Sudakov Resummation

In this lecture note, we discuss the physical origin of the Sudakov factor[1] by using the thrust distribution as an example and the application of the Sudakov resummation in hadronic collisions. Sudakov factors, especially the double logarithmic terms, can appear in many physical processes as a result of the incomplete cancellation of soft (soft-collinear) divergences between real and virtual contributions.

This lecture note is intended to be an informal and intuitive note on the Sudakov resummation in QCD for graduate students to learn some basics on this topic and for the purpose of learning only. It is by no means complete or rigorous. Please use it with caution.

In the lecture, I presume that you are familiar with QFT and Peskin's book on QFT. From time to time, I will refer to certain chapters in Peskin in case that you are interested in more details. A lot of material in this note is based on what I learnt over the past few years from Prof. Al Mueller, Prof. J. Owens, Dr. F. Yuan and other colleagues, as well as from some QCD textbooks and material published online.

1 Introduction

1.1 Resummation

Resummation is a vague and broad concept, which implies that summation needs to be done once more. The question is that what summation has been performed in the first place. In perturbative QCD, we expand cross section σ in terms of powers of α_s

$$\sigma = \sigma_0 + \alpha_s\sigma_1 + \alpha_s^2\sigma_2 + \cdots = \sum_{i=0}^{\infty}\alpha_s^i\sigma_i, \tag{1}$$

$$\sigma_0 \sum_{i=0}^{\infty}\alpha_s^i\left(L^i + C^{(i)}\right) \quad \text{ideal QCD expansion}$$

$\sigma_0 \sum_{i=0}^{n-1}\alpha_s^i L^i$	$\sigma_0 \sum_{i=0}^{n-1}\alpha_s^i C^{(i)}$	$\Leftarrow$ pQCD
$\sigma_0 \sum_{i=n}^{\infty}\alpha_s^i L^i$	$\sigma_0 \sum_{i=n}^{\infty}\alpha_s^i C^{(i)}$	
$\Uparrow$ resummation	$\nwarrow$ negligible	

where $\sigma_i = \sigma_0\left(L^i + C^{(i)}\right)$ represents the cross section computed from the i-th order perturbative calculation. Sometimes, this expansion is convergent without the appearance of large logarithms L, then the above series can be truncated at certain order. Often, this series is not convergent, since large logarithms can appear in higher order expansions $\sigma_n \sim L^n$, where L stands for the logarithmic term.

More specifically, for inclusive total cross section in e^+e^- annihilation, we will see that higher order corrections are just constants without large logarithms. In this case, the pQCD expansion is convergent. For the thrust distribution, as we will see, the appearance of large logarithms can cause the breakdown of pQCD expansion, since $\alpha_s^n\sigma_n$ can increase as n increases. This means that we can no longer truncate the above series in terms of pQCD expansion, and we no longer have reliable predictions if we only rely on finite fixed order results.

Obviously, we can not do all order calculations exactly in pQCD. However, we can systematically resum $\alpha_s^n L^n$ terms up to all order and neglect the constant higher order corrections. Examples: the well-known

[1] See Peskin for discussions on conventional Sudakov double logarithmic form factor.

DGLAP evolution equation resums $\left(\alpha_s \ln Q^2/\mu^2\right)^n$ and BFKL evolution equation resums $(\alpha_s \ln 1/x)^n$ at the leading logarithmic (LL) level. Usually we categorize the level of the resummation of $\sum_n \alpha_s^k (\alpha_s L)^n$ as follows

- LL: leading log $\Leftrightarrow \sum_n (\alpha_s L)^n$ with $k=0$;
- NLL: next to leading log $\Leftrightarrow \alpha_s \sum_n (\alpha_s L)^n$;
- N^kLL: $\sum_n \alpha_s^k (\alpha_s L)^n$.

The purpose of resummation is to restore predictive power in theoretical calculations and describe the relevant physics better from an overall perspective.

1.2 Infrared safety

In higher order caculations in QFT, we often encounter two kinds collinear divergence and soft divergence. Both of them are of the Infrared divergence type. That is to say, they both involve long distance.

- According to uncertainty principle, soft $\longleftrightarrow$ long distance;
- Also one needs an infinite time in order to specify accurately the particle momenta, and therefore their directions.

However, physical observables, due to infrared safety, are always finite when we measure them. Infrared safety is the property of any experimental observable, which can be computed reliably in perturbative QCD (order by order perturbative expansion of α_s with finite coefficients at every order).

- **Kinoshita-Lee-Nauenberg theorem**: For a suitable defined inclusive observable (e.g., $\sigma_{e^+e^-\to \text{hadrons}}$), there is a cancellation between the soft and collinear singularities occurring in the real and virtual contributions. Physical observables always requires the cancellation.
- Any new observables must have a definition which does not distinguish between

$$\text{parton} \longleftrightarrow \text{parton + soft gluon}$$
$$\text{parton} \longleftrightarrow \text{two collinear partons}$$

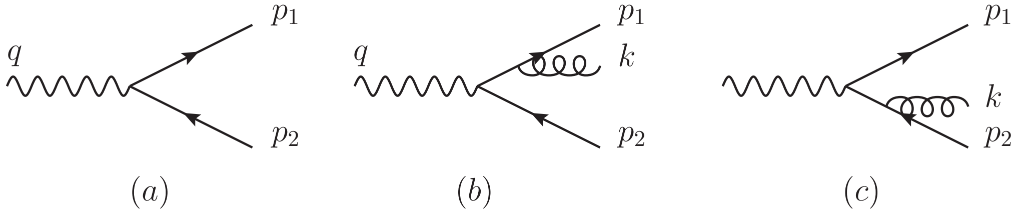
- Observables that respect the above constraint are called infrared safe observables. Infrared safety is a requirement that the observable is calculable in pQCD.
- Other infrared safe observables, for example, jets and the event shape observable thrust: $T = \max \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$.

1.3 Inclusive total cross section in e^+e^- annihilation

Consider the process (for detailed derivations, please read my lecture note on the e^+e^- annihilation)

$$e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{2}$$

To simplify the calculation we can compute this process as the decay of the virtual photon γ with the four momentum $q^\mu = (Q, 0, 0, 0)$ into $q\bar{q}$ and $q\bar{q}g$ as shown in the following figure.



The main results of this process are summarized as the following bullet points.

(a) Born diagram gives $\sigma_0 = \alpha_{em}\sqrt{s}N_c\sum_q e_q^2\left(\frac{4\pi\mu^2}{s}\right)^\epsilon \frac{\Gamma[2-\epsilon]}{\Gamma[2-2\epsilon]}$ in dimensional regularization with $d = 4-2\epsilon$. In this lecture note, we always use this notation together with the MS bar scheme.

(b) NLO: real contribution (3 body final state). $x_i \equiv \frac{2E_i}{Q}$ with $Q = \sqrt{s}$

$$\frac{d\sigma_3}{dx_1dx_2} = C_F\frac{\alpha_s}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$\text{with} \quad \frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left[\frac{1}{(1-x_1)}+\frac{1}{(1-x_2)}\right]$$

(c) Energy conservation $\Rightarrow x_1+x_2+x_3 = 2$. Momentum conservation $\Rightarrow \vec{p}_1+\vec{p}_2+\vec{p}_3 = 0$.

(d) From $(p_1+p_3)^2 = 2p_1\cdot p_3 = (Q-p_2)^2 = Q^2(1-x_2)$, we see that $x_2 \to 1$ ($(p_1+p_3)^2 \to 0$) $\Rightarrow \vec{p}_3 \,||\, \vec{p}_1$ Collinear Divergence (Similarly one has collinear divergence when $x_1 \to 1$).

(e) When $x_3 \to 0 \Rightarrow x_1 \to 1$ and $x_2 \to 1$, we encounter the double pole as shown above, which is the indication of the Soft Divergence.

(f) Final results for the NLO real and virtual contributions are

$$\sigma_r = \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}-\frac{2\pi^2}{3}\right],$$

$$\sigma_v = \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right],$$

respectively. Therefore, summing over the LO and NLO contributions to the total cross section yields

$$\lim_{\epsilon\to 0}\sigma^{\text{tot}}_{\gamma^*\to X} = \sigma_0\left[1+\frac{3}{4}C_F\frac{\alpha_s(\mu)}{\pi}+\mathcal{O}(\alpha_s^2)\right], \tag{3}$$

which is finite in 4-dimension when we take $\epsilon \to 0$.

Homework: Read through the lecture note and reproduce the above total cross-section.

(g) We have seen "almost complete" cancellation between real and virtual contributions for the total cross section as suggested by the KLN theorem with a small constant NLO correction of the order $\frac{3}{4}C_F\frac{\alpha_s(\mu)}{\pi}$. Here I used the loose term "almost complete" cancellation to describe the situation that only a small constant correction survives the cancellation in the total cross section at NLO.

In contrast, for other more differential observables, due to additional cuts made to the real contributions (or due to different constraints made to the real and virtual contributions), the cancellation between the real and virtual diagrams often is "incomplete" in the sense that large logarithms can appear as the result of the soft and collinear divergence cancellation between real and virtual graphs. In general, large logarithms can show up as the expansion in terms of ϵ

$$\frac{1}{\epsilon}a^\epsilon = \frac{1}{\epsilon}e^{\epsilon\ln a} = \frac{1}{\epsilon}+\ln a+\cdots \tag{4}$$

$$\frac{1}{\epsilon^2}a^\epsilon = \frac{1}{\epsilon^2}e^{\epsilon\ln a} = \frac{1}{\epsilon^2}+\ln a\frac{1}{\epsilon}+\frac{1}{2}\ln^2 a+\cdots. \tag{5}$$

Next, let us take a look at the example of the thrust distribution in e^+e^- annihilation where large logarithms start to appear.

2 Sudakov Resummation for Thrust Distribution

Thrust is an event shape observable reflecting the structure of the hadronic events in e^+e^- annihilation. The thrust T[1] is defined as

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|},$$

where $\vec{p}_i$ are the final-state hadron (or parton) momenta in the center of mass frame of e^+e^- collisional system and $\vec{n}$ is an arbitrary unit vector which maximizes $\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$. The direction of $\vec{n}$ vector is given by the direction of the largest momentum particle. It is straightforward to find $T = 1$ for back-to-back pencil-like events and $T = 1/2$ for spherically symmetric events.

Furthermore, T is infrared safe, i.e. insensitive to the emission of soft or collinear gluons, since T is invariant under the branching $\vec{P}_i \to \vec{P}_j + \vec{P}_k$, whenever $\vec{P}_j \parallel \vec{P}_k$ or one of them is soft.

2.1 Thrust in pQCD

Now let us study the thrust distribution in terms of the pQCD expansion.

(a) At zeroth order, we have the born process $e^- + e^- \to q + \bar{q}$, which gives the pencil like events, it is easy to show that $T = 1$ in this case (this is true for NLO virtual graph as well). Due to momentum conservation, $p_1 = p_2$, therefore, $T = 1$ by definition when $\vec{n}$ is chosen along $\vec{p}_1$ or $\vec{p}_2$. Therefore, we can write the normalized differential cross section as

$$\frac{1}{\sigma_0}\frac{d\sigma_0}{dT} = \delta(T-1).$$

(b) At the first order, we consider the $2 \to 3$ ($e^- + e^- \to q + \bar{q} + g$) process, which generates three particle final state events. We have derived the cross section in class which reads

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)},$$

where $x_1 = \frac{2p_1}{Q}$ and $x_2 = \frac{2p_2}{Q}$ are for the quark and antiquark, respectively. The energy conservation indicates $x_1 + x_2 + x_3 = 2$ where $x_3 = \frac{2p_3}{Q}$. Using geometry and momentum conservation, we should be able to find $\underline{T = \max[x_1, x_2, x_3]}$ in this case. For three-particle events, minimum value of T is 2/3 when all three momentum are equal, while the maximum value of T is 1.

(c) Use the delta function trick, we can write the differential cross section of thrust as

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{C_F\alpha_s}{2\pi}\int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\delta(T - \max[x_1, x_2, x_3]),$$

where $x_3 = 2 - x_1 - x_2$. We can perform the above integrations and find $\frac{1}{\sigma_0}\frac{d\sigma}{dT}$ as the function of T as follows

$$\frac{d\sigma}{\sigma_0 dT} = 2\frac{C_F\alpha_s}{2\pi}\int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1>x_2>x_3;\,\mathrm{or},\,x_2>x_1>x_3} \tag{6}$$

$$+2\frac{C_F\alpha_s}{2\pi}\int_{2-2T}^{1-T/2} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1>x_3>x_2;\,\mathrm{or},\,x_2>x_3>x_1} \tag{7}$$

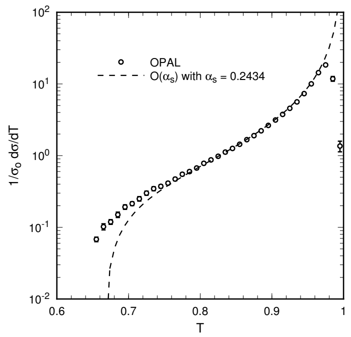
$$+2\frac{C_F\alpha_s}{2\pi}\int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{(2-T-x_2)^2 + x_2^2}{(T+x_2-1)(1-x_2)}\right]_{x_3>x_2>x_1;\,\mathrm{or},\,x_3>x_1>x_2} \tag{8}$$

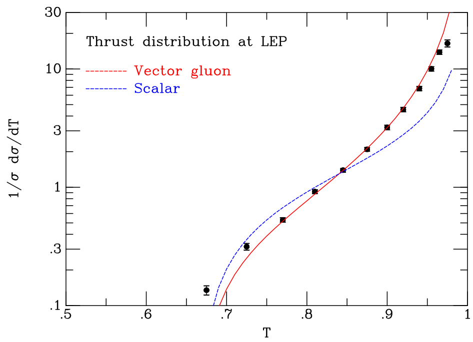
As shown above, first we consider the region in which $x_1 > x_2 > x_3$, and note that the delta function sets $T = x_1$ in this region, then determine the range of the integration for x_2 according to energy momentum conservation before we integrate over x_2. Next we can consider the other five different regions and sum all of them together which gives the following final expression at the first non-trivial order

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2-3T+2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right]. \tag{9}$$

Homework: Derive the above thrust distribution in Eq. (9) for $T < 1$.

(d) We can compare the above expression to the experimental data as shown below.





- Left figure: Deficiency at low T due to kinematics. $T > 2/3$ at this order. By continuing the pQCD expansion to higher order, we expect that the agreement at low T will get improved.
- Left figure: Miss the data when $T \to 1$ due to divergence as seen below.

$$\left.\frac{d\sigma}{\sigma_0 dT}\right|_{T\to 1} \sim \frac{C_F \alpha_s}{2\pi} \left[\frac{4}{(1-T)} \ln \frac{1}{1-T} - \frac{3}{1-T} \right] \to \infty.$$

  In this region, pQCD expansion fails due to the divergent behavior. This is to say that we have to resum the large logarithms up to all order: Sudakov factor!
- Right figure: Indication of gluon being a vector boson instead of a scalar. [2]

2.2 Resummation of the thrust distribution

To perform the resummation near $T = 1$, we can use the following intuitive steps.

(a) Let us include the Born and virtual as well as real contributions, and write

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2-3T+2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right] + C\delta(1-T), \tag{10}$$

where C is a divergent constant, which can be determined by the following integral according to Eq. (3)

$$\int_{T_{\min}}^{1} dT \frac{d\sigma}{\sigma_0 dT} = 1 + C_F \frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2). \tag{11}$$

Homework:
From Eq. (9), show that the following exact expression of the thrust distribution at one-loop satisfies Eq. (11) with $T_{\min} = 2/3$

$$\begin{aligned} \frac{d\sigma}{\sigma_0 dT} &= \delta(1-T) + \frac{C_F \alpha_s}{2\pi} \left[\delta(1-T)\left(\frac{\pi^2}{3} - 1\right) - \frac{3(3T-2)(2-T)}{(1-T)_+} \right] \\ &\quad + \frac{C_F \alpha_s}{2\pi} \frac{2(3T^2-3T+2)}{T} \left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T} \right)_+ \right], \end{aligned} \tag{12}$$

where the plus distribution is defined as $\int_a^1 dx f(x)(g(x))_+ = \int_a^1 dx f(x) g(x) - f(1) \int_0^1 dx g(x)$.

Since we are only interested in the large logarithm resummation, we can neglect the $\frac{\alpha_s}{\pi}$ term in the above expression as far as the resummation is concerned in our later derivations.

(b) With the help of the cumulative distribution method, we can first define $F(T) = \int_T^1 dT' \frac{d\sigma}{\sigma_0 dT'}$, and find

$$1 = \int_{T_{\min}}^1 dT \frac{d\sigma}{\sigma_0 dT} = \int_{T_{\min}}^T dT \frac{d\sigma}{\sigma_0 dT} + \int_T^1 dT \frac{d\sigma}{\sigma_0 dT} = \int_{T_{\min}}^T dT \frac{d\sigma}{\sigma_0 dT} + F(T) \quad \Rightarrow \tag{13}$$

$$. \quad F(T) \simeq 1 - \frac{C_F \alpha_s}{\pi} \ln^2(1-T) - \frac{C_F \alpha_s}{\pi} \frac{3}{2} \ln(1-T), \tag{14}$$

where the double log term comes from soft gluon region while the single log term is due to collinear gluon emissions.

(c) Assuming that soft and collinear contributions factorize (as we will show later in this lecture notes), we can resum these logarithms and obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \qquad \Rightarrow \qquad F(T) = \exp\left[-\frac{C_F \alpha_s}{\pi} \ln^2(1-T) - \frac{C_F \alpha_s}{\pi} \frac{3}{2} \ln(1-T)\right], \tag{15}$$

which is also known as the Sudakov factor for the thrust distribution. The exponentiation of the Sudakov factor can be understood as the result of resumming arbitrary number n soft gluon emission.

Alternatively, one can derive the above result by solving a differential equation involving $F(T)$. The cumulative distribution $F(T)$ represents the probability of thrust T in the interval $[T, 1]$. From fixed order result in Eq. (13), we find that $F(T)$ must satisfy

$$\frac{dF(T)}{dT} = -\frac{d\sigma}{\sigma_0 dT} = -\frac{C_F \alpha_s}{2\pi} \left[\frac{4}{(1-T)} \ln \frac{1}{1-T} - \frac{3}{1-T}\right]. \tag{16}$$

However, physically $F(T)$ should always be positive and therefore the above differential equation must be modified to insure its positivity. Since the gluon emission described by the right hand side of the above equation, can only decrease T (More gluon radiation tends to make the event shape more spherical), we expect that the change of $F(T)$ should also depend on itself. Therefore, we obtain

$$\frac{dF(T)}{dT} = -\frac{C_F \alpha_s}{2\pi} \left[\frac{4}{(1-T)} \ln \frac{1}{1-T} - \frac{3}{1-T}\right] F(T), \tag{17}$$

which can also be interpreted as the result of iteration due to multiple gluon emission. The solution of the above equation gives the resummed result in Eq. (15).

In the end, taking a derivative respect to T yields the Sudakov resummed thrust distribution

$$\frac{d\sigma}{\sigma_0 dT} = -\frac{dF(T)}{dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{4}{(1-T)} \ln \frac{1}{1-T} - \frac{3}{1-T}\right] \exp\left[-\frac{C_F \alpha_s}{\pi} \ln^2(1-T) - \frac{C_F \alpha_s}{\pi} \frac{3}{2} \ln(1-T)\right]. \tag{18}$$

This result can describe the measured thrust distribution reasonably well in the $T \to 1$ limit by taming the divergence. Modern technique such as the renomalization group equation (RGE) method in soft collinear effective theory (SCET) has been developed not too long ago, this allow us to perform the Sudakov resummation systematically. (See more discussion on the thrust distribution in Ref. [3].)

(d) The above resummation techniques will be used repeatedly in the following discussions. You will be able to also observe the pattern of Sudakov factors when they appear.

3 Sudakov Resummation in Drell-Yan Process

The simplest example in hadronic collisions is the reaction in which a high-invariant-mass (Q) lepton pair is created from $q\bar{q}$ annihilation in a proton-proton collision. This is known as the Drell-Yan process with the following LO cross section

$$Q^2 \frac{d\sigma}{dQ^2 dY} = \sum_q x_1 q(x_1) x_2 \bar{q}(x_2) \sigma(q\bar{q} \to l^+ l^-), \tag{19}$$

where $\sigma(q\bar{q}\to l^+l^-)=\frac{1}{N_c}e_q^2\frac{4\pi\alpha^2}{3Q^2}$ and kinematical variables are defined as $Y=\frac{1}{2}\ln\frac{x_1}{x_2}$ and $x_{1,2}=\frac{Q}{\sqrt{S}}e^{\pm Y}$.

Interestingly this process receives large higher order corrections which needs to be resummed from time to time. There are actually two types of Sudakov resummations related to this process. They are both soft gluon resummation w.r.t. different observables in DY processes, and they resemble a lot of similarities with the original Sudakov factor in QED.

(a) For the invariant mass distribution $\frac{d\sigma}{dQ^2}$ with $Q^2=x_1x_2S$, when $Q^2\to S$, x_1 and x_2 both approach 1, namely the threshold of this production. Due to the kinematical threshold constraint, the radiated gluons are always soft, therefore this type of resummation is also called the threshold resummation.

(b) For the transverse momentum Q_T distribution $\frac{d\sigma}{d^2Q_T}$ with fixed large mass Q^2, when $Q_T\ll Q$, the emission of soft gluons with small $k_\perp$ dominates the higher order correction. Also, this is known as Q_T (TMD) resummation.

3.1 NLO correction to the DY process

Let me briefly mention the NLO correction of DY lepton pair production which is quite similar to the NLO calculation of e^+e^- annihilation.

(a) First, we redo the LO calculation in dim-reg, and fix the normalization for the Born cross section σ_0.

(b) Second, we can easily include the virtual contribution and write the sum of LO and NLO virtual contribution as

$$\sigma_0\delta(1-z)\left\{1+\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right]\right\}, \tag{20}$$

where $z\equiv\frac{Q^2}{x_1x_2S}$, which indicates how much center of mass energy are taken by the virtual photon. The virtual contribution is exactly the same as the one computed in e^+e^- annihilation, thus we do not have to compute it again.

(c) The NLO partonic real contribution coming from one gluon emission can be written as[2]

$$\begin{aligned}\frac{d\hat{\sigma}}{dQ^2}(real) &= \frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{z^\epsilon(1-z)^{1-2\epsilon}}{\Gamma(1-\epsilon)}\\ &\int_0^1 dy\,[y(1-y)]^{-\epsilon}\left[(1-\epsilon)\frac{y^2+(1-y)^2}{y(1-y)}+\frac{2z}{(1-z)^2y(1-y)}-2\epsilon\right],\end{aligned} \tag{21}$$

where $y\equiv\frac{1}{2}(1+\cos\theta)$ with the virtual photon momentum $q^\mu=(E,q\sin\theta,0,q\cos\theta)$. For real graphs, we find the virtual photon energy and momentum are $E=\sqrt{S}(1+z)/2$ and $q=\sqrt{S}(1-z)/2$, respectively. When $z=1$, $Q^2=x_1x_2S$ which means all the energy of $q\bar{q}$ goes into the virtual photon and the recoil gluon is soft. The divergence occurred at $y=0$, 1 in the above expression correspond to the collinear divergence since the polar angle $\theta=\pi$ or 0 accordingly.

To calculate the real part, it is customary to use the following identity involving the so-called plus-function.

$$\frac{1}{(1-z)^{1+\epsilon}}=-\frac{1}{\epsilon}\delta(1-z)+\frac{1}{(1-z)_+}-\epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+, \tag{22}$$

where the plus-function should always be understood in an integral and it is defined as

$$\int_a^1 dzf(z)_+g(z)\equiv\int_a^1 dzf(z)g(z)-g(1)\int_0^1 dzf(z)=\int_a^1 dzf(z)[g(z)-g(z)]-g(1)\int_0^1 dzf(z). \tag{23}$$

[2]You are encouraged to show this by considering the Feynman graphs for the $q\bar{q}\to g+\gamma^*$ process in dim-reg and integrating over two body final states phase space together with the energy-momentum conservation constraints.

To show the identity is true, we consider the following integral

$$\begin{aligned}\int_a^1 dz\frac{g(z)}{(1-z)^{1+\epsilon}} &= g(1)\int_a^1 dz\frac{1}{(1-z)^{1+\epsilon}}+\int_a^1 dz\frac{g(z)-g(1)}{(1-z)^{1+\epsilon}}\\ &= \frac{-1}{\epsilon}(1-a)^{-\epsilon}g(1)+\int_a^1 dz\frac{g(z)-g(1)}{(1-z)}-\epsilon\int_a^1 dz\frac{\ln(1-z)[g(z)-g(1)]}{(1-z)}+\mathcal{O}(\epsilon^2)\\ &= \frac{-1}{\epsilon}g(1)+\int_a^1\frac{1}{(1-z)_+}g(z)-\epsilon\int_a^1 dz\left(\frac{\ln(1-z)}{(1-z)}\right)_+ g(z)+\mathcal{O}(\epsilon^2).\end{aligned} \tag{24}$$

Therefore, using the above identity together with the integral form of the beta function

$$\int_0^1 dx x^{u-1}(1-x)^{v-1} = B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \tag{25}$$

we can cast the real contribution into the following form

$$\begin{aligned}\frac{d\hat{\sigma}}{dQ^2}(real) &= \frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\\ &\left[\frac{2}{\epsilon^2}\delta(1-z)-\frac{2}{\epsilon}\frac{1+z^2}{(1-z)_+}-2\ln z\frac{1+z^2}{(1-z)}+4(1+z^2)\left(\frac{\ln(1-z)}{(1-z)}\right)_+\right],\end{aligned} \tag{26}$$

The full contribution including the Born, virtual and real parts is then

$$\begin{aligned}&\delta(1-z)-\frac{2}{\epsilon}\underbrace{\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right)}_{\equiv P_{qq}(z)}\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\\ &+\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\left(-8+\frac{2\pi^2}{3}\right)\delta(1-z)-2\ln z\frac{1+z^2}{(1-z)}+4(1+z^2)\left(\frac{\ln(1-z)}{(1-z)}\right)_+\right].\end{aligned} \tag{27}$$

The above expression still contains collinear divergence which is proportional to $P_{qq}(z)$, in contrast to $e^+e^-\to q\bar{q}$ total cross section, which is free of any divergence. This is due to the fact that we introduce quark distributions which distinguish the following two degenerate states

$$q \text{ state} \quad \text{and} \quad q + \text{collinear gluon state}.$$

Therefore, we should treat them as the same states in our calculation by absorbing this collinear divergence $-\frac{1}{\epsilon}P_{qq}(z)$ into quark (antiquark) distributions. The amount of finite term subtracted from the total contribution together with the $\frac{1}{\epsilon}$ pole is arbitrary.

The most popular scheme is the $\overline{\text{MS}}$ scheme which absorbs $-\frac{1}{\epsilon}+\gamma_E-\ln 4\pi+\ln\frac{M_f^2}{\mu^2}$ into the bare quark distribution $q(x)$ and defines a new renormalized quark distribution $q(x,M_f)$ at the factorization scale M_f.

$$-\frac{1}{\epsilon}\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = \underbrace{-\frac{1}{\epsilon}+\gamma_E-\ln 4\pi+\ln\frac{M_f^2}{\mu^2}}_{\overline{\text{MS}}\,\text{subtraction}}+\underbrace{\ln\frac{Q^2}{M_f^2}}_{\text{hard part}}+\mathcal{O}(\epsilon) \tag{28}$$

$$\Rightarrow \quad q(x,M_f) = q(x)-\frac{1}{\epsilon}\frac{\alpha_s}{2\pi}C_F\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\int dydz\delta(x-yz)P_{qq}(z)q(y) \tag{29}$$

$$\Rightarrow \quad \frac{dq(x,M_f)}{d\ln M_f^2} = \frac{\alpha_s}{2\pi}C_F\int dydz\delta(x-yz)P_{qq}(z)q(y) = \frac{\alpha_s}{2\pi}C_F\int\frac{dz}{z}P_{qq}(z)q(x/z). \tag{30}$$

The above equation indicates that quark distribution with momentum fraction x can be derived from a quark with momentum fraction $y=x/z>x$ through $q\to qg$ splittings. Due to iteration in the presence of higher order graphs, we can replace $q(x/z)$ by $q(x/z,M_f)$ which make the above equation close. This equation is known as the DGLAP equation for the quark-quark channel. The solution to this equation resums the $\alpha_s \ln M_f^2/\mu^2$ type of logarithm.

The full expression at NLO after restoring the normalization is then

$$
\begin{aligned}
\frac{d\sigma}{dQ^2} &= \sum_q \frac{\sigma(q\bar{q})}{S} \int_\tau^1 \frac{dx_1}{x_1} q(x_1, M_f) \int_{\tau/x_1}^1 \frac{dx_2}{x_2} \bar{q}(x_2, M_f) \mathcal{H}(z, Q, M_f), \\
\mathcal{H}(z, Q, M_f) &\equiv \left\{ \delta(1-z) + \frac{\alpha_s}{2\pi} C_F \left[2P_{qq}(z) \ln \frac{Q^2}{M_f^2} + \delta(1-z)\left(-8 + \frac{2\pi^2}{3}\right)\right] \right. \\
&\quad \left. + \frac{\alpha_s}{2\pi} C_F \left[-2\ln z \frac{1+z^2}{(1-z)} + 4(1+z^2)\left(\frac{\ln(1-z)}{(1-z)}\right)_+ \right] \right\},
\end{aligned} \tag{31}
$$

where $\tau = x_1 x_2 z$ and $z = \dfrac{Q^2}{x_1 x_2 S}$. Let me make some comments on the above NLO results.

(a) Now the NLO cross section $\dfrac{d\sigma}{dQ^2}$ is finite, and it is independent of M_f^2 in principle. But the factorization scale M_f^2 should not be too far away from Q^2 to ensure small NLO in correction the hard factor $\mathcal{H}(z, Q, M_f)$. In practice, we use the DGLAP evolved PDFs which resum $\alpha_s \ln M_f^2/\mu^2$, therefore, the above NLO cross section $\dfrac{d\sigma}{dQ^2}$ has the uncanceled M_f dependence at α_s^2 order.

(b) Interesting phenomenon starts to show up when $\tau \to 1$, which is known as the threshold limit. In this situation, $z \to 1$ and $x_{1,2} \to 1$ which generates large logarithms such as $\ln^2(1-\tau)$ and $\ln(1-\tau)$ coming from the plus-functions (You can see this from Eq. (24) after identifying a as τ in the DY process). Due to convolutions in the above expression, it is more convenient to go to Mellin space to perform the threshold resummation ($\ln^2(1-\tau) \longleftrightarrow \ln^2 N$), then transform back numerically. See Ref. [4] for more detail. It is worth noting that, instead of the $\overline{\text{MS}}$ scheme, a different scheme (for example the DIS scheme) may be used in literatures.

In the Mellin space, it is interesting to note that everything factorizes as illustrated below

$$
\begin{aligned}
&\int_0^1 d\tau \tau^{N-1} \int dz \int dx_1 \int dx_2 q(x_1) \bar{q}(x_2) \delta(\tau - z x_1 x_2) f_+(z) \\
&= \int_0^1 dz z^{N-1} f_+(z) \int_0^1 dx_1 x_1^{N-1} \int_0^1 dx_2 x_2^{N-1} q(x_1)\bar{q}(x_2) = f_+(N) q(N) \bar{q}(N),
\end{aligned} \tag{32}
$$

where $f_+(N) \equiv \int_0^1 dz z^{N-1} f_+(z)$. Here $f_+(z)$ could be the terms such as $P_{qq}(z)$ or $\left(\dfrac{\ln(1-z)}{(1-z)}\right)_+$ in $\mathcal{H}(z, Q, M_f)$. It is useful to note that the Mellin transform can be done with the help of the following identities and tricks.

$$
\int_0^1 dz z^{N-1} \frac{1}{(1-z)_+} = \lim_{e\to 0} \int_0^1 \frac{z^{N-1}-1}{(1-z)^{1+e}} = \lim_{e \to 0} \frac{1}{e}\left[1 - \frac{\Gamma(1-e)\Gamma(N)}{\Gamma(N-e)}\right] \simeq -\gamma_E - \ln N + \mathcal{O}\left(\frac{1}{N}\right), \tag{33}
$$

$$
\int_0^1 dz z^{N-1} \left(\frac{\ln(1-z)}{1-z}\right)_+ \simeq \frac{1}{2}(\gamma_E + \ln N)^2 + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{N}\right), \tag{34}
$$

where the second line can be derived by using the same trick as in the first identity with an additional derivative w.r.t. e. For sufficiently large N, the Mellin transform is dominated by the end point around $z \sim 1 - 1/N$.

In the Mellin space, the resummation w.r.t. to $\alpha_s \ln^2 N$ and $\alpha_s \ln N$ gives rise to the Sudakov factor which reads

$$
\Delta_N = \exp\left(2\frac{\alpha_s C_F}{\pi} \ln^2 N + \mathcal{O}(\ln N) + \cdots\right). \tag{35}
$$

(c) The above approach for the threshold resummation is quite intuitive and straightforward. However, the Landau pole problem in the Sudakov factor may occur in the case of running coupling (see Ref. [5] for a different modern formulation of the threshold resummation based on SCET).

3.2 Transverse momentum (Q_T) distribution resummation in DY process

Now let us consider Q_T distribution of the lepton pair (γ^*) in hadronic collisions in the $Q_T \ll Q$ limit. We choose the kinematic region where Q^2 is not too close to S, so that threshold logs are not important.

At leading order, $Q_T = 0$ since q and $\bar{q}$ carries no k_T in collinear factorization. This can be cast into the following form

$$\frac{d\sigma_{DY}^{(LO)}}{dQ^2 d^2Q_T} = \sum_q \int \frac{dx_1}{x_1} q(x_1) \int \frac{dx_2}{x_2} \bar{q}(x_2) \frac{\sigma(q\bar{q})}{S} \delta(1-z) \frac{1}{(2\pi)^2} \int d^2 b_\perp e^{-iQ_T \cdot b_\perp}, \tag{36}$$

where the last integral yields $\delta^{(2)}(Q_T)$ as expected.

3.2.1 Momentum space analysis

Additional gluon radiation can generate non-zero Q_T. In particular, soft-gluon emissions are the dominant contribution in the $Q_T \ll Q$ limit. To simplify the calculation, let us consider the emission of a gluon from an energetic quark, which allows us to write the rate of the $q \to qg$ splitting as

$$P = \frac{\alpha_s C_F}{2\pi^2} \int \frac{d^2 k_\perp}{k_\perp} \int d\xi \frac{1+\xi^2}{1-\xi}\bigg|_{\xi \neq 1}, \tag{37}$$

where ξ is the longitudinal momentum fraction of the final state quark w.r.t. the parent quark. When $\xi = 1$, which implies that the radiated gluon carries zero momentum $(1-\xi)$, we need to include the virtual contribution and replace $\frac{1+\xi^2}{1-\xi}$ by $\left(\frac{1+\xi^2}{1-\xi}\right)_+$. The detail derivation of the above splitting function can be found in Peskin and other textbook on the DGLAP equation.

If we consider the gluon radiation at given $k_\perp$ on top of the Born process $q\bar{q} \to \gamma^*$ together with the kinematic constraint on the soft gluon radiation ($\frac{k_\perp^2}{(1-\xi)p^+ x_1} \leq x_2 p^- \to \xi \leq 1 - \frac{k_\perp^2}{Q^2}$), we can ontain

$$\frac{dP}{d^2 k_\perp} = \frac{\alpha_s C_F}{2\pi^2} \frac{1}{k_\perp} \int_0^{1-\frac{k_\perp^2}{Q^2}} d\xi \frac{1+\xi^2}{1-\xi} = \frac{\alpha_s C_F}{\pi^2} \frac{1}{k_\perp} \ln \frac{Q^2}{k_\perp^2} + \cdots. \tag{38}$$

Next, let us use the same trick as employed in the thrust calculation by considering a partially integrated rate

$$F(Q_T) = \int_0^{Q_T} d^2 k_\perp \frac{dP}{d^2 k_\perp} \simeq 1 - \int_{Q_T}^{Q} d^2 k_\perp \frac{dP}{d^2 k_\perp} \simeq 1 - \frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}. \tag{39}$$

Again we have used the fact that $\int_{\text{full space}} d^2 k_\perp \frac{dP}{d^2 k_\perp} = 1 + \mathcal{O}(\alpha_s)$ after taking the Born contribution into account as in the thrust case. Now the above result should look pretty familiar to you.

Furthermore, I wish to convince you that soft gluon radiations factorize kinematically and factorize in color space. The former factorization in kinematics can be easily seen by applying the Dirac equation as in Page 202 of Peskin if the soft gluon is radiated from an energetic quark. (If the soft gluon is radiated from an energetic gluon, then we need to apply the Ward identity and Eikonal approximation to the triple gluon vertex. The rest of the derivation is identical to the quark case.) The latter factorization can be understood if you try to compute the color factor of two gluon radiations and find the color factor of the leading logarithmic contribution is C_F^2 as a simple exercise. For more complete discussion on this issue, see Refs. [6, 7].

In the end, after summing over arbitrary number of identical gluon emissions, we obtain

$$F(Q_T) = \sum_0^\infty \frac{(-1)^n}{n!} \left[\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]^n = \exp\left[-\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]. \tag{40}$$

Similar to the thrust distribution, we can differentiate $F(Q_T)$ and get

$$\frac{dP}{dQ_T^2} = \frac{1}{\sigma} \frac{d\sigma}{dQ_T^2} = \frac{\alpha_s C_F}{\pi} \frac{\ln \frac{Q^2}{Q_T^2}}{Q_T^2} \exp\left[-\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]. \tag{41}$$

Several comments are in order.

- $F(Q_T)$ is usually referred as the Sudakov form factor for which can be interpreted as the probability for emitting no gluons with transverse momentum greater than Q_T.
- When $Q_T \to 0$, $F(Q_T) \to 0$ means that lepton pairs always have non-zero transverse momentum since gluon radiation is inevitable. When $q\bar{q}$ annihilate into a virtual photon, the gluon clouds as part of the quark wave function have to be release due to the annihilation.
- Although this result is qualitatively correct, there is one problem with it! We should take into account the transverse momentum conservation for arbitrary number of gluon radiations. This can be achieved in coordinate space.

3.2.2 Resummation in coordinate space

The key problem now is to transform everything to the coordinate space which naturally conserves the transverse momentum as shown below.

$$\begin{aligned}
&\text{0th order} \quad \frac{dP_0}{d^2Q_T} = \delta^{(2)}(Q_T) \\
&\text{1th order} \quad \frac{dP_1}{d^2Q_T} = \frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp \ln\frac{Q^2}{k_\perp^2}\delta^{(2)}(Q_T - k_\perp) \\
&\text{2th order} \quad \frac{dP_2}{d^2Q_T} = \frac{1}{2!}\left(\frac{\alpha_s C_F}{\pi^2}\right)^2 \int d^2k_{1\perp}\ln\frac{Q^2}{k_{1\perp}^2}\int d^2k_{2\perp}\ln\frac{Q^2}{k_{2\perp}^2}\delta^{(2)}(Q_T - k_{1\perp} - k_{2\perp}) \\
&\cdots,
\end{aligned}$$

where we have taken the soft gluon limit and assumed that soft gluons factorize. Now use the identity

$$\delta^{(2)}(Q_T - k_{1\perp} - k_{2\perp} - \cdots - k_{n\perp}) = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-i(Q_T - k_{1\perp} - k_{2\perp} \cdots - k_{n\perp})\cdot b_\perp}, \tag{42}$$

and define the Sudakov factor $S(b_\perp) = -\frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp \frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2}e^{ik_\perp\cdot b_\perp}$, eventually one can arrive at

$$\frac{dP}{d^2Q_T} = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-iQ_T\cdot b_\perp}\sum_{n=0}^{\infty}\frac{1}{n!}[-S(b_\perp)]^n = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-iQ_T\cdot b_\perp}e^{-S(b_\perp)}. \tag{43}$$

The above result is one of the main results which shows the exponentiation of the Sudakov factor in the coordinate space together with the consideration of transverse momentum conservation.

The rest of the task is then to compute the Sudakov factor $S(b_\perp)$. Immediately, we can realize that the above definition of $S(b_\perp)$ is not convergent at $k_\perp \to 0$ limit when we perform the Fourier transform. The divergence is simply due to the fact that we forget to include the virtual contribution. By adding the virtual contribution, we arrive at

$$S(b_\perp) = -\frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp e^{ik_\perp\cdot b_\perp}\left[\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2} - \delta^{(2)}(k_\perp)\int^{Q^2} d^2l_\perp \frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\right], \tag{44}$$

where the upper limit of the virtual contribution is set by Q^2 due to the cancellation of UV divergences between two types of virtual diagrams in Drell-Yan processes. The evaluate the above integrals, we can adopt the dimensional regularization and use the following identities

$$\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{1}{2}\ln^2\frac{Q^2 b_\perp^2}{c_0^2} - \frac{\pi^2}{12}\right], \tag{45}$$

$$\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}l_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\bigg|_{l_\perp<Q} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{\pi^2}{12}\right], \tag{46}$$

where more details regarding this can be found in the appendix of Ref. [8]. Thus, due to the incomplete cancellation between real and virtual contributions (The real part has constraints due to the measurement at fixed Q_T, while no constraint is imposed on the virtual diagram.), a potentially large logarithmic contribution remains

$$S(b_\perp) = \frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2 b_\perp^2}{c_0^2}, \tag{47}$$

where $c_0 = 2e^{-\gamma_E}$. At the end of the day, we can then modify Eq. (36) and obtain the Sudakov resummation formula for DY processes at leading double logarithmic level

$$\frac{d\sigma_{DY}^{(LL)}}{dQ^2 d^2 Q_T} = \sum_q \frac{\sigma(q\bar{q})}{S} \frac{1}{(2\pi)^2} \int d^2 b_\perp e^{-iQ_T \cdot b_\perp} e^{-S(b_\perp)} \int \frac{dx_1}{x_1} q(x_1, \mu) \int \frac{dx_2}{x_2} \bar{q}(x_2, \mu) \delta(1-z). \tag{48}$$

Again, a few remarks are in order regarding the Sudakov resummation in DY processes:

- The original and more complete derivation at next-to-leading-logarithmic level by Collins, Soper and Sterman can be found in Ref. [9]. What we have done above is just an intuitive way of understanding the Sudakov double logarithm term which is also known as the A term. Again the physics behind the A term is the incomplete cancellation of soft divergences due to the imposed constraint.

- We can also include the single log term known as the B term $-\frac{3\alpha_s C_F}{2\pi} \ln \frac{Q^2 b_\perp^2}{c_0^2}$, which arises due to the incomplete cancellation of the collinear divergence. In addition, to simplify the above resummed expression, it is the common practice to set $\mu = c_0/b_\perp$ in quark distributions $q(x_1, \mu)$ and $\bar{q}(x_2\, \mu)$.

- Putting everything altogether including the running coupling effect, the Sudakov factor usually is written as

$$S(b_\perp) = \int_{c_0^2/b_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\ln \frac{Q^2}{\bar{\mu}^2} A(\alpha_s) + B(\alpha_s) \right], \tag{49}$$

  where $A(\alpha_s) = \sum_n \left(\frac{\alpha_s}{2\pi}\right)^n A_n$ and $B(\alpha_s) = \sum_n \left(\frac{\alpha_s}{2\pi}\right)^n B_n$ with $A_1 = 2C_F$ and $B_1 = -3C_F$ for the quark channel. Higher order (for example two-loop) calculations yield the coefficients A_n and B_n with $n > 1$.

  The above expression is usually derived from the CSS evolution[9] of the $W(Q, b_\perp)$ function

$$\frac{\partial}{\partial \ln Q^2} W(Q, b_\perp) = [K(\mu, b_\perp) + G(Q, \mu)]\, W(Q, b_\perp), \tag{50}$$

  where $K + G = -\frac{\alpha_s C_F}{\pi}\left[\ln \frac{Q^2 b_\perp^2}{c_0^2} - \frac{3}{2}\right]$ at one-loop level with $K = -\frac{\alpha_s C_F}{\pi} \ln \frac{\mu^2 b_\perp^2}{c_0^2}$ the soft part of the evolution kernel and $G = -\frac{\alpha_s C_F}{\pi}\left[\ln \frac{Q^2}{\mu^2} - \frac{3}{2}\right]$ the hard part of the evolution kernel. These two kernels are both related to the cusp anomalous dimension γ_K

$$\frac{\partial}{\partial \ln \mu} K(\mu, b_\perp) = -\gamma_K = -\frac{\partial}{\partial \ln \mu} G(Q, \mu). \tag{51}$$

  In fact, the solutions to the above evolution equations resums the Sudakov logarithms.

- One interesting pattern regarding the Sudakov resummation is that it always appear in the case of multiple distinct scales (Q and Q_T in this example). If one integrates over Q_T by considering the total rate, then the Sudakov factor disappears. As discussed before, the Sudakov logs are due to the incomplete cancellation of real and virtual graphs when Q_T is fixed. If one integrates Q_T, such constraint imposed on the real graphs is then removed, thus we expect the “complete" cancellation as in inclusive observables.

- The full one-loop calculation also produce the constant correction known as the C term $\frac{\alpha_s}{2\pi} C_F \left[\pi^2 - 8\right]$. Usually the C term is not resummed.

- Non-perturvative Sudakov factor is also employed in phenomenology in order to regularize the large $b_\perp$ (small momentum) region. We have computed only the perturbative part in the above calculation which corresponds to the small $b_\perp$ region. When $b_\perp$ is as large as $1/\Lambda_{QCD}$, we should turn on the non-perturbative Sudakov factor and adopt the so-called $b_* \equiv \frac{b_\perp}{\sqrt{b_\perp^2/b_{max}^2+1}}$ prescription[9]. Therefore, in practice, Sudakov factor is usually the sum of the perturbative and NP part as follows

$$S^{tot}(b_\perp) = S_P(b_*) + S_{NP}(b_\perp), \tag{52}$$

where $S_P(b_*)$ is computed from perturbations while $S_{NP}(b_\perp)$ is usually fitted from DY and SIDIS experimental data. Nevertheless, $S_{NP}(b_perp)$ is not important in very high energy collisions.

One important related question: Can we trust the perturbative Sudakov factor in the $Q_T \sim 0$ region? The answer is yes. The reason is that the integrand before we perform the Fourier transform is strongly peaked at small-$b_\perp$ region, which implies that the dominant contribution comes from the small-$b_\perp$ region, i.e., the perturbative region.

- Sudakov resummation becomes insufficient and unnecessary in the region $Q_T \sim Q$, where the pQCD expansion is sufficient and accurate.

4 Jet productions and Sudakov Resummation

A jet is a narrow cone (of size R) of hadrons and other particles produced by a quark or a gluon in high energy collisions. It is supposed to reflect the properties of the original quark or gluon as the surrogate.

4.1 Jet mass resummation

It is also worth noting that the resummation of jet mass distribution is also Sudakov type. Detailed discussion can be found in Ref. [10, 11]. Similar to our previous discussion on the thrust distribution, we can also define a new physical observable, i.e., the jet mass $M_J^2 = (\sum_{i\in \text{jet}} p_i)$. It is very useful in distinguishing quark jets from gluon jets as we shall see below.

First of all, jet mass M_J of a single parton is trivially zero. For two partons, $M_J = 0$ only when one of the two partons is either soft or collinear. To obtain non-trivial mass, we need to consider the branching of gluons from a high energy quark or gluon. By definition, this yields

$$M_J^2 = (l_1 + l_2)^2 = 2 l_1 \cdot l_2 = l_{1\perp} l_{2\perp} \left(\Delta y^2 + \Delta\phi^2\right), \tag{53}$$

where Δy and $\Delta\phi$ are the rapidity and azimuthal angle difference between two branches and $\theta^2 = \Delta y^2 + \Delta\phi^2$. Without losing much generosity, one can suppose the original parton has the transverse momentum $P_\perp$ with rapidity $y = 0$, then $M_J^2 = z(1-z)P_\perp^2\theta^2$ where $l_{1\perp} = zP_\perp$ and $l_{2\perp} = (1-z)P_\perp$. Here we have also assumed that the branching angle $\theta \ll 1$.

Recall that the branching of $q \to qg$ is given by (see Chapter 17 of Peskin)

$$\frac{\alpha_s}{2\pi} C_F \int \frac{dk_\perp^2}{k_\perp^2} \int dz \frac{1+(1-z)^2}{z}, \tag{54}$$

therefore, by using the δ—function trick in the above equation, we can obtain the jet mass distribution for nonzero mass m as follows

$$\frac{m^2 d\sigma}{\sigma dm^2} = \frac{\alpha_s}{2\pi} C_F m^2 \int_0^{R^2} \frac{d\theta^2}{\theta^2} \int dz \frac{1+(1-z)^2}{z} \delta\left(m^2 - z(1-z)\theta^2 P_\perp^2\right) \quad \Rightarrow \tag{55}$$

$$\frac{\rho d\sigma}{\sigma d\rho} \simeq \frac{\alpha_s C_F}{2\pi}\left(2\ln\frac{1}{\rho} - \frac{3}{2}\right) \quad \text{with} \quad \rho \equiv \frac{m^2}{P_\perp^2 R^2}. \tag{56}$$

It is also interesting to recall that the above distribution should include the virtual and LO contribution at $m=0$. Again using the same trick discussed in previous sections, we can define

$$\Sigma = \frac{1}{\sigma_0}\int_0^\rho \frac{d\sigma}{\sigma d\rho'}d\rho' = 1-\frac{1}{\sigma_0}\int_\rho^1 \frac{d\sigma}{\sigma d\rho'}d\rho' = 1-\frac{\alpha_s C_F}{2\pi}\left(\ln^2\frac{1}{\rho}-\frac{3}{2}\ln\frac{1}{\rho}\right). \tag{57}$$

The rest of the discussion then follows closely the Sudakov resummation of the thrust distribution which gives

$$\Sigma^{\text{quark}}_{\text{resummed}} = \exp\left[-\frac{\alpha_s C_F}{2\pi}\left(\ln^2\frac{1}{\rho}-\frac{3}{2}\ln\frac{1}{\rho}\right)\right]. \tag{58}$$

In addition, it is straightforward to compute jet mass distribution for the gluon channel and find that the only two changes are $C_F \to C_A$ and $\frac{3}{2} \to 2\beta_0 = \frac{11}{6} - \frac{N_f}{9}$. This implies that quark and gluon jets have numerically different jet mass distributions.

4.2 Sudakov Resummation in Dijet Productions

At last, I would like to briefly mention the Sudakov resummation in dijet angular (azimuthal) correlation in hadronic collisions[12].

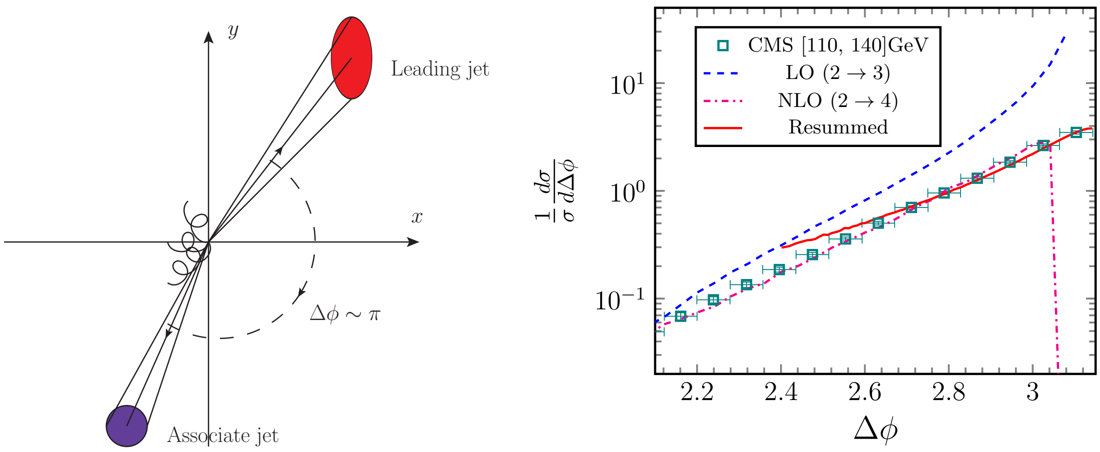


Figure 1: The angular correlation of inclusive dijet data compared with theoretical calculations.

As shown above, the angular correlation between the leading jet (the jet with the largest P_T) and the associate jet (the jet with the second largest P_T) can be measured in proton-proton collisions at the LHC. In the right plot of Fig. 2, one can see that the perturbative QCD framework, i.e., the collinear factorization can describe the large angle region. However, due to the appearance of large logarithms such as $L \sim \ln^2\frac{P_\perp^2}{q_\perp^2}$ with $P_\perp \gg q_\perp$, pQCD expansion eventually breaks down in the back-to-back region. Here $P_\perp$ is approximately the leading jet P_T, while $q_\perp$ is the transverse momentum imbalance between the leading jet and the associate jet. One needs to employ the Sudakov resummation formalism in order to resum those large logarithms and describe the data in the back-to-back region.

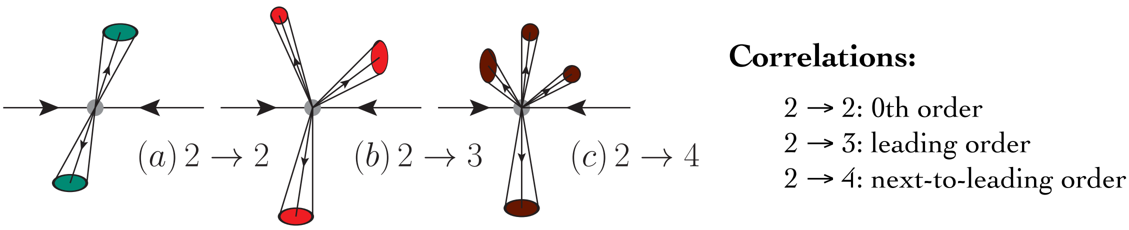


Figure 2: Inclusive dijet productions in terms of perturbative expansions.

In the collinear factorization, the incoming partons carry no transverse momentum, therefore the $2 \to 2$ process as shown in Fig. 2 gives no contribution to dijet angular correlations other than a delta function.

The first non-trivial contribution comes from the $2 \to 3$ process and the NLO correction to dijet correlations starts at the $2 \to 4$ order together with $2 \to 3$ virtual graphs. However, no matter how high order we go, we always get divergent results for dijet productions in the back-to-back region due to soft (and/or collinear) gluon emissions. Let us consider the $2 \to 3$ case in which the leading jet and the associate jet are exactly back-to-back, i.e., $\Delta\phi = \pi$. It is easy to see that the unobserved parton (the third parton in the final state) can be soft and therefore introduce large double logarithms. The large Sudakov logarithms in the $\Delta\phi \sim \pi$ region make the pQCD expansion insufficient and eventually break down.

There are two types of soft gluon emission in this process, i.e., the initial state and final state gluon radiations. If a gluon is radiated by the incoming parton before the hard collision, it is considered as an initial state gluon. The final state gluon is radiated from the outgoing final state parton after the hard collision. The divergent behavior is caused by the soft and collinear gluon emission in the initial state as well as the soft gluon emission in the final state subject to the jet-cone regularization. It is important to notice that the collinear final state gluon contribution is removed due to the jet cone regularization.

For simplicity, let us consider the $q_i(k_1) + q_j(k_2) \to q_i(p_1) + q_j(p_2)$ channel in dijet productions, and define momentum imbalance $\vec{q}_\perp \equiv \vec{p}_{1\perp} + \vec{p}_{2\perp}$, jet momenta $P_\perp \sim p_{1\perp} \sim p_{2\perp}$. In the limit $P_\perp \gg q_\perp$, we need to resum multiple soft gluon emission and use the following resummed formula

$$\frac{d\sigma^{ij}_{\text{dijet}}}{dy_1 dy_2 d^2p_{1\perp} d^2p_{2\perp}} = \sigma_{ij} \int \frac{d^2b_\perp}{(2\pi)^2} e^{-iq_\perp \cdot b_\perp} W(Q, b_\perp), \tag{59}$$

$$\text{with} \quad W(Q, b_\perp) = x_1 f_i(x_1, \mu_b) x_2 f_j(x_2, \mu_b) e^{-S(Q, b_\perp)}, \tag{60}$$

$$S(Q, b_\perp) = S_{pert}(Q, b_*) + S_{NP}(Q, b_\perp) \tag{61}$$

$$S_{pert}(Q, b_*) = \int_{\mu_b^2 = c_0^2/b_*^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A \ln \frac{Q^2}{\mu^2} + B + (D_1 + D_2) \ln \frac{1}{R^2} \right]. \tag{62}$$

Several comments regarding the above Sudakov resummed formula are in order.

- As before, soft gluon emissions factorize from the Born cross section $\sigma_{ij} = \frac{\alpha_s^2 \pi}{s} \frac{4}{9} \frac{s^2 + u^2}{t^2}$, where s, t, u are the normal partonic Mandelstam variables. For other channels, the Born cross section can be found in Chapter 17 of Peskin.
- The kinematics give $Q^2 = x_1 x_2 S$ and $x_{1,2} = \frac{P_\perp}{S}\left(e^{\pm y_1} + e^{\pm y_2}\right)$ with the produced jet rapidity y_1 and y_2.
- All the $A = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n A^{(n)}$, $B = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n B^{(n)}$, $D = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n D^{(n)}$ coefficients can be computed perturbatively. One-loop calculation gives the contribution of $n = 1$ coefficients, etc. A and B coefficients are associated with initial state Sudakov radiations. For each initial state quark, there is a contribution of $\frac{\alpha_s}{2\pi} C_F$ to A ($A^{(1)} = C_F$) and a contribution of $-\frac{\alpha_s}{2\pi}\frac{3}{2} C_F$ to B ($B^{(1)} = -\frac{3}{2} C_F$). For each incoming gluon, the corresponding contributions to A and B are $\frac{\alpha_s}{2\pi} C_A$ and $-\frac{\alpha_s}{2\pi} 2\beta_0 C_A$.
- The D term is due to final state radiation and it is $\frac{\alpha_s}{2\pi} C_F$ for final state quark jets and $\frac{\alpha_s}{2\pi} C_A$ for final state gluon jets. In deriving this result, the so-called narrow cone approximation has been made which requires $R^2 \ll 1$.
- The C term, which is not written in above equation, is just constant higher order α_s correction and it is not resummed usually.
- Other complication such as the soft factor will not be discussed here. For more information, see Ref. [12]. Again, we use b_* prescription to separate perturbative and non-perturbative regions.

The procedure of deriving the above result is very similar to what we have done before for DY process. The main new ingredient in this calculation is the final state gluon radiation. First of all, the virtual contribution to the Born diagrams, which is proportional to $\frac{1}{\epsilon^2}$ in dim-reg, is universal and it has been computed in Ref. [13]. Second, the contribution from the final state gluon radiation inside the jet cone is similar to the virtual contribution since it is also proportional to the Born cross section and $\frac{1}{\epsilon^2}$. Third, soft gluon radiations also has soft divergence $\frac{1}{\epsilon^2}$ as expected. Last, we can also have gluon radiation collinear to the incoming partons. It is worth noting that the final state collinear divergence is regularized by the jet cone algorithm.

At the end of the day, the $\frac{1}{\epsilon^2}$ divergences from the first three contributions exactly cancel and the finite remaining terms yield the Sudakov logarithms in the A and D terms. The collinear divergence in the last contribution should be renormalized into the incoming parton distribution and the leftover single logarithm gives contribution to the B term. Again, these logarithms arise due to the incomplete cancellation between the real and virtual contributions.

Recently, in order to probe the properties of the cold and hot nuclear medium, there have been increasing interests in applying Sudakov resummation to hard processes in heavy ion collisions. The joint resummation of small-x logarithms and Sudakov logarithms is discussed in Refs. [8, 14]. Dijets can be used to probe the quark gluon plasma created in heavy ion collision[15, 16]. The Sudakov factor which takes vacuum parton shower into account plays an important role, since it helps to establish the baseline in proton-proton collisions.

References

[1] E. Farhi, Phys. Rev. Lett. **39**, 1587 (1977).

[2] J. R. Ellis and I. Karliner, Nucl. Phys. B **148**, 141 (1979).

[3] M. D. Schwartz, “Quantum Field Theory and the Standard Model.”

[4] S. Catani and L. Trentadue, Nucl. Phys. B **327**, 323 (1989).

[5] T. Becher and M. Neubert, Phys. Rev. Lett. **97**, 082001 (2006) [hep-ph/0605050].

[6] J. G. M. Gatheral, Phys. Lett. **133B**, 90 (1983).

[7] J. Frenkel and J. C. Taylor, Nucl. Phys. B **246**, 231 (1984).

[8] A. H. Mueller, B. W. Xiao and F. Yuan, Phys. Rev. D **88**, no. 11, 114010 (2013) [arXiv:1308.2993 [hep-ph]].

[9] J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **250**, 199 (1985).

[10] A. J. Larkoski, arXiv:1709.06195 [hep-ph].

[11] S. Marzani, G. Soyez and M. Spannowsky, Lect. Notes Phys. **958**, pp. (2019) [arXiv:1901.10342 [hep-ph]].

[12] P. Sun, C.-P. Yuan and F. Yuan, Phys. Rev. D **92**, no. 9, 094007 (2015) [arXiv:1506.06170 [hep-ph]].

[13] R. K. Ellis and J. C. Sexton, Nucl. Phys. B **269**, 445 (1986).

[14] A. H. Mueller, B. W. Xiao and F. Yuan, Phys. Rev. Lett. **110**, no. 8, 082301 (2013) [arXiv:1210.5792 [hep-ph]].

[15] A. H. Mueller, B. Wu, B. W. Xiao and F. Yuan, Phys. Lett. B **763**, 208 (2016) [arXiv:1604.04250 [hep-ph]]; Phys. Rev. D **95**, no. 3, 034007 (2017) [arXiv:1608.07339 [hep-ph]].

[16] L. Chen, G. Y. Qin, S. Y. Wei, B. W. Xiao and H. Z. Zhang, Phys. Lett. B **773**, 672 (2017) [arXiv:1607.01932 [hep-ph]]; Phys. Lett. B **782**, 773 (2018) [arXiv:1612.04202 [hep-ph]].