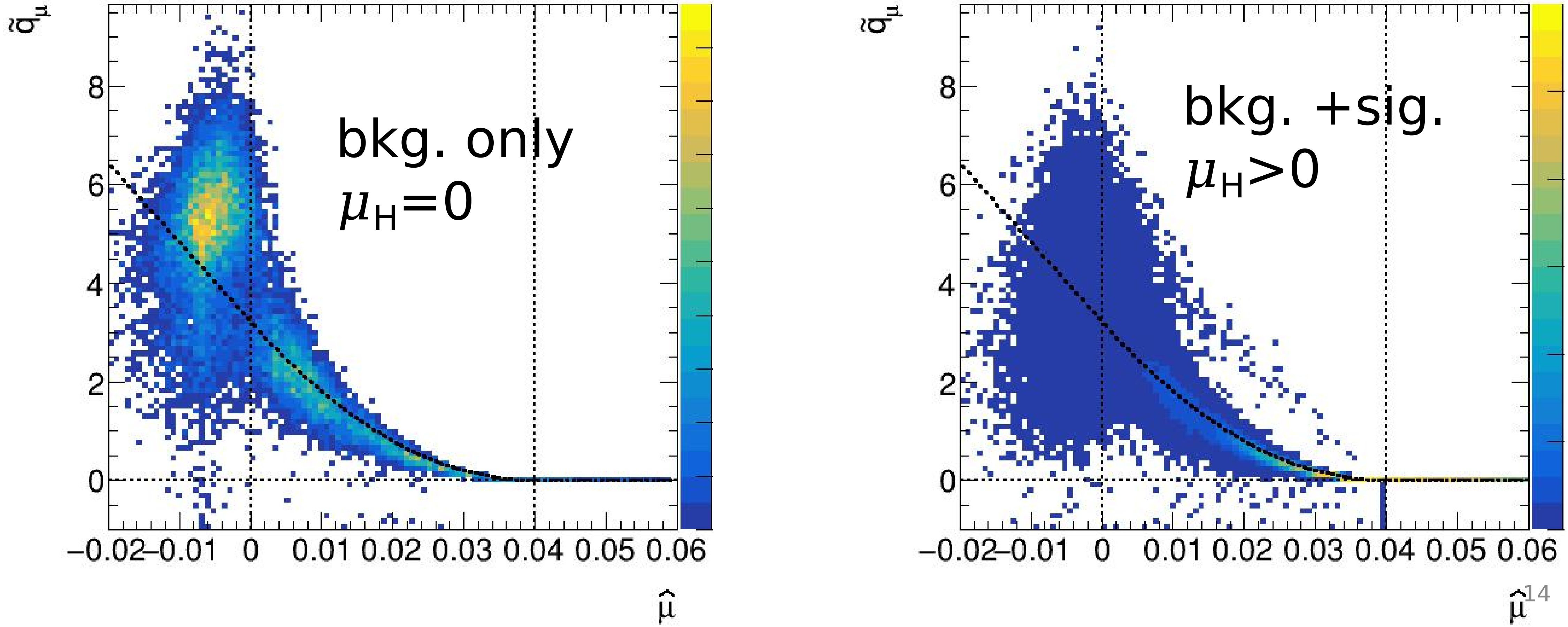
Signal like ← → Background like





Why “bumps” in PDF of

$$\tilde{q}_\mu = \begin{cases} 0 & \hat{\mu} > \mu\,, \\ \frac{(\hat{\mu}-\mu)^2}{\sigma^2} & \mu \geq \hat{\mu} \geq 0\,, \\ \frac{\mu^2 - 2\mu\hat{\mu}}{\sigma^2} & \hat{\mu} < 0\,. \end{cases}$$

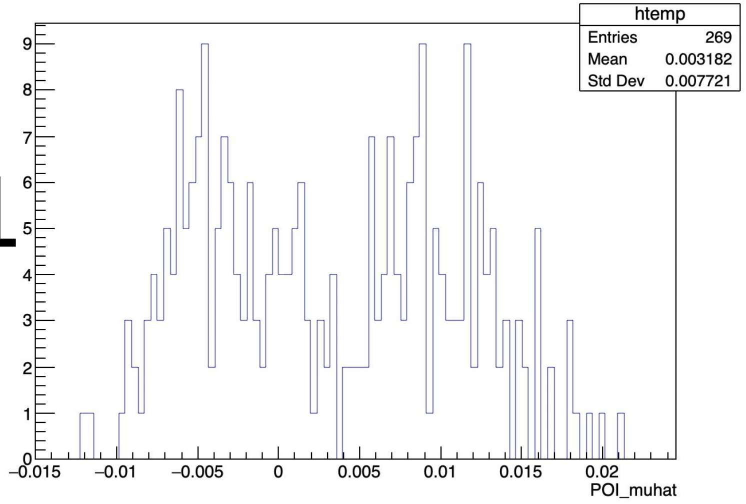


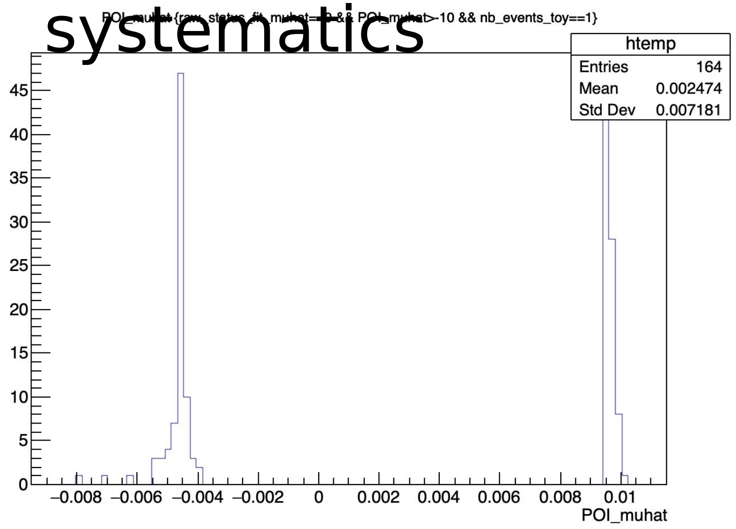
Why “bumps” in PDF of $\widetilde{q_{\mu}}$?

switch off systematics

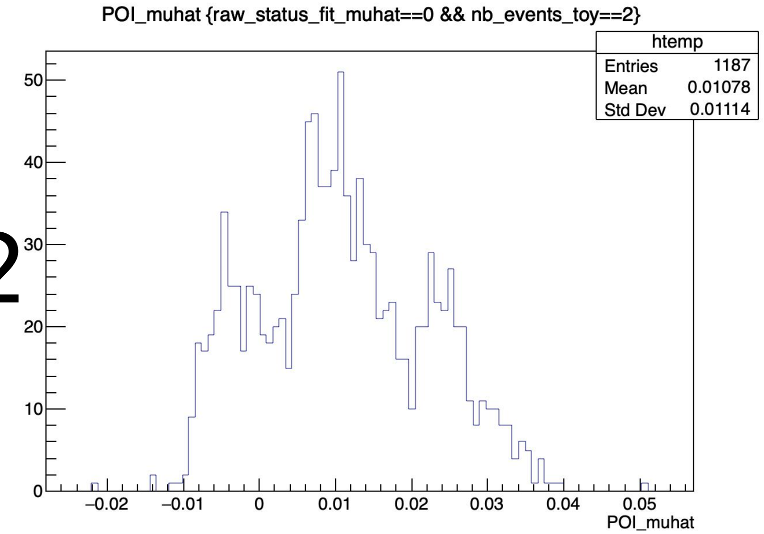
Things are clear if looking at the $\hat{\mu}$ distribution in **toys** for the number of events = 1, 2, …

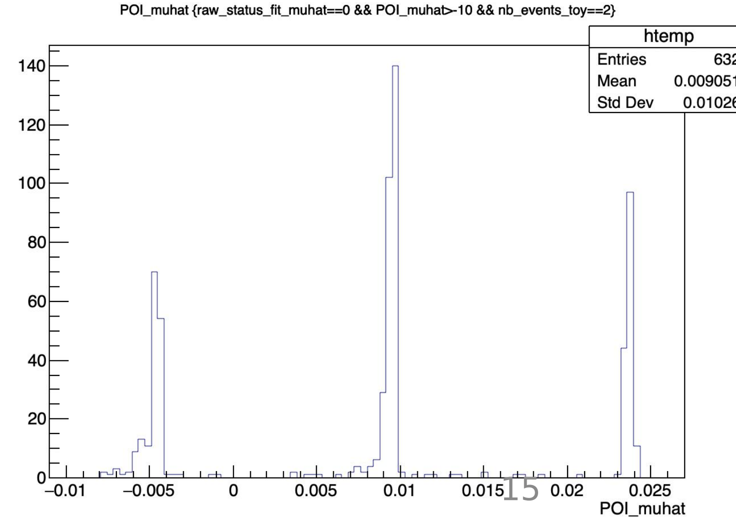
n=1



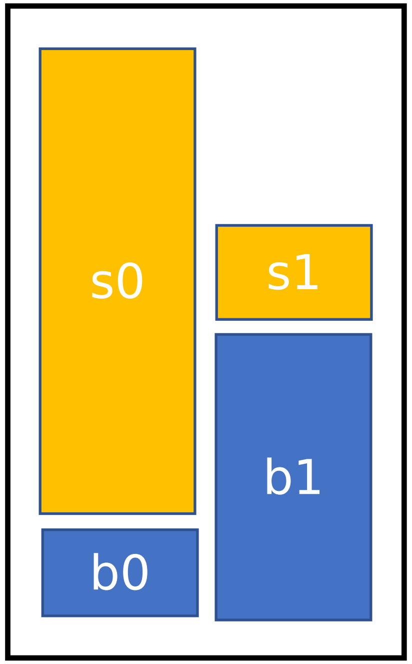


n=2





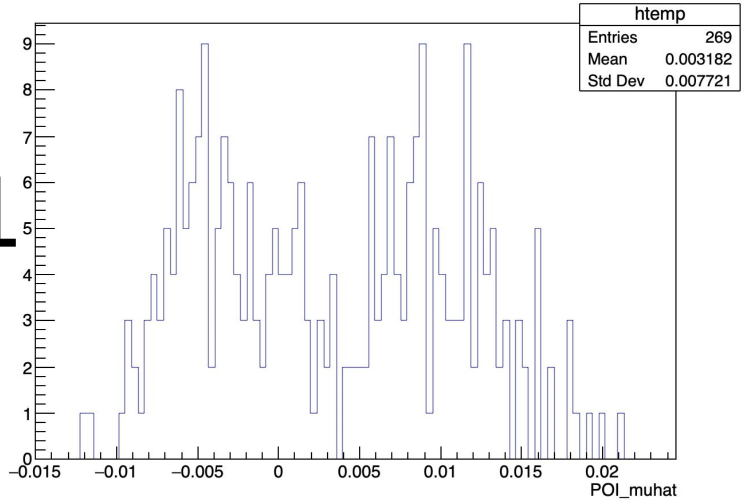
Why “bumps” in PDF of $\widetilde{q_\mu}$?

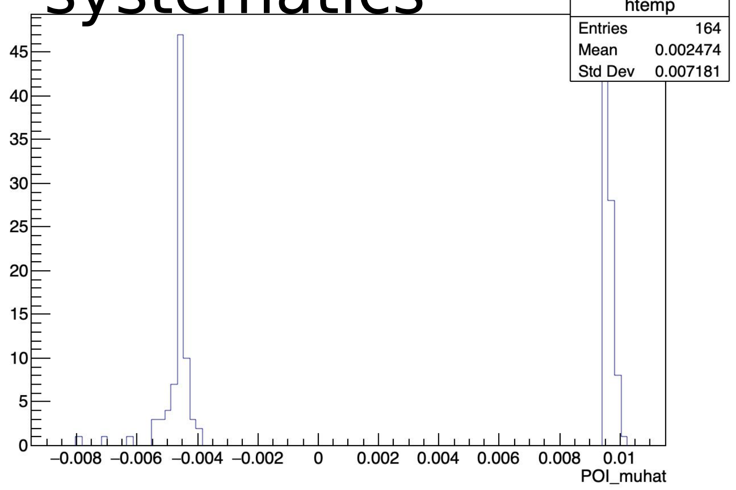


supposing a **2-bin** observable distribution

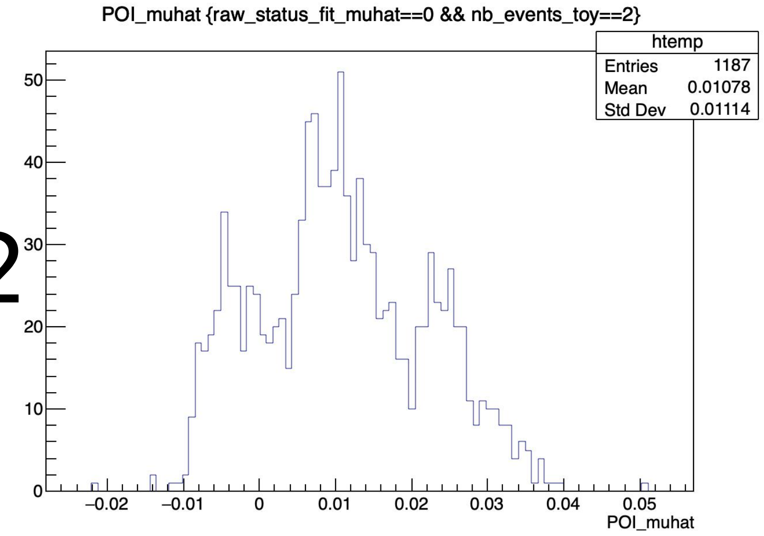
switch off systematics

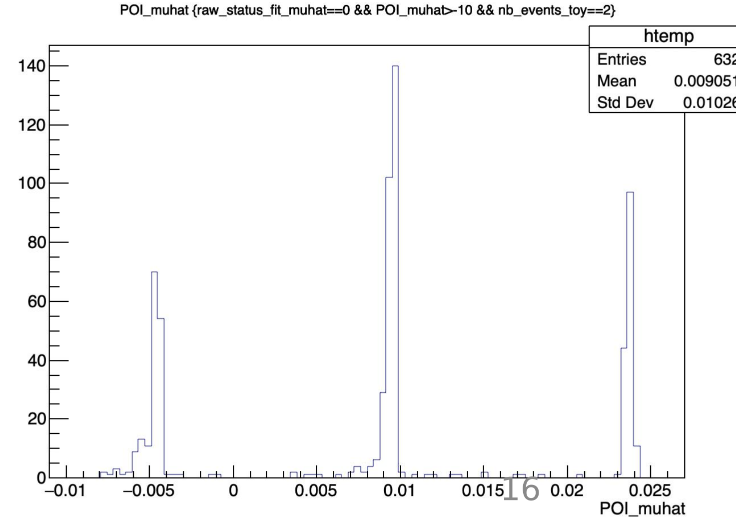
n=1





n=2

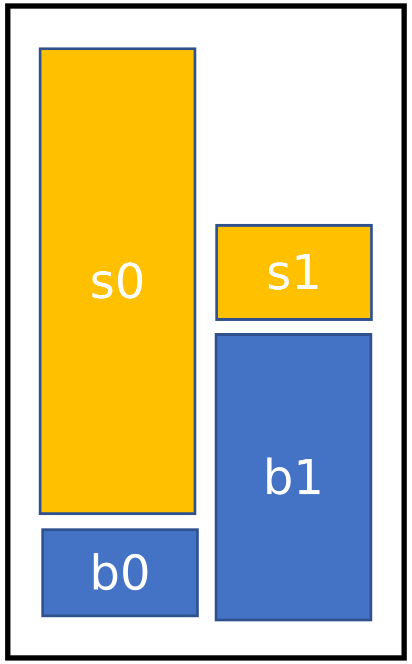




Why “bumps” in PDF of $\tilde{\mu}$?

$$\mathcal{L} = \Pi_{i=0}^{N} P(n_i | b_i + \mu s_i)$$

$$\log \mathcal{L} = \sum_{i=0}^{N} n_i \log(b_i + \mu s_i) - (b_i + \mu s_i)$$



supposing a **2-bin** observable distribution

$$\partial \log \mathcal{L} / \partial \mu = 0$$

$$A = 2 s_0 s_1$$

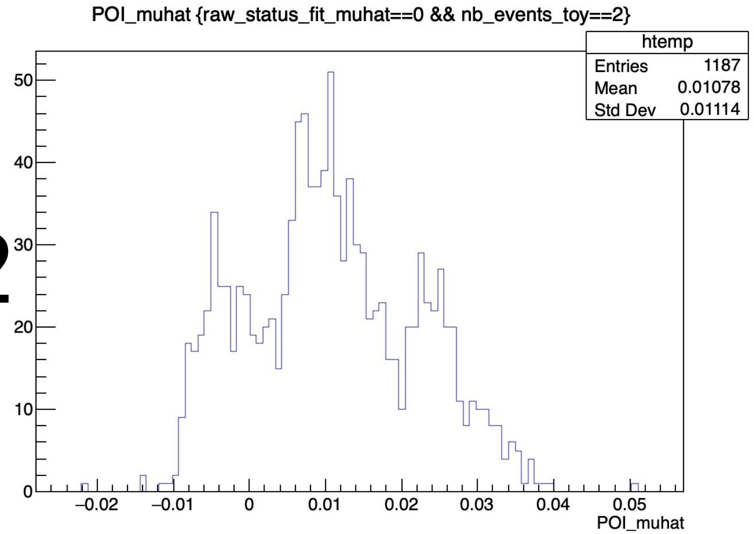
$$B = s_0 b_1 + s_1 b_0 - \frac{n_0 + n_1}{s_0 + s_1} s_0 s_1$$

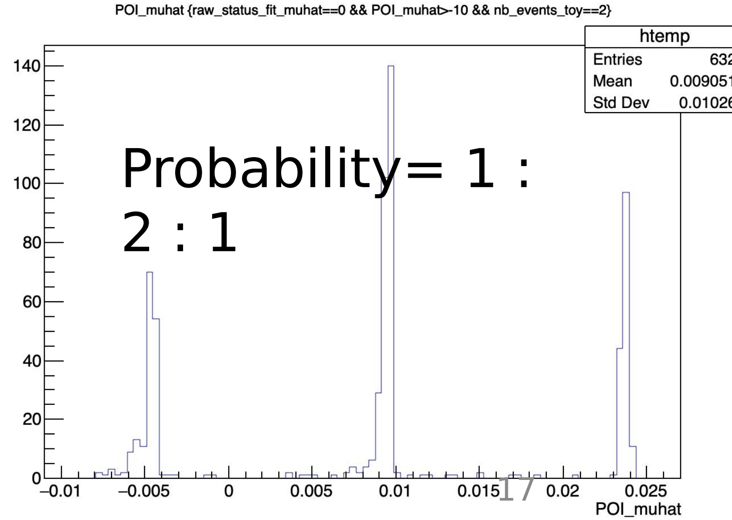
$$C = b_0 b_1 - \frac{n_0 s_0 b_1 + n_1 s_1 b_0}{s_0 + s_1}$$

$$\hat{\mu}(n_0, n_1) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\hat{\mu}(k, n-k) \approx -\frac{b_0}{s_0} + \frac{k}{s_0 + s_1}$$

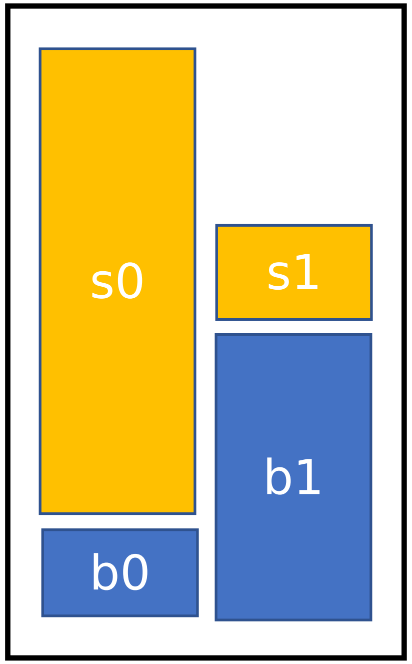
n=2





Probability= 1 : 2 : 1

New Asymptotic Formulae



- supposing a **6-bin** observable distribution
- Supposing $n_{small}=5$

$$f(T_\mu|\mu_H) = \sum_{n=0}^{+\infty} f(T_\mu|n,\mu_H)P(n|b+\mu_H s)$$
$$= \sum_{n=0}^{n_{\rm small}} f(T_\mu|n,\mu_H)P(n|b+\mu_H s) + \sum_{n>n_{\rm small}} f(T_\mu|n,\mu_H)P(n|b+\mu_H s)$$
$$\approx \sum_{n=0}^{n_{\rm small}} f_{\rm SS}(T_\mu|n,\mu_H)P(n|b+\mu_H s) + (1-\sum_{n=0}^{n_{\rm small}} P(n|b+\mu_H s))f_{\rm LS}(T_\mu|n,\mu_H)$$

n_{small}: threshold between large statistics and small statistics

Small-Statistics part:
- discrete
- analytical/numerical calculation

Large-Statistics part:
- continual
- classical asymptotic formulae

$$f_{\rm SS}(T_\mu|n,\mu_H) = \sum_{k_0+k_1+k_2+k_3+k_4+k_5=n} \frac{n!}{k_0!k_1!\cdots k_5!}\left(\frac{b_5}{b+\mu_H s}\right)^{k_5}\Pi_{i=0}^{4}\left(\frac{b_i+\mu_H s_i}{b+\mu_H s}\right)^{k_i}$$
$$\times f_{\rm binned}(T_\mu|n_i=k_i, i=0,1,2,3,4;\mu_H)\ ,$$

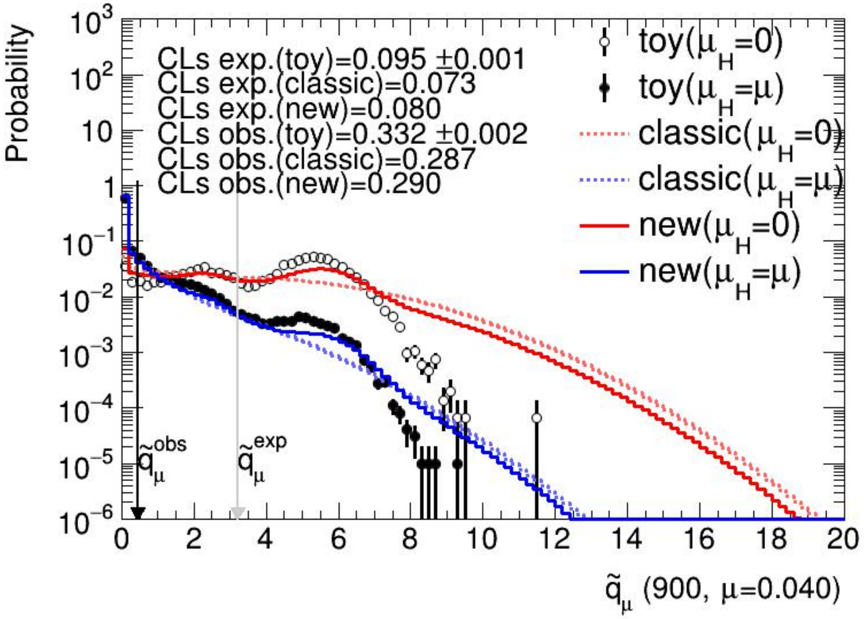
New Asymptotic Formulae

n_{small}: threshold between large statistics and small statistics

$$f(T_\mu|\mu_H) = \sum_{n=0}^{+\infty} f(T_\mu|n,\mu_H)P(n|b+\mu_H s)$$

$$\sum_{n=0}^{n_{\mathrm{small}}} f(T_\mu|n,\mu_H)P(n|b+\mu_H s) + \sum_{n>n_{\mathrm{small}}} f(T_\mu|n,\mu_H)P(n|b+\mu_H s)$$

$$\sum_{n=0}^{n_{\mathrm{small}}} f_{\mathrm{SS}}(T_\mu|n,\mu_H)P(n|b+\mu_H s) + (1-\sum_{n=0}^{n_{\mathrm{small}}} P(n|b+\mu_H s))f_{\mathrm{LS}}(T_\mu|n,\mu_H)$$



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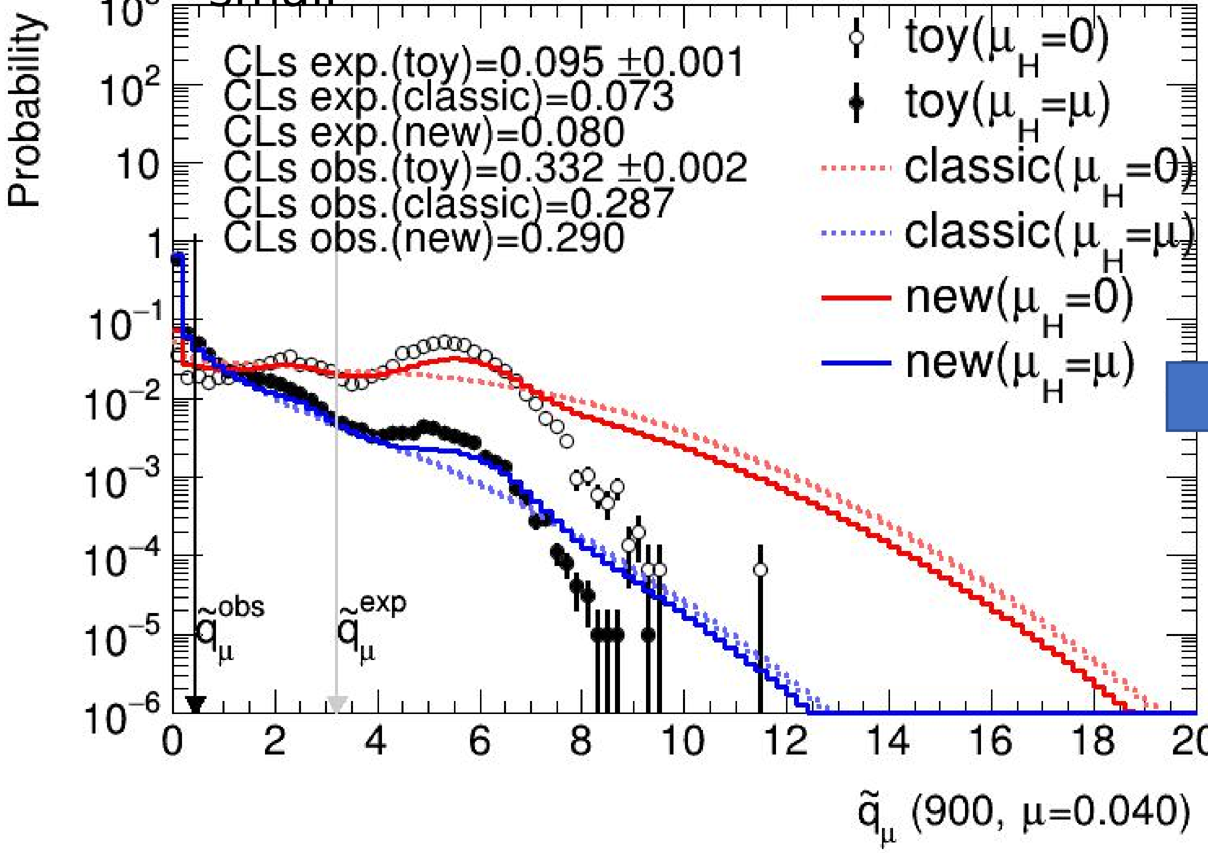
$$f_{\mathrm{SS}}(T_\mu|n,\mu_H) = \sum_{k_0+k_1+k_2+k_3+k_4+k_5=n} \frac{n!}{k_0!k_1!\cdots k_5!}\left(\frac{b_5}{b+\mu_H s}\right)^{k_5}\Pi_{i=0}^{4}\left(\frac{b_i+\mu_H s_i}{b+\mu_H s}\right)^{k_i}$$
$$\times f_{\mathrm{binned}}(T_\mu|n_i=k_i, i=0,1,2,3,4;\mu_H)\ ,$$

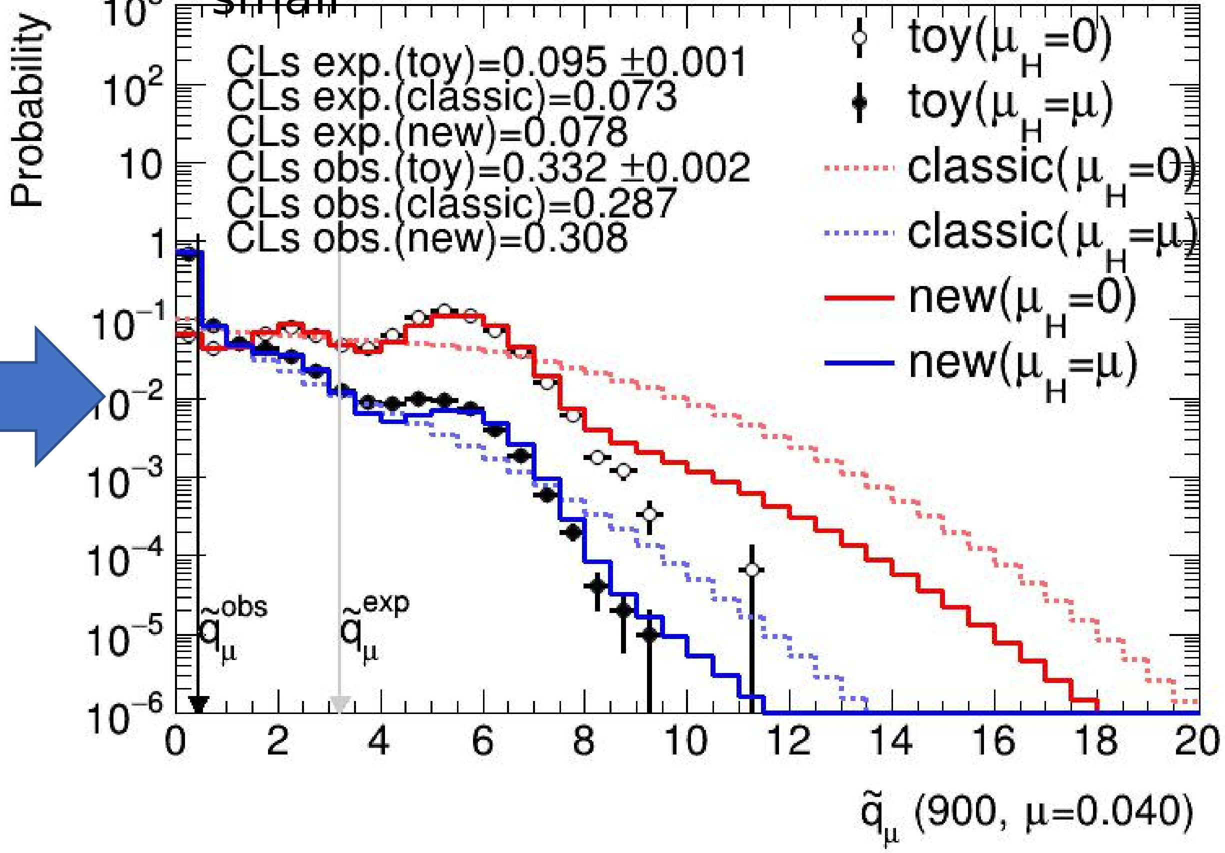
New Asymptotic Formulae

n_{small}: threshold between large statistics and small statistics

A conservative choice of n_{small}



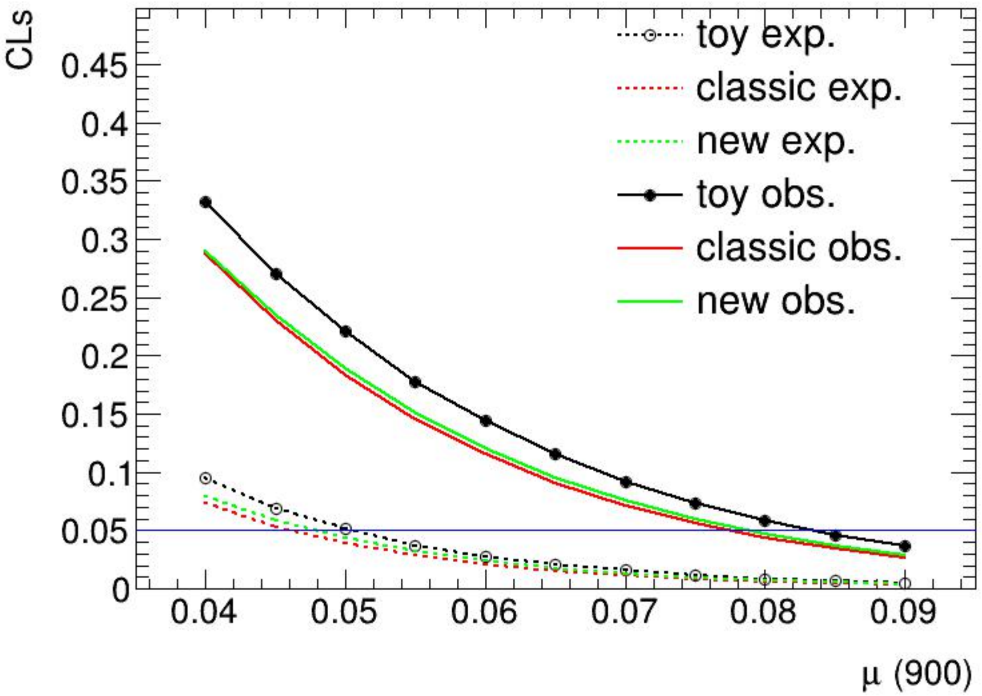
An aggressive choice of n_{small}



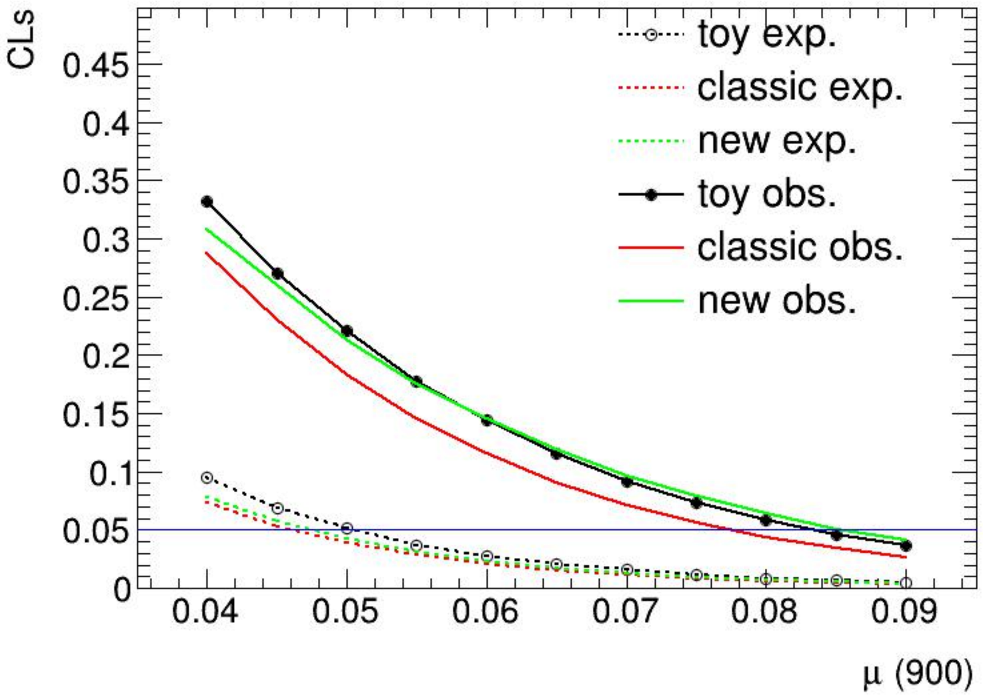
New Asymptotic Formulae

n_{small}: threshold between large statistics and small statistics

A conservative choice of n

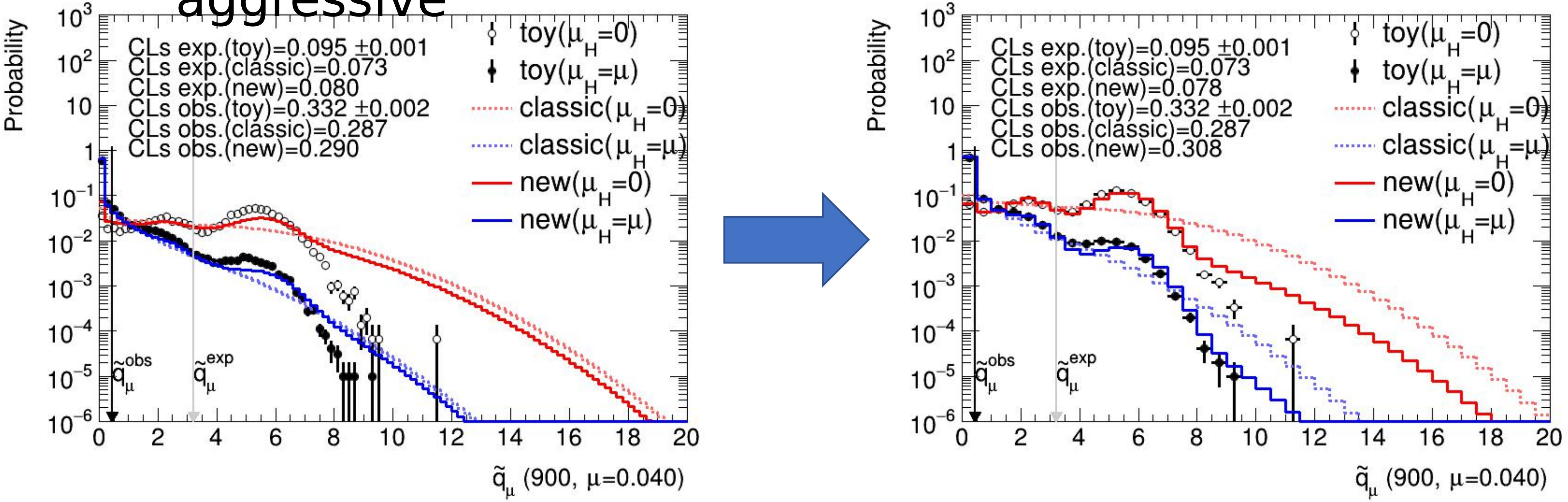


An aggressive choice of n



Comments on New Asymptotic Formulae

- Still more of education value due to
  - How to choose n_{small}?
  - How many bins in the Small-Statistics part?
  - How to correct the Large-Statistics part if we want to be aggressive



Summary

- New asymptotic formulae proposed (work in progress)
  - Large-statistics part: described by classical asymptotic formulae
  - Small-statistics part: described by analytic/numerical calculation

- Application to a real case: di-higgs resonance search X HH bbyy
  - Look promising
  - Still some issues to be fixed
  - More of education value for the moment

Thank you for your attention!